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Renewal theorems, products of random matrices, and toral endomorphisms

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Dedicated to Martine Babillot

Abstract.

We consider a subsemigroup T of the linear group G of the d-dimensional Euclidean space V, which is "sufficiently large". We study the orbit closures of T in V and we apply the results to semi-groups of endomorphisms of the d-dimensional torus. The method uses the knowledge of the potential kernel of the Markov chain on V defined by a probability measure supported on T. The condition of being "large" is satisfied for example by a subsemigroup of $SL(V)$, Zariski-dense in $SL(V)$.

§1. Introduction

We denote by G the linear group of the Euclidean space $V = \mathbb{R}^d$, by $\mathbb{T}^d$ the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the lattice of integer points in $\mathbb{R}^d$. We denote by $M_{inv}(d, \mathbb{Z})$ the subsemigroup of elements g in G such that $g\mathbb{Z}^d \subset \mathbb{Z}^d$. These elements are $d \times d$ matrices with integer coefficients with non zero determinant. We observe that such a matrix defines a surjective endomorphism of $\mathbb{T}^d$.

Let us consider, to begin, the simplest situation $d = 1$. A basic fact of Diophantine approximation, is that, for given irrational $\alpha \in [0, 1[$ and $\varepsilon \in]0, 1[$, there exists relatively prime integers p, q such that: $q|\alpha - p/q| = \{q\alpha\} < \varepsilon$. Also, the set $\{\{q\alpha\}; q \in \mathbb{N}\}$ is dense in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If α is rational then $\{\{q\alpha\}; q \in \mathbb{N}\}$ is finite. In [19], Hardy and Littlewood have considered such properties when q belongs to a proper subset Q of $\mathbb{N}$. In particular they have shown that $Q = \{n^2; n \in \mathbb{N}\}$ is such a set. We observe that $\{n^2; n \in \mathbb{N}\}$ is a multiplicative subsemigroup of $\mathbb{N}$. Hence one can ask for the validity of such properties for various subsets of $\mathbb{N}$. In [7], H. Furstenberg has proved that any "non lacunary" subsemigroup

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T of $\mathbb{N}$ satisfies the required properties. In general, non lacunarity of T means that T is not contained in a multiplicative semigroup of the form $q^{\mathbb{N}} = \{q^n; n \in \mathbb{N}\}$ where q is an integer. The simplest example of such a semigroup is $Q = \{2^m 3^n; m, n \in \mathbb{N}\}$. In this case, non lacunarity follows from the irrationality of $Log2/Log3$. This type of property has been considered for $d > 1$ and $T \subset M_{inv}(d, \mathbb{Z})$ with T a commutative semigroup, by D. Berend in particular [5]. Also, it can be considered in the larger setting of multiparameter hyperbolic actions of groups or semigroups on manifolds (see for example [21]). In particular, G.A. Margulis has asked in [22] for necessary and sufficient conditions on a semigroup $T \subset M_{inv}(d, \mathbb{Z})$ for the dichotomy of density or finiteness of the T-orbits in $\mathbb{T}^d$.

Here, we give a brief exposition of some recent results on this problem; we concentrate mostly on the case where T is non commutative and "large". The condition of T being "large" can be made precise in terms of Zariski closure of T. We recall that the Zariski closure of a set X in G is the set of zeros of the polynomials on G which vanish on X. The Zariski closure of T is a group with a finite number of connected components. As this time two approaches on this problem have been developed. A direct one, by A. Muchnik [23], extends the arguments of D. Berend to the non necessarily commutative situation. A different approach uses properties of potential kernels of random walks on semi-simple groups ([17], [18]); it can be sketched as follows.

We denote by μ a finitely supported probability measure on $M_{inv}(d, \mathbb{Z})$ and we assume that its support generates a "large" semigroup of G. In this context we can obtain informations on orbit closures of T in $\mathbb{R}^d$ from properties of μ-invariant measures and potential measures on $\mathbb{R}^d$. Then, using projection on $\mathbb{T}^d$ and topological dynamics arguments, we can describe the orbit closures of T on $\mathbb{T}^d$. Let us consider in more detail the case $d = 1$. Let T be the multiplicative semigroup of $\mathbb{N}$ generated by $a = 2, b = 3$, and let $\mu = \frac{1}{2}(\delta_a + \delta_b)$. Then a basic fact, due to the irrationality of $Log2/Log3$, is that the semigroup $\{mLoga + nLogb; m, n \in \mathbb{N}\}$ is "more and more dense " in $\mathbb{R}$ at $+\infty$. This fact can be quantified by considering the potential measure $\sum_0^\infty \overline{\mu}^k$ of the additive random walk on $\mathbb{R}$ defined by $\overline{\mu}$, the image of μ by the map $x \to \mathrm{Log}x$. Then, according to the renewal theorem, this measure looks like Lebesgue measure at $+\infty$. Then, projecting on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and using arguments of topological dynamics, one can prove that, for every irrational $\alpha \in \mathbb{T}$, the orbit $T\alpha$ accumulates at zero, hence is dense in $\mathbb{T}$. This line of reasoning remains valid for $d > 1$; one has to use matricial analogues of the renewal theorem; useful results in this direction can be found in [2], [12], [14], [15], [17]. A noteworthy fact, in this context is that the non lacunarity condition, is automatically satisfied , if T is non commutative and "large" (see Proposition 2 and Corollary 5). In order to describe a typical result in case $d > 1$, in a simplified situation, we introduce some notations.

Definition 1. *An element g in G is said to be proximal if we can write*

$$V = \mathbb{R}v_g \oplus V_g^<,$$

where $gv_g = \lambda_g v_g, \lambda_g \in \mathbb{R}$, $gV_g^< = V_g^<$ and the spectral radius of g in $V_g^<$ is strictly less than $|\lambda_g|$.

Definition 2. *An element g in G is said to be quasi-expanding if its spectral radius is strictly greater than 1.*

Definition 3. *A semigroup $T \subseteq G$ is said to be strongly irreducible if there do not exist a finite union of proper subspaces which is T-invariant.*

For short, we will say that T satisfies condition $(i-p)$ if T is strongly irreducible and contains a proximal element. We will say that T satisfies $(i-p-e)$ if T satisfies $(i-p)$ and moreover, T contains a quasi-expanding element. The condition $(i-p)$ is satisfied if the Zariski closure of T contains $SL(V)$. The set of proximal elements in T will be denoted by T^{prox}. In this paper we sketch a proof of the following

Theorem 1 ([18]). *Assume T is a subsemigroup of $M_{inv}(d, \mathbb{Z})$, which satisfies $(i-p-e)$. Then the T-orbits on $\mathbb{T}^d$ are finite or dense.*

Example 1. *Let $d = 2$, $a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and let T be the subsemigroup of $SL(2, \mathbb{Z})$ generated by a, b. Then the conditions of the theorem are clearly satisfied. Hence the T-orbits in $\mathbb{T}^2$ are finite or dense.*

Remark 1. *a) The theorem gives a partial answer to the question of G.A. Margulis [22]. For the general case, see section 3 below.*

b) The theorem will follow from a description of the T-orbits in V, where T is a general subsemigroup of G which satisfies $(i-p-e)$. Compactness of $\mathbb{T}^d$ allows us to restrict the study to T-orbits which accumulate at zero. The expansion property allows us to conclude that such an orbit is "large" according to a well known principle in hyperbolic dynamics.

We need to consider the action of the semi-group T on $\mathbb{P}(V)$, the projective space of V, as well as on the factor space $V/_{\{\pm Id\}}$. We denote by π the projection of V onto $V/_{\{\pm Id\}}$ and by $\pi(V)$ the image of $V \setminus \{0\}$. The projection of $v \in V \setminus \{0\}$ on $\mathbb{P}(V)$ will be denoted $\overline{v}$. The set of accumulation points of a subset X of a topological space will be denoted by X^{ac}.

Definition 4. *Let T be a subsemigroup of G which satisfies condition $(i-p)$. We denote by L_T the subset of $\mathbb{P}(V)$ defined by $L_T =$ closure $\{\overline{v}_g \in \mathbb{P}(V); g \in T^{prox}\}$. We denote by $\widetilde{L}_T$ the inverse image of L_T in $V \setminus \{0\}$.*

It is easy to show (see for example [15]) that L_T is the unique T-minimal subset of $\mathbb{P}(V)$. Then theorem 1 will be a consequence of the

Theorem 2 ([18]). *Assume T is a subsemigroup of G which satisfies $(i-p-e)$, Φ is a closed T-invariant subset of $V \setminus \{0\}$ such that $0 \in \Phi^{ac}$. Then $\pi(\Phi) \supset \pi(\widetilde{L}_T)$.*

Let μ be a probability measure on G, T_μ the closed subsemigroup of G generated by its support S_μ. We observe that, if X is a G-space, the action of G on X can be extended to probability measures as follows:

$$\mu * \rho = \int \delta_{gx} d\mu(g) d\rho(x).$$

We will consider in particular the Markov operators P and $\widetilde{P}$ on $\mathbb{P}(V)$ and $\pi(V)$ respectively defined by

$$P(\overline{v}, .) = \mu * \delta_{\overline{v}}, \quad \widetilde{P}(v, .) = \mu * \delta_{\pi(v)}.$$

The space $\pi(V)$ can be written, using polar coordinates, in the form:

$$\pi(V) = \mathbb{P}(V) \times \mathbb{R}_+^*.$$

Furthermore, the group $\mathbb{R}_+^*$ acts naturally on the right on $\pi(V)$ by dilations. We denote by ℓ the Lebesgue measure on the group $\mathbb{R}$ or $\mathbb{R}_+^*$ (which is isomorphic to $\mathbb{R}$). We know (see [15]) that, if T_μ satisfies $(i-p)$, there exists on $\mathbb{P}(V)$ a unique μ-stationary measure $\nu (\nu = \mu * \nu)$. The support of ν is L_{T_μ}. If S_μ is compact, the limit of the sequence $\frac{1}{n} \int Log\|g\| d\mu^n(g)$ is finite and it will be denoted $\gamma(\mu)$. With these notations, theorem 2 will be a consequence of:

Theorem 3. *Assume μ is a probability measure on G such that S_μ is compact, T_μ satisfies $(i-p-e)$, $\gamma(\mu) > 0$. Then, for any $v \in V \setminus \{0\}$,* *we have the following weak convergence on $\pi(V)$:*

$$\lim_{t\to 0}\sum_{0}^{\infty}\mu^k * \delta_{t\pi(v)} = \frac{1}{\gamma(\mu)}\nu\otimes\ell.$$

If $d = 1$, this statement is called the renewal theorem (see [6], p 300-309). It leads to Theorem 2 and to the following corollary, once a convenient measure μ on T has been chosen. It has the following purely topological corollary;

Corollary 1. *Assume T is a subsemigroup of G which satisfies $(i-p-e)$. Then, for any $v \in V\setminus\{0\}$, we have the convergence:*

$$\liminf_{t\to 0}\quad t\pi(\overline{Tv}) \supset \pi(\widetilde{L}_T).$$

The convergence above is taken as the Hausdorff convergence on compact subsets of $\pi(V)$. The corollary says that T-orbits on V are "large at the infinity".

§2. Some extensions of the renewal theorem

We describe here a basic tool of the proof of Theorem 3.

The following is the classical renewal theorem;

Theorem 4. *Let μ be a probability measure on $\mathbb{R}$ such that the closed subgroup generated by its support is $\mathbb{R}$. Assume that μ has finite mean $\gamma(\mu) > 0$. Then we have the following weak convergence:*

$$\lim_{t\to -\infty}\sum_{0}^{\infty}\mu^k * \delta_t = \frac{1}{\gamma(\mu)}\ell,$$

where ℓ is Lebesgue measure.

In order to illustrate its meaning in dynamics, we consider the following special flow $(X, \widehat{\theta}^t)$. Assume $A = S_\mu \subset \mathbb{R}_+$ is finite and let $\Omega = A^{\mathbb{N}}, \rho = \mu^{\otimes\mathbb{N}}$. We consider the subset X of $\Omega\times\mathbb{R}$:

$$X = \{(\omega, x)\ \ 0 \le x \le \omega_0\},$$

and endow X with the probability measure $\widehat{\rho}$ which is the restriction of $\frac{1}{\gamma(\mu)}\rho\otimes\ell$ to X. We denote:

$$S_k(\omega) = \omega_0 + \cdots + \omega_{k-1},$$

and we observe that, for any $x, t \in \mathbb{R}_+$, there exists a unique $k \in \mathbb{N}$ such that $S_k(\omega) \leq x + t < S_{k+1}(\omega)$, we denote by θ the shift transformation on Ω, and the flow $\widehat{\theta}^t$ on X is defined by

$$\widehat{\theta}^t(\omega, x) = (\theta^k \omega, x + t - S_k(\omega)),$$

where k is as above.

Then $\widehat{\theta}^t$ is the so called special flow under the function $\omega \to \omega_0$; the measure $\widehat{\rho}$ is $\widehat{\theta}^t$-invariant. The renewal theorem, for μ as above, is closely related to the following

Proposition 1. *The flow $\widehat{\theta}^t$ on $(X, \widehat{\rho})$ is mixing.*

Proof. We denote by T the transformation on $X \times \mathbb{R}$ defined by

$$T(\omega, x) = (\theta\omega, x - \omega_0).$$

We observe that its adjoint with respect to $\rho \otimes \ell$ is the Markov kernel $\widetilde{P}$ defined by

$$\widetilde{P}\varphi(\omega, x) = \sum_{a \in A} \varphi(a\omega, x + a)\mu(a).$$

Furthermore $t \in \mathbb{R}$ acts by translation on $X \times \mathbb{R}$. In particular, for a function $\psi \in C(X \times \mathbb{R})$, we have: $(\delta_t * \psi)(\omega, x) = \psi(\omega, x - t)$.

On the other hand, for any $\varphi \in C(X)$ we have, with k as above:

$$\varphi(\widehat{\theta}^t(\omega, x)) = \varphi o T^k(\omega, x + t),$$

$$\varphi(\widehat{\theta}^t(\omega, x)) = \sum_0^\infty \varphi o T^n(\omega, x + t).$$

Then, for any $\psi \in C(X)$, we have, using duality:

$$< \varphi \circ \widehat{\theta}^t, \psi >_{\widehat{\rho}} = < \varphi \circ \widehat{\theta}^t, \psi >_{\rho \otimes \ell} = \sum_0^\infty < \varphi, (\widetilde{P}^n \delta_{-t})(\psi) >_{\rho \otimes \ell} .$$

If ψ is of the form $\psi' \otimes u$ with $\psi' \in C(\Omega)$ depending only of the first r coordinates and u is continuous with compact support on $\mathbb{R}$, we have for $n \geq r$:

$$(\widetilde{P}^n \delta_{-t})(\psi) = (\mu^{n-r} * \delta_{-t})(\widetilde{P}^r \psi),$$

where $\widetilde{P}^r \psi$ depends only of the x-coordinate. Also:

$$\sum_0^\infty (\widetilde{P}^n \delta_{-t})(\psi) = \sum_0^{r-1} (\widetilde{P}^n \delta_{-t})(\psi) + \sum_0^\infty (\mu^k * \delta_{-t})(\widetilde{P}^r \psi).$$

From the renewal theorem:

$$\lim_{t\to+\infty}\sum_0^\infty(\widetilde{P}^n\delta_{-t})(\psi)=\frac{1}{\gamma(\mu)}(\rho\otimes\ell)(\psi).$$

Hence, we have

$$\lim_{t\to+\infty} <\varphi\circ\widehat{\theta}^t,\psi>_{\widehat{\rho}}=\widehat{\rho}(\varphi)\widehat{\rho}(\psi),$$

for any $\varphi\in C(X)$, $\psi\in C(X)$ depending of finitely many coordinates. A density argument in $\mathbb{L}^2(X)$ allows to conclude. □

The renewal theorem admits a natural extension to "Gibbsian random walks" defined in terms of a mixing subshift of finite type (Ω,θ) endowed with a Gibbs measure ρ and a Hölder function f on Ω (See [11] [12] [27]).

Definition 5. *We say that the Hölder function f on the subshift (Ω,θ) is arithmetic if there exists a Hölder function u on Ω such that $f+u\circ\theta-u$ takes its values in $c\mathbb{Z}$ for some $c\in\mathbb{R}$.*

We consider only a unilateral subshift (Ω,θ) where $\Omega\subset A^{\mathbb{N}}$ and A is finite. We denote by $p(\omega,a)$ the conditional probability of $\omega_{-1}=a$, given $\omega=(\omega_0,\omega_1,\cdots)$. The Markov kernel $\widetilde{P}$ on $\Omega\times\mathbb{R}$ is defined by:

$$\widetilde{P}\varphi(\omega,x)=\sum_{a\in A}\varphi(a\omega,x+f(\omega))p(\omega,a).$$

Then we have the following extension of the renewal theorem (See [12]):

Theorem 5. *With the above notations, assume $\gamma(f)=\int f(\omega)d\rho(\omega)>0$ and f is non arithmetic. Then for any function φ on $X\times\mathbb{R}$, continuous with compact support we have the following convergence:*

$$\lim_{t\to+\infty}\sum_0^\infty\widetilde{P}^k\varphi*\delta_t=\frac{1}{\gamma(f)}(\rho\otimes\ell)(\varphi).$$

This theorem is also closely related to the mixing of special flows over subshifts of finite type endowed with Gibbs measures.

Coming back to the matricial setting described in the introduction, we give below some important tools of the proof of Theorem 3.

Proposition 2. *Let $T\subset G$ be a subsemigroup which satisfies condition $(i-p)$. We denote*

$$\textstyle\sum_T=\{Log|\lambda_g|\ \ g\in T^{prox}\}.$$

Then $\sum_T$ generates a dense subgroup of $\mathbb{R}$.

This result is proved in [18]. It is an analogue of the arithmetical condition assumed in the above theorem.

Keeping with the analogy of the above theorem, we consider the action of G on $\mathbb{P}(V)$ and we denote $g \cdot b$ the result of the projective action of g on $b \in \mathbb{P}(V)$. Then we write the action of G on $\pi(V)$ as

$$g(b,x) = (g \cdot b\ ,\ \|gb\|x)$$

The Markov operator on $\mathbb{P}(V)$ (resp $\pi(V)$) associated with μ then can be written as:

$$P\varphi(b) = \int \varphi(g \cdot b)d\mu(g), \quad \text{resp.} \quad \widetilde{P}\varphi(b,x) = \int \varphi(g.b,\ x\|gb\|)d\mu(g).$$

It is convenient to "decompose" the operator $\widetilde{P}$, using the Fourier operators P^s $(s \in \mathbb{R})$ which are defined on $\mathbb{P}(V)$ by:

$$P^s\varphi(b) = \int \varphi(g.b)\|gb\|^{is}\ d\mu(g).$$

For $\varepsilon \in]0,1]$, denote by $H_\varepsilon(\mathbb{P}(V))$ the space of ε-Hölder functions φ on $\mathbb{P}(V)$ defined by the condition:

$$[\varphi]_\varepsilon = \underset{b,b' \in \mathbb{P}(V)}{Sup} \frac{|\varphi(b) - \varphi(b')|}{d^\varepsilon(b,b')} < +\infty,$$

where $d(b,b')$ is the distance on $\mathbb{P}(V)$ defined by

$$d(b,b') = \|b \wedge b'\| = |sin(b,b')|.$$

This space is a Banach space with respect to the norm defined by:

$$\|\varphi\|_\varepsilon = [\varphi]_\varepsilon + |\varphi|_\infty.$$

The operators P^s have nice spectral properties, due to the following:

Proposition 3. *Assume that the probability measure μ has compact support and T_μ satisfies condition $(i-p)$. Then, for ε small, there exists $c_0 \in [0,1[$ and $c(\varepsilon) \in \mathbb{R}_+$ such that*

$$[P\varphi]_\varepsilon \leq c_0[\varphi]_\varepsilon,$$

$$[P^s\varphi]_\varepsilon \leq c_0[\varphi]_\varepsilon + c(\varepsilon)|\varphi|_\infty.$$

This allows, using [20], to define a principal eigenvalue $k(s) \in \mathbb{C}$ which plays the role of the Fourier transform in classical Probability Theory. Theorem 3 can be proved along the lines of the classical analytic proof of the renewal theorem (see [12]). On the other hand, Proposition 3 is a consequence of the properties of random walks on G, namely of the fact that the top Lyapunov exponent of μ is is simple in the Lyapunov spectrum of μ (see [9], [15]). In order to prove the corollary 1 , we make use of two facts. If T is compactly generated, we can find μ such that $T_\mu = T$ and $\gamma(\mu) > 0$. Then the corollary follows from the fact that the support of $\nu \otimes \ell$ is $\pi(\widetilde{L}_T)$. If T is not compactly generated, we proceed by exhaustion.

§3. Orbit closures of semigroups of toral endomorphisms

We consider here natural extensions of Theorem 1 and we describe some of the tools required in their proofs. We refer to [16] [17] [23] for detailed proofs.

Theorem 6 ([23]). *Assume T is a subsemigroup of $M_{inv}(d, \mathbb{Z})$ which satisfy the conditions:*

1) There is no T-invariant finite union of rational proper subspaces in V.

2) There is no relatively compact T-orbit in $V \setminus \{0\}$.

3) The group generated by T is not a finite extension of a cyclic group.

Then the T-orbits in $\mathbb{T}^d$ are finite or dense.

This result answers the question of Margulis in case of $\mathbb{T}^d (d \geq 1)$, since it is not difficult to show that the conditions of the theorem are necessary for the dichotomy of density or finiteness of T-orbits in $\mathbb{T}^d$. It can also be proved following the lines developed in section 2. We observe that condition (1) implies that the Zariski closure of T in G is reductive. An important component of the proof sketched here is the study of T-orbits on V, for a general subsemigroup of G such that its Zariski closure, denoted $Zc(T)$, is semi-simple (see [17]). We sketch below some of the corresponding tools and simplify some technical aspects of [17], using the recent results of [16].

Let S be an $\mathbb{R}$-algebraic and connected semi-simple group, μ a probability measure on S such that the semigroup T_μ generated by its support is Zariski dense in S. We consider an Iwasawa decomposition of $S : S = KAN$; we denote by M the centraliser of A in K, by [M,M] its commutator subgroup. We choose a Weyl chamber $\mathcal{A}^+$ in the Lie algebra $\mathcal{A}$ of A and we denote $A^+ = \exp \mathcal{A}^+ \subset A$. Then we have the polar decomposition $S = K\overline{A}^+K$ and we denote by $a(g)$ the $\overline{A}^+$-component of g. If μ has compact support, we can define (see [15]) the Lyapunov vector of μ by

$$\lambda(\mu) = \lim \frac{1}{n} \int Log\ a(g) d\mu^n(g) \in \overline{\mathcal{A}}^+.$$

We denote by $F = K/M$ the so-called Furstenberg boundary of S, and we observe that the S-homogeneous space S/MN can be written as a product $(K/M) \times A$. Also the S-homogeneous space S/N can be written as $K \times A$. The group A acts on the right on these spaces. The result of the action of $a \in A$ on $v \in S/MN$ (resp $x \in S/N$) will be simply denoted va (resp xa). We denote by ℓ the Lebesgue measure of A and write $r = dimA$. It is known that there exists on F a unique μ-stationary measure ν, and its support Λ_{T_μ} is the unique T_μ-minimal subset of F. Also we have $\lambda(\mu) \in \mathcal{A}^+$ (see [15]) and [9]. The following is proved in [17], using results of [2].

Theorem 7. *With the above notations, for any $v \in S/MN$, we have the following weak convergence:*

$$\lim_{t \to +\infty} t^{\frac{r-1}{2}} \sum_0^\infty \mu^k * \delta_{ve^{-t\lambda(\mu)}} = c_\mu \nu \otimes \ell,$$

where c_μ is a positive constant. Furthermore the measure $\nu \otimes \ell$ is μ-invariant extremal .

We need also to consider the potential kernel of μ on $S/N = K \times A$ and the μ-stationary measures on $K = S/AN$.

Theorem 8 ([16]). *With the above notations, the action of μ on $K = S/AN$ has only a finite number $p \leq 2^r$ of extremal μ-stationary probabilities $\sigma_i (1 \leq i \leq p)$. Each of them is invariant under the right action of the connected component of M and has projection ν on K/M. The T_μ-minimal subsets of K are the supports $\widehat{\Lambda}_T^i$ of the measures σ_i $(1 \leq i \leq p)$.*

In case $S = SO(n,1)$ and T is a Zariski dense sub semigroup of S, the theorem implies that the action of T on $SO(n) = K$ has a unique minimal subset which is the inverse image in K of the unique T-minimal subset Λ_T of the boundary $SO(n)/SO(n-1)$ of $SO(n,1)$.

Corollary 2. *Each measure $\sigma_i \otimes \ell$ on S/N $(1 \leq i \leq p)$ is μ-invariant and extremal. For any $x \in S/N$, the cluster values, when*

$t \to +\infty$, of the family of potentials $t^{\frac{r-1}{2}} \sum_{0}^{\infty} \mu^k * \delta_{xe^{-t\lambda(\mu)}}$ *are positive linear combinations of the measures* $\sigma_i \otimes \ell$ $(1 \le i \le p)$.

Let T be a Zariski dense semigroup of S. Then it is known (see for example [9], [24]) that T contains $\mathbb{R}$-regular elements *i.e.* elements g conjugate to elements of MA^+. We denote by T^{prox} the set of $\mathbb{R}$-regular elements in T. For such a $g \in T^{prox}$ we denote

$$\lambda(g) = \lambda(\delta_g) \in \mathcal{A}^+,$$

and by C_T we denote the closed subcone of $\overline{\mathcal{A}^+}$ generated by such elements $\lambda(g), g \in T^{prox}$. Also it is known that C_T is convex and has non empty interior C_T° (see [3]). If T is compactly generated, it is possible to find μ, as above, with $T_\mu = T$ and $\lambda(\mu)$ proportional to $\lambda(g)$. Furthermore, for $g \in T^{prox}$, the projection modulo [M,M] of a conjugate of g in MA^+ is uniquely defined. We denote this projection in $MA/[M,M]$ by $\sigma(g)$ and we consider the closed subgroup $\Delta_T \subset MA/[M,M]$ generated by the elements $\sigma(g)(g \in T^{prox})$. Then we have the

Theorem 9 ([16]). *Let T be a Zariski dense subsemigroup of the semi-simple group S. Then, the closed subgroup $\Delta_T \subset MA/[M,M]$ generated by the elements $\sigma(g)(g \in T^{prox})$ has finite index in $MA/[M,M]$.*

This result is a natural extension of Proposition 2. It extends also an analogous, result of [4], where $\sigma(g)$ is replaced by $\lambda(g) \in \mathcal{A}$. Furthermore, we observe that a deep study of analogous properties of Zariski dense subgroups of semi-simple groups has been developed by G. Prasad and A.S Rapinchuk (see for example [25], Theorem 2). This approach, based on embedding in linear groups over p-adic fields gives also, as a by-product, the statement of Theorem 9, in the case where, for some embedding j in $GL(n,\mathbb{R})$, the coefficients of $j(T)$ belong to a number field (G. Prasad, personal communication). Using Theorem 9, we get the following corollaries.

Corollary 3. *Let T be a Zariski dense subsemigroup of the semi-simple group S, g an $\mathbb{R}$-regular element of T such that $\lambda(g) \in C_T^\circ$. Let $\Phi \subset G/N$ be a closed T-invariant subset, and $x \in \widehat{\Lambda}_T^i \subset K$ such that*

$$g \in xMA^+x^{-1}, \quad g^{-\mathbb{N}}x \subset \Phi.$$

Then Φ contains $\widehat{\Lambda}_T^i A$.

This corollary leads to a description of the closed T-invariant subsets Φ of $V(dimV > 1)$, such that $0 \in \Phi^{ac}$, without assuming irreducibility of T, as in Theorem 2.

Corollary 4. *Let T be a subsemigroup of $GL(V)$ such that its Zariski closure is semi-simple, Φ a closed T-invariant subset of V such that $0 \in \Phi^{ac}$. Then there exists a non zero vector $u \in V$, a conjugate A_u of A in S, and an index $i \in [1,p]$ such that:*

$$0 \in \overline{A_u u} \subset \Phi,$$

$$\Phi \supset \widehat{\Lambda}^i_T A_u u.$$

From this we deduce the

Corollary 5 ([17]). *Let T be a subsemigroup of $M_{inv}(d,\mathbb{Z})$ such that its Zariski closure in $GL(V)$ is semi-simple. Assume that T does not preserve a finite union of rational subspaces. Then every T-orbit in $\mathbb{T}^d$ is finite or dense.*

It can be shown (see [17]) that any infinite closed T-invariant subset X of $\mathbb{T}^d$ contains 0 as an accumulation point. Hence, lifting X to $\mathbb{R}^d$, Corollary 5 follows from Corollary 4.

As already said before, a proof of Theorem 6 can also be obtained along these lines. In this more general case, the Zariski closure of T is reductive, hence one needs to show a renewal theorem for reductive groups.

Conditions (2) and (3) in Theorem 6 need to be added to the hypothesis of Theorem 8, in view of the presence of a non trivial center in this reductive group.

§4. Final comments

Let G be a connected Lie group, G/H a non compact homogeneous space of G, μ a probability measure on G such that the semi-group T_μ generated by the support of μ is "large". For example if μ has a density which is positive and continuous at e, one has $T_\mu = G$. If G is algebraic, one can require T_μ to be Zariski dense in G.

It would be interesting to describe, in some special cases, the Radon measures on G/H which are extremal solutions of the equation $\mu * \lambda = \lambda$. This is closely related to the description of the Martin boundary of the Markov chain on G/H defined by μ, *i.e* to the limits of the normalized potentials associated with $\sum_0^\infty \mu^k * \delta_x (x \in G/H)$ when x goes to the infinity. It should be possible to obtain from these informations properties of the T-orbit closures in G/H at infinity. In the situation considered in this paper one has $G = GL(V)$, $G/H = V \setminus \{0\}$. The case where G is the affine group of V and $G/H = V$ is also of interest for geometrical reasons. For a study of recurrence relations with random coefficients in such a setting, see [10].

From the analytic point of view, if μ has a density, very little seems to be known on the Martin boundary of $(G/H, \mu)$ even if $H = \{e\}$. If G/H is a symmetric space and μ is defined by the heat kernel at time one, the Martin boundary has been calculated in ([13]). For $G/H = G$, the extremal solutions of the equation $\mu * \lambda = \lambda$ have been calculated in some cases (see [1], [2], [7], [11], [13], [26]). The knowledge of such measures for $G/H \neq G$, should also be useful for the study of orbit closures $\overline{Tx}$ $(x \in G/H)$ at the infinity, since it corresponds to a simpler situation.

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