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Densities and harmonic measure

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Abstract.

Several notions of densities related to zero sequences, interpolating sequences and sampling sequences of holomorphic functions are presented. Some ties with harmonic measure estimates are shown.

§1. Introduction

In this survey we will present several notions of densities and its relation to some classical problems in function theory. We show how some of these densities can be computed through precise estimates of the harmonic measure on conveniently crafted domains. This new interpretation of the densities may be useful in the extension of the classical function theory in the disk to other domains or Riemann surfaces.

The results that we present here are not new, and we will point to the sources along the exposition. There exists a nice book [16] with the state of the art on the problems of interpolation and sampling sequences. If one is interested in an (elementary) survey on motivation of these problems and its conection to signal analysis see for instance [3] and the references therein.

§2. Different densities

Given a sequence of points Λ in $\mathbb{R}$ or $\mathbb{C}$ we will define different quantities $\mathcal{D}(\Lambda)$ that try to provide a mathematical definition to the intuitive concept of the density of the sequence. There are several possibilities as we will see, and each of these appeared in the literature to deal with

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different problems of function theory. The heuristic principle behind all the results that we present is that the density of the zero sequence of an holomorphic function is controlled by the growth of such function. Of course this is what lies behind Nenvanlinna theory but to begin with we would like to recall a classical result, the Beurling-Malliavin theorem:

2.1. The Beurling-Malliavin density

The Paley-Wiener PW_τ space consists of entire functions of exponential type lower or equal than τ ($|f(z)| \leq Ce^{\tau|\Im z|}$) and $f \in L^2(\mathbb{R})$.

Let us first discuss uniqueness sets $\Lambda \subset \mathbb{R}$ for PW_τ, that is, sets for which $f \in PW_\tau$ and $f(\lambda) = 0 \ \forall \lambda \in \Lambda$ implies $f \equiv 0$. Since every $f \in PW_\tau$ is entire, it is clear that every set Λ with a finite accumulation point is a uniqueness set; it is also transparent that a finite set cannot be a uniqueness set, so we assume from now on that Λ is an infinite sequence without acumulation points. It is intuitively clear that Λ must be dense in some sense, so that $f_{|\Lambda} = 0$ implies $f = 0$. Now, if $f \in PW_\tau$ and $f(\alpha) = 0$, then the function $g(z) = f(z)\frac{(z-\beta)}{(z-\alpha)}$ is again in PW_τ and $g(\beta) = 0$; this means that we can move arbitrarily any finite number of points of Λ without changing the problem. Consequently, the control on the density of the sequences Λ should be asymptotic, depending just on how Λ behaves "at infinity". In a series of deep and very celebrated papers, Beurling and Malliavin (see for instance [6]) proved some results giving an almost complete description of uniqueness sets for PW_τ. They introduced a density $\mathcal{D}_{BM}(\Lambda)$, called now the *Beurling-Malliavin density*, and proved the following

Theorem 2.1. *If a real sequence Λ satisfies $\mathcal{D}_{BM}(\Lambda) > 2\tau$ then Λ is a uniqueness set for PW_τ. Conversely if Λ is a uniqueness set for PW_τ then $\mathcal{D}_{BM}(\Lambda) \geq 2\tau$.*

The definition of $\mathcal{D}_{BM}(\Lambda)$ is complicated, but geometric in nature. It is called a density because the number $\mathcal{D}_{BM}(\Lambda)$ depends on how many points does Λ have in big intervals. It is closely related to the classical density

$$\overline{\mathcal{D}}(\Lambda) = \overline{\lim_{r\to 0}} \frac{n_\Lambda(r)}{2r},$$

where $n_\Lambda(r)$ indicates the number of points of Λ in $[-r, r]$. In particular, $\mathcal{D}_{BM}(\Lambda) \geq \overline{\mathcal{D}}(\Lambda)$. The precise definition is the following: Let Λ be a sequence of real numbers contained in $(0, +\infty)$. We fix $A > 0$ and we let

$$S_A(\Lambda) = \left\{ t > 0; \ \frac{n_\Lambda(\tau) - n_\Lambda(t)}{\tau - t} > A \text{ for some } \tau > t \right\}.$$

The set $S_A(\Lambda)$ is of the form $\bigcup_k (a_k, b_k)$ and we define

$$\|S_A(\Lambda)\| = \sum_k \frac{(b_k - a_k)^2}{a_k^2}.$$

Finally the density $\mathcal{D}_{BM}(\Lambda)$ is the infimum of all A such that $\|S_A(\Lambda)\| < \infty$. If the sequence Λ is real but not strictly positive we define $\Lambda_+ = \Lambda \cap (0, +\infty)$, $\Lambda_- = -(\Lambda \cap (-\infty, 0))$ and

$$\mathcal{D}_{BM}(\Lambda) = \max(\mathcal{D}_{BM}(\Lambda_+), \mathcal{D}_{BM}(\Lambda_-)).$$

The exact description of the uniqueness sets for PW_τ remains however unsolved.

2.2. The Beurling-Nyquist density

If one is interested instead in a uniqueness problem with stability, then the problem becomes the following. Describe the sequences $\Lambda \subset \mathbb{R}$ such that

$$\sum_\Lambda |f(\lambda)|^2 \lesssim \int_\mathbb{R} |f|^2 \lesssim \sum_\Lambda |f(\lambda)|^2, \tag{2.1}$$

for all functions $f \in PW_\tau$. For simplicity we will assume that Λ is *separated*, i.e. $\inf_{\lambda \neq \lambda'} |\lambda - \lambda'| > 0$. The separated sequences that satisfy (2.1) are called sampling sequences for the Paley-Wiener space and they are very important in signal analysis because these are the sequences that allow a stable discretization of band-limited and finite energy signals. Their description can almost be achieved with a density very much like in the case of uniqueness sequences with the Beurling-Malliavin theorem. The following result was proved by Beurling (see [1]):

Theorem 2.2. *If Λ is a uniformly separated real sequence and $\mathcal{D}^-_{BN}(\Lambda) > \tau$ then Λ is a sampling sequence for PW_τ. Conversely if Λ is a sampling sequence for PW_τ then $\mathcal{D}^-_{BN}(\Lambda) \geq \tau$.*

The lower Beurling-Nyquist density $\mathcal{D}^-_{BN}(\Lambda)$ is defined as

$$\mathcal{D}^-_{BN}(\Lambda) = \lim_{r\to\infty} \min_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x, r+x))}{r}.$$

The corresponding upper Beurling-Nyquist density $\mathcal{D}^+_{BN}(\Lambda)$ is defined as

$$\mathcal{D}^+_{BN}(\Lambda) = \lim_{r\to\infty} \max_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x, r+x))}{r},$$

which is related to the following interpolation problem. For which separated $\Lambda \subset \mathbb{R}$, the restriction $PW_\tau \to \ell^2$, $f \to \{f(\lambda)\}$ is onto? This sequences are called interpolating sequences and they are relevant in one wants to codify a discrete signal over a continuous band-limited signal. The corresponding theorem by Beurling states

Theorem 2.3. *If Λ is a separated real sequence such that $\mathcal{D}^+_{BN}(\Lambda) < \tau$ then Λ is an interpolating sequence for PW_τ. Conversely if Λ is an interpolating sequence then $\mathcal{D}^+_{BN}(\Lambda) \leq \tau$.*

Again the critical case where $\mathcal{D}^+_{BM}(\Lambda) = \mathcal{D}^-_{BM}(\Lambda) = \tau$ is not covered by the theorems. Sampling and interpolating sequences for the Paley-Wiener space have been recently described in [10] and [16] but the notions involved are more delicate.

2.3. The Bergman space, Seip's density

There are similar notions of sampling and interpolation for the Bergman space. The weighted Bergman space B_τ is defined as the holomorphic functions in the disk such that

$$\|f\|^2_{B_\tau} := \int_{\mathbb{D}} |f|^2(1-|z|)^{2\tau-1} < +\infty.$$

A sequence $\Lambda \subset \mathbb{D}$ is *separated* in this context if $\inf_{\lambda\neq\lambda'} \rho(\lambda,\lambda') > 0$, where $\rho(z,w) = |z-w|/|1-\bar{w}z|$ is the pseudohyperbolic distance.

The sampling sequences for the Bergman space are those sequences Λ such that

$$\|f\|^2_{B_\tau} \simeq \sum |f(\lambda)|^2(1-|\lambda|)^{2\tau+1}.$$

and Λ is an interpolating sequence for the Bergman space whenever for any sequence of values $\{v_\lambda\}$ such that $\sum |v_\lambda|^2(1-|\lambda|)^{2\tau+1} < +\infty$ there is a function in B_τ such that $f(\lambda) = v_\lambda$. Again there is a corresponding notion of density that describes the uniformly separated sampling (or interpolating) sequences in the Bergman space. This density was introduced by Seip in [13] and it is defined as

$$\mathcal{D}^+_S(\Lambda) = \limsup_{r\to 1} \sup_{z\in\mathbb{D}} \frac{\sum_{\rho(\lambda,z)<r} 1-\rho(z,\lambda)}{\log 1/(1-r)},$$
$$\mathcal{D}^-_S(\Lambda) = \liminf_{r\to 1} \inf_{z\in\mathbb{D}} \frac{\sum_{\rho(\lambda,z)<r} 1-\rho(z,\lambda)}{\log 1/(1-r)}.$$

The corresponding theorem is

Theorem 2.4. *A separated sequence Λ is interpolating for B_τ if and only if $\mathcal{D}^+_S(\Lambda) < \tau$ and it is sampling if and only if $\mathcal{D}^-_S(\Lambda) > \tau$.*

2.4. Korenblum density

We introduce now a new density that almost describes the zeros in the Bergman space in the same sense that the Beurling-Malliavin almost describes the zeros of the Paley-Wiener space. This density was introduced by Korenblum in [7].

Given a finite set E of points in $\mathbb{T}$ we define the Beurling-Carleson entropy of E as

$$\hat{\kappa}(E) = \sum_k \frac{|I_k|}{2\pi}\left(\log\frac{2}{\pi}|I_k| + 1\right),$$

where I_k are the arcs complementary to E in $\mathbb{T}$. To each set E we associate to it the Korenblum flower $F(E)$ as the union of Stolz regions with vertex on E, i.e. $\{z \in \mathbb{D}; d(\frac{z}{|z|}, E) \le 1 - |z|\}$. Finally let $\sigma(\Lambda, E)$ be the Blashke sum of the points of Λ that are inside the Korenblum flower, i.e.

$$\sigma(\Lambda, E) = \sum_{\lambda\in\Lambda\cap F(E)} \log 1/|\lambda|.$$

The density $\mathcal{D}_K(\Lambda)$ is defined as the infimum of all $A > 0$ such that

$$\sup_{E\subset\mathbb{T}} (\sigma(\Lambda, E) - A\hat{\kappa}(E)) < +\infty.$$

The following theorem is a refinement of Seip [15] and [14] of a previous work by Korenblum [7]

Theorem 2.5. *If Λ is a sequence in the unit disk such that $\mathcal{D}_K(\Lambda) > \tau$ then Λ is a uniqueness set for B_τ and conversely if Λ is a uniqueness set for B_τ then $\mathcal{D}_K(\Lambda) \ge \tau$.*

In this context it should be noted that the original paper by Korenblum studied the zeros of functions in $\mathcal{A}^{-\infty} = \cup_{\tau>0} B_\tau$. The zeros, in view of Theorem 2.5 are the sequences such that $\mathcal{D}_K(\Lambda) < \infty$. The necessity of the density condition in the work of Korenblum was proved with a delicate study of the distortion of certain conformal mappings. There is a more elementary proof due to Bruna and Massaneda [2] that uses some estimates of the harmonic measure and that allows them to work in higher dimensions.

We will sketch this potential theoretic proof. Suppose that $f \in B_\tau$ with $f(0) = 1$ and $Z(f) = \Lambda$. Denote by $u = \log|f|$, u is a subharmonic function in the disk with the growth $u^+ \le C\log 1/(1 - |z|^2)$. Take any of the star shaped regions of Korenblum $F(E)$. Then

$$0 = u(0) = \int_{\partial F(E)} u(\zeta)d\omega(0,\zeta) - \int_{F(E)} g(0,\zeta,F(E))\Delta u(\zeta), \tag{2.2}$$

where $g(0,\zeta,F(E))$ is the Green function of $F(E)$ with pole at 0 and $\omega(0,\zeta)$ the hamonic measure evaluated at the origin. With a careful estimate of the harmonic measure it follows that

$$\int_{\partial F(E)} \log \frac{1}{1-|z|^2} d\omega(0,\zeta) \leq c\hat{\kappa}(E),$$

and more easily $g(0,y,F(E)) \geq c(1-|y|)$ for $|y| > 1/2$. Thus the necessary condition of Korenblum follows from (2.2) because $\Delta u = C\sum_{\lambda\in\Lambda}\delta_\lambda$.

This is part of a more general scheme, where the study of the zeros sequences of holomorphic functions is seen to be equivalent to the study of the Poisson equation $\Delta u = \mu$, where μ is a positive measure. We are interested in finding solutions u to the equation without any boundary restriction but with some growth estimates. The conection is clear since for any holomorphic function f, $u = \log|f|$ is a subharmonic function and viceversa for any solution u of $\Delta u = \sum_\Lambda \delta_\lambda$ there is an holomorphic function f such that $u = \log|f|$ and f vanishes on Λ. This connection has been exploited in many situations, see for instance [4].

2.5. Weighted densities

The study of these densities suggests the following pattern: The functions in the Paley-Wiener space are charactrized by the growth $e^{\tau|\Im z|}$ and the functions in the Bergman-space by $e^{\tau \log 1/(1-|z|^2)}$. In all cases the growth is of the type $e^{\phi(z)}$ where ϕ is a subharmonic function. This is very natural since $\log|f|$ is subharmonic whenever f is holomorphic, but the striking point is that the densities in both cases are related to $\Delta\phi$, in the case of the Paley-Wiener case this corresponds to τ times the Lebesgue meaure on the real line and in the weighted Bergman space to τ times the invariant measure on the disk. This is no coincidence, in general the density of the sampling, interpolating and zero sequences must be measured in the geometry of the manifold endowed with a metric related to the Laplacian of the weight.

Consider for instance the following situation. Take ϕ a subharmonic function in $\mathbb{C}$ with some mild regularity (doubling Laplacian, i.e. there is a $C > 0$ such that for all disks D, $\mu(2D) \leq C\mu(D)$ where μ denotes the positive measure $\mu = \Delta\phi$. Let $\rho(z)$ be the radius such that $\mu(D(z,\rho(z))) = 1$ (one has to think of ρ^2 as a sort of regularized $\Delta\phi$). Let $\mathcal{F}_\phi$ be the space of entire functions f such that $fe^{-\phi} \in L^\infty(\mathbb{C})$. The problem of describing interpolating and sampling sequences is the natural one in this setting. To solve it one has to introduce some densities tied to the metric in $\mathbb{C}$ induced by ρ.

Definition 2.1. A sequence Λ is *ρ-separated* if there exists $\delta > 0$ such that

$$|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda')) \qquad \lambda \neq \lambda'.$$

Definition 2.2. Assume that Λ is a ρ-separated sequence and recall that we denote $\mu = \Delta\phi$.

The *upper uniform density of* Λ with respect to $\Delta\phi$ is

$$\mathcal{D}^+_{\Delta\phi}(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#\big(\Lambda \cap \overline{D(z, r\rho(z))}\big)}{\mu(D(z, r\rho(z)))}.$$

The *lower uniform density of* Λ with respect to $\Delta\phi$ is

$$\mathcal{D}^-_{\Delta\phi}(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#\big(\Lambda \cap \overline{D(z, r\rho(z))}\big)}{\mu(D(z, r\rho(z)))}.$$

The following theorem proved in [8] is

Theorem 2.6. *Let ϕ be a subharmonic function with a doubling Laplacian.*

(i) *A sequence Λ is sampling for $\mathcal{F}_\phi$, if and only if Λ contains a ρ-separated subsequence Λ' such that $\mathcal{D}^-_{\Delta\phi}(\Lambda') > 1/2\pi$.*

(ii) *A sequence Λ is interpolating for $\mathcal{F}_\phi$, if and only if Λ is ρ-separated and $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$.*

§3. Riemann surfaces

This section is more especulative. All these results concern the study of function spaces defined on the whole $\mathbb{C}$ or in a disk with different growths. It is also possible to study the same problems in Riemann surfaces or in several complex variables. We will not deal with the multidimensional situation, although there has been some recent progress (see [9]), but we will rather concentrate on the Riemann surfaces. There are two (at least) possible approaches to define the right density that governs the interpolating or sampling sequences for holomorphic L^2 functions in the surface. Both use some potential theory to define them. In the first one as developed in [12] they compute the density of a sequence using instead of disks, the sublevel sets of the Green function. With these densities they have obtained some sufficient conditions (although not necessary) for a sequence to be sampling or interpolating in the Riemann surface. When restricted to the disk one reobtains Seip's characterization for the Bergman space.

The second approach consists in using some harmonic measure estimates to provide an alternative definition of the densities. We will present the result in the disk. By its invariant nature this new definition can be transported to any Riemann surface. We are inspired by a following result [5] due to Garnett Gehring and Jones. We need some notation. For a $z \in \mathbb{D}$, let $D(z,r)$ be the pseudohyperbolic disk $D(z,r) = \{w \in \mathbb{D}; \rho(z,w) < r\}$. As usual if A is a portion of the boundary of an open set Ω and $z \in \Omega$, then $\omega(z, A, \Omega)$ denotes the harmonic measure of A from the point z.

Theorem 3.1. *A separated sequence Λ is an interpolating sequence for H^∞ if and only if*

$$\inf_{\lambda\in\Lambda} \omega(\lambda, \partial\mathbb{D}, \mathbb{D} \setminus \bigcup_{\lambda'\neq\lambda} D(\lambda', c)) > 0$$

for some $0 < c < 1$.

To obtain a counterpart of this result, we define the following densities. Set

$$\Omega(z,r) = \Omega(\Lambda; z, r) = \mathbb{D} \setminus \bigcup_{1/2<\rho(\lambda,z)<r} D(\lambda, 1-r),$$

which is a finitely connected domain. We see that the uniform pseudohyperbolic radius of the little disks tends to 0 as $r \to 1$. This decay is tuned with the growth of r in such a way that the numbers

$$\mathcal{D}_h^-(\Lambda) = \liminf_{r\to 1^-} \inf_{z\in\mathbb{D}} \log \frac{1}{\omega(z, \partial\mathbb{D}, \Omega(z,r))}$$

and

$$\mathcal{D}_h^+(\Lambda) = \limsup_{r\to 1^-} \sup_{\lambda\in\Lambda} \log \frac{1}{\omega(\lambda, \partial\mathbb{D}, \Omega(\lambda,r))}$$

are positive when Λ is uniformly dense. In fact, we have the following precise characterization that is proved in [11]

Theorem 3.2. *For a separated sequence Λ in $\mathbb{D}$ we have*

$$\mathcal{D}_S^-(\Lambda) = \mathcal{D}_h^-(\Lambda) \quad \textit{and} \quad \mathcal{D}_S^+(\Lambda) = \mathcal{D}_h^+(\Lambda).$$

This theorem is proved with a direct proof that the harmonic measure density is comparable to the "geometric" density. It will be interesting to prove that whenever $\mathcal{D}_h^+(\Lambda) < \tau$ then Λ is interpolating for B_τ, directly without using Seip's characterization of interpolating sequences. This will possibly allow the generalization of such notions to Riemann surfaces.

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