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Brownian motion and harmonic measure in conic sections

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Abstract.

This is a survey of results on the exit time and the exit position of Brownian motion from cones and parabola-shaped regions in Euclidean space. The paper begins with a section on harmonic measure.

§1. Harmonic measure

1.1. The Dirichlet Problem and harmonic measure

Harmonic measure has long been a central theme of Potential Theory: that this is as true today as it was in the past is confirmed by the recent publication of the major book *Harmonic measure* by Garnett and Marshall [13].

The Dirichlet Problem is the boundary value problem for the Laplace equation: given a region D in $\mathbf{R}^n$ and a bounded continuous function f on the boundary of D, one is to find a function u on the closure $\overline{D}$ of the region with the properties that

(i) u is continuous on $\overline{D}$,
(ii) u is harmonic in D, that is $\Delta u \equiv 0$ in D,
(iii) $u|_{\partial D} = f$

This boundary value problem arises in a number of physical contexts, for example that of determining the steady state temperature inside a region when the temperature on the boundary of the region is specified. This physical interpretation also sheds light on a defining characteristic of harmonic functions: they are the functions that satisfy the mean-value property, in that the average value of a harmonic function over

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a sphere is its value at the centre of the sphere [9, Theorem 1.9][1]. If the average steady state temperature on the sphere was, say, greater than the temperature at its centre there would then be a net flow of heat from the hotter sphere to its cooler centre. The temperature at the centre would then increase and this would not, after all, be a steady state temperature distribution.

The Dirichlet Problem as stated does not always have a solution: and even when it does it may not be unique, depending on how one treats points at infinity. Regions for which a solution exists, no matter what the specified boundary function may be, are called *regular* for the Dirichlet Problem. In the classical approach to characterizing such domains, a boundary point ζ is said to be *regular* if there is a barrier at that point, this being a function that is superharmonic and positive in D near ζ and that tends to zero on approach to the boundary at ζ. A domain is then proved to be regular if all of its boundary points are regular [15, Theorem 2.10] [1, Chapter 6]. The approach taken in Hayman and Kennedy's book is to deal first with bounded regular domains, and to consider unbounded possibly irregular domains later when the extra machinery needed, in particular that of polar sets, is in place [15, Section 5.7.1]. This is a dichotomy that may profitably be kept in mind.

A solution of the Dirichlet Problem corresponding to the continuous boundary function f is called a harmonic extension of f. Such a harmonic extension may be considered for more general boundary data (with a suitable reformulation of condition (i)) [15, Theorems 2.10 and 2.17]. In particular, corresponding to a Borel measurable subset E of the boundary of D, the boundary data $f = 1_E$ (so that f takes the value 1 on the part E of the boundary and 0 on the remaining boundary) has a harmonic extension into D. This solution is called the *harmonic measure* of E and is denoted by $\omega(x, E; D)$. As a function of x, therefore, it is harmonic. If x is held fixed and E varies, then $\omega(x, E; D)$ is a measure on the boundary of D [15, Theorem 3.10]. The harmonic extension of a general Borel measurable function can then be constructed by integration with respect to harmonic measure on the boundary. Intuitively, it helps to think of the harmonic measure of E as the solution of the Dirichlet Problem with boundary data $f = 1_E$. From a technical point of view, it is best to construct harmonic measure as the measure

[1]This is but one of many possible references where a precise statement of this theorem may be found. I have chosen not to give perfectly precise statements of results in this article, with the excuse that specialists will know these results and will not need to read this article, while the rest of us are now forewarned not to take statements too literally and to consult at a minimum the cited references.

$\omega(x, E; D)$ on the boundary of D for which

$$\int_{\partial D} f(\zeta)\, d\omega(x, \zeta; D)$$

returns the value at x of the harmonic extension of f into D for any continuous bounded function f on the boundary of D.

The study of harmonic measure in planar domains is facilitated by conformal mapping, as harmonic measure is *conformally invariant*. By conformal invariance of harmonic measure we mean that, if f is analytic and one-to-one in the planar domain D, then

$$\omega(z, E; D) = \omega\left(f(z), f(E); f(D)\right).$$

To see why this holds true, we write h for the function $\omega\left(\cdot, f(E); f(D)\right)$. Then $h \circ f$ satisfies

$$\Delta(h \circ f)(z) = (\Delta h)(f(z))\, |f'(z)|^2, \quad z \in D.$$

Thus $h \circ f$ is harmonic in D, since h is harmonic in $f(D)$, and its boundary values are 1_E. Hence $(h \circ f)(z) = \omega(z, E; D)$.

For example, to compute the harmonic measure $\omega(z, E; D)$ for a simply connected planar domain D, one maps D conformally onto the unit disk U by a map f for which $f(z) = 0$. Then

$$\omega(z, E; D) = \omega(0, F; U), \quad F = f(E),$$

the latter being the normalized angular measure of F on the unit circle. While this 'solves' the problem in principle, in practice relatively few explicit conformal mappings are known.

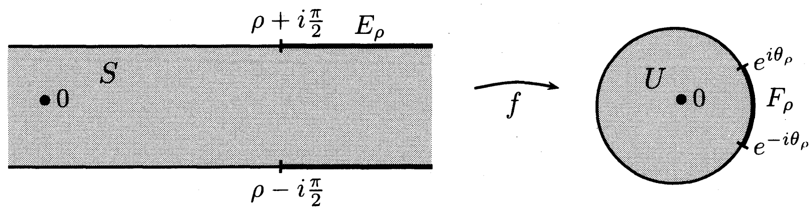
As a simple example that is relevant to the subject matter of this paper, we will compute the rate of decay of harmonic measure in the infinite strip $S = \{z\colon |\operatorname{Im} z| < \pi/2\}$. We set $E_\rho = \{z\colon |\operatorname{Im} z| = \pi/2 \text{ and } \operatorname{Re} z > \rho\}$ and set $\omega(\rho) = w(0, E_\rho; S)$.

$$f(z) = \frac{e^z - 1}{e^z + 1}$$

is a conformal map of S onto the unit disk with $f(0) = 0$. For $\rho > 0$, the image F_ρ of E_ρ under f is the shorter arc of the unit circle lying between $e^{-i\theta_\rho} = f(\rho - i\pi/2)$ and $e^{i\theta_\rho} = f(\rho + i\pi/2)$. The harmonic measure of this arc at 0 is its normalized angular measure, which is θ_ρ/π.

Since

$$f\left(\rho + i\frac{\pi}{2}\right) = \frac{e^{\rho + i\pi/2} - 1}{e^{\rho + i\pi/2} + 1} = \frac{e^{2\rho} - 1 + 2ie^\rho}{1 + e^{2\rho}},$$



we see that

$$\theta_\rho = \arg\left[e^{2\rho} - 1 + 2ie^\rho\right] = \arctan\left(\frac{2e^\rho}{e^{2\rho}-1}\right).$$

Thus, in summary,

$$\omega(\rho) = w(0, E_\rho; S) = w(0, F_\rho; U) = \frac{1}{\pi}\arctan\left(\frac{2e^\rho}{e^{2\rho}-1}\right),$$

from which it follows that

$$\omega(\rho) = \frac{2}{\pi}\, e^{-\rho} + \mathrm{O}\left(e^{-3\rho}\right) \text{ as } \rho \to \infty.$$

1.2. Brownian motion and harmonic measure

Brownian motion in $\mathbf{R}^n$ is a mathematical model of the position of a particle that is subject to random buffeting with no preferred direction and whose intensity is independent of position. The possible paths of the particle are the continuous functions $\omega : [0, \infty) \to \mathbf{R}^n$. Brownian motion may be viewed as a measure, known as Wiener measure, on this space of continuous paths. We write $B_t(\omega) = \omega(t)$ for the position of the particle at time t if it follows the path ω. Then each B_t is a random variable on path space. Wiener measure on path space is constructed so that (i) the net displacements in disjoint time intervals are independent and (ii) the net displacement $B_t - B_s$ between time s and t (with $s < t$) is normally distributed with mean zero and covariance matrix $t-s$ times the identity matrix. As is customary, we write P_x to denote the probability (Wiener measure) of events (measurable sets of paths) that gives full measure to the paths with initial point x, and we write E_x for the expectation (integral) with respect to P_x.

For a region D, we write

$$\tau_D = \inf\{t > 0 \colon B_t \notin D\}.$$

This is the first exit time of Brownian motion from the region D, and plays a key role in this story. The exit time is a random variable, as its value depends on the particular path. The first exit position of Brownian motion from D is then B_{τ_D}.

The connection between Brownian motion and harmonic measure, first elucidated by Kakutani, is simply this: harmonic measure at x in D is the exit distribution from D of Brownian motion with initial point x [9, Section 1.4]. That is,

$$\omega(x, E; D) \;=\; P_x(\tau_D < \infty, B_{\tau_D} \in E),$$

for each Borel measurable subset E of the boundary of D. In fact, for any domain D and any bounded function f on the boundary of D

$$h(x) \;=\; E_x\left[f(B_{\tau_D})\, 1_{\{\tau_D<\infty\}}\right], \quad x \in D,$$

is harmonic. This becomes 'obvious' once we remember that harmonic functions are those with the mean-value property. To see why, let's consider a sphere S_x in D with centre x, and consider those paths that first hit the sphere at a point y on the sphere, before then going on to exit D. When we delete the initial sections between x and y, the new paths constitute a new Brownian motion starting from y, and this doesn't change the exit position. Therefore the paths from x that exit the sphere S_x at y contribute $h(y)$ times the probability of first exiting the sphere at y, and

$$h(x) \;=\; E_x\left[f(B_{\tau_D})\, 1_{\{\tau_D<\infty\}}\right] \;=\; \int_{S_x} h(y)\, d\sigma(y),$$

where $d\sigma$ is the distribution of the first hitting position on the sphere of Brownian motion with initial point at its centre. But σ just has to be the uniform distribution on the sphere, which gives the mean-value property for h.

The probabilistic characterization of the regularity of a boundary point for the Dirichlet Problem is more intuitive than that involving the barrier function. A boundary point ζ of a region D is regular if $P_\zeta(\tau_D = 0) = 1$ [20, Section 9.2]: there must be enough boundary near the boundary point ζ so that a Brownian motion with initial point ζ will immediately hit the boundary with probability one. If ζ is a regular boundary point, it is then not too hard to see that a Brownian motion, whose initial point x is in D and is near ζ, will exit D near ζ with high probability. If f is continuous at ζ, it will follow that $f(B_{\tau_D})$ will be close to $f(\zeta)$ with high probability. In effect, the function h will be continuous at ζ. This is the probabilistic solution of the Dirichlet problem [20, Section 9.2], [9, Section 1.6]. The case $f = 1_E$ is the assertion that harmonic measure and hitting probabilities of Brownian motion are one and the same.

1.3. Distortion Theorems

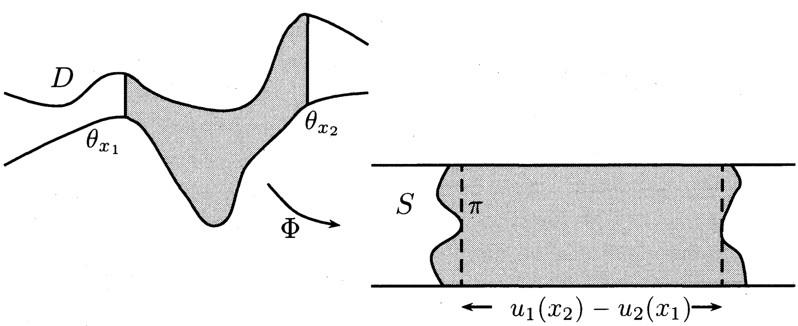
In his 1930 thesis, Ahlfors proved a distortion theorem for conformal mappings with which he settled the Denjoy conjecture and which has since proved to be useful and influential. We follow the eminently readable account in [2].

A domain D is said to be *strip like* if it is simply connected and contains a curve $\beta(t)$, $0 < t < 1$, with $\operatorname{Re}\beta(t) \to -\infty$ as $t \to 0$ and $\operatorname{Re}\beta(t) \to \infty$ as $t \to 1$. Thus the curve determines two prime ends, that we call $-\infty$ and ∞. The domain D is then mapped conformally onto the strip S by a map Φ with $\operatorname{Re}\Phi(\beta(t)) \to -\infty$ as $t \to 0$ and $\operatorname{Re}\Phi(\beta(t)) \to \infty$ as $t \to 1$. For each x, the intersection of the vertical line $\operatorname{Re} z = x$ with D consists of open line segments, one of which separates the two prime ends determined by the curve β. This crosscut of D is traditionally labelled θ_x and its length is written as $\theta(x)$. Finally, we write

$$u_2(x) = \sup_{z\in\theta_x} \operatorname{Re}\Phi(z) \quad \text{and} \quad u_1(x) = \inf_{z\in\theta_x} \operatorname{Re}\Phi(z).$$

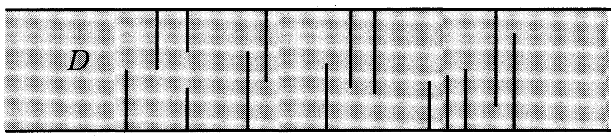
Ahlfors' Distortion Theorem *If* $\displaystyle\int_{x_1}^{x_2} \frac{dx}{\theta(x)} > 2\pi$, *then*

$$\frac{1}{\pi}\int_{x_1}^{x_2} \frac{dx}{\theta(x)} \le u_1(x_2) - u_2(x_1) + 4.$$

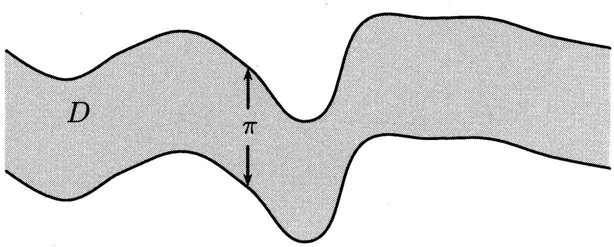


In geometric terms, this theorem is a lower bound on the area $\pi\,[u_1(x_2) - u_2(x_1)]$ of the largest rectangle contained in the image under Φ of that part of D between the crosscuts θ_{x_1} and θ_{x_2}. The distortion theorem shows that if the strip like domain is narrow, the conformal map Φ must stretch the distance, in the image strip, between the images of crosscuts in D.

There are essentially two situations in which Ahlfors' Distortion Theorem underestimates the rate of growth of the mapping Φ. If, say, vertical slits are removed from D then the integral $\int_{x_1}^{x_2} dx/\theta(x)$ will not detect them, yet one knows that the mapping Φ will then grow much faster. For example, in the case of the strip S with as many slits removed as one may wish, the distortion theorem treats Φ as if it was the identity map. The second situation in which Φ grows faster than



predicted by the Ahlfors Distortion Theorem is typified by taking the strip S and bending it into the shape of a snake or the crenellations on a castle, while leaving the lengths of the crosscuts unchanged.



In 1942, Warschawski [24] proved an upper bound on the area $\pi\,[u_2(x_2) - u_1(x_1)]$ of the smallest rectangle that contains the image under Φ of that part of Ω between the crosscuts θ_{x_1} and θ_{x_2}. His theorems involve extra terms that measure the oscillation of the central line of the strip like domain and the oscillation of the width of the domain. We will state a special and slightly weaker case of his results that will be sufficient for our purposes.

Distortion Theorem for certain symmetric domains *Suppose that D is a strip like domain that takes the form*

$$D = \left\{ z \colon |\mathrm{Im} z| < \frac{1}{2}\theta(\mathrm{Re} z) \right\},$$

where θ is a non negative function on the real line with

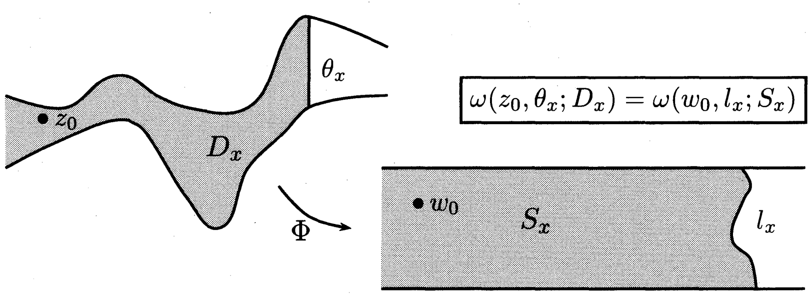
$$\int_{-\infty}^{\infty} \frac{\theta'(x)^2}{\theta(x)}\, dx \;<\; \infty.$$

Then, for $z = x + iy \in D$ and fixed x_0,

$$\operatorname{Re}\Phi(z) \;=\; \pi \int_{x_0}^{x} \frac{ds}{\theta(s)} + \mathrm{O}(1) \quad \text{as } x \to \infty.$$

1.4. Higher dimensional distortion theorems

The distortion theorems of Ahlfors and Warschawski may be viewed as harmonic measure estimates. Suppose that we wish to estimate the harmonic measure of the crosscut θ_x, with respect to that part D_x of D to the left of θ_x, at some fixed point z_0. Having mapped D onto the strip S by the map Φ, we need the harmonic measure of the curve $l_x = \Phi(\theta_x)$ with respect to that part S_x of the strip to the left of l_x, evaluated at the fixed point $w_0 = \Phi(z_0)$.



We infer the position of l_x from the distortion theorems, in which we fix x_1 and take x_2 to be a varying x. Ahlfors lower bound on $u_1(x_2) - u_2(x_1)$ becomes, in effect, a lower bound on $u_1(x)$, and implies that l_x must lie at least a certain distance to the right in the strip S. This gives an upper bound on the harmonic measure of l_x, as harmonic measure of a vertical cross cut in S decreases as the cross cut is moved to the right. Warschawski's upper bound on $u_2(x)$ shows that l_x cannot be too far to the right in the strip S, and therefore leads to a lower bound on the harmonic measure of l_x. We can then estimate the harmonic measure, by using the example at the end of Section 1.1. One will ideally end up with an estimate involving

$$\exp\left[-\pi \int_{x_0}^{x} \frac{ds}{\theta(s)}\right].$$

There are harmonic measure versions of these distortion theorems, particularly upper bounds, that work in any finite dimension, and for non simply connected domains in the plane. Tsuji, building on work of Carleman, proved just such an estimate from above involving a term analogous to that arising in the simply connected planar case. For the many developments in this area, the reader need go no further than the books by Tsuji [23] and Ohtsuka [19], forgetting neither Baernstein's account [2], nor Haliste's complete and careful exposition [14], nor Section 8.1.7 in Hayman [16].

§2. Cones

2.1. Burkholder's 1977 paper on exit times of Brownian motion

In [7], Burkholder studies Brownian motion in $\mathbf{R}^n$ with starting point $x \in \mathbf{R}^n$ and an accompanying stopping time τ. With

$$B^*_\tau \;=\; \sup_t |B_{t\wedge\tau}|,$$

he proves that if one of the random variables $\sqrt{n\tau + |x|^2}$, $|B_\tau|$ or B^*_τ is p^{th}-power integrable, with $p \in (0,\infty)$, then so are they all. To deduce that B^*_τ is p^{th}-power integrable if $|B_\tau|$ is p^{th}-power integrable, it is assumed that $\mathrm{E}_x \log \tau < \infty$ if the dimension is 2 and that $\mathrm{P}_x(\tau < \infty) = 1$ in higher dimensions. The norms of these three random variables are then comparable, with constants that depend only on p and n.

These results are applicable when τ is the first exit time from a domain D in $\mathbf{R}^n$, in which case he proves the additional result that $\tau^{1/2}$ is p^{th}-power integrable if and only if the function $|x|^p$ has a harmonic majorant in D (this being a function u that is harmonic in D, and for which $|x|^p \le u(x)$ for all x in D). In Section 4 of the paper, Burkholder specializes to the case when D is the image of a conformal map F of the unit disk in two dimensions, and brings H^p spaces into play.

2.2. Integrability of exit time and exit place for a cone

As a 'simple' application of his results, Burkholder works out everything for a right circular cone

$$\Gamma_\alpha = \{x \in \mathbf{R}^n \setminus \{0\}, 0 \le \theta(x) < \alpha\}$$

where $\theta(x)$ is the angle between x and $(1,0,\ldots,0)$. Let us write T_α for the exit time from Γ_α, and go through the argument in two dimensions. If $p\alpha < \pi/2$ the function

$$u(x) = |x|^p \cos(p\theta)/\cos(p\alpha)$$

is harmonic in Γ_α, and $|x|^p \leq u(x)$ there. In this case, $|x|^p$ has a harmonic majorant and $T_\alpha^{1/2}$ is in L^p. In the case $p\alpha = \pi/2$,

$$u(x) = |x|^p \cos(p\theta)$$

is a harmonic function in Γ_α, vanishes on the boundary of Γ_α, and satisfies $0 < u(x) \leq |x|^p$ in Γ_α. From this Burkholder deduces that $T_\alpha^{1/2}$ is not in L^p. In fact, fixing any x in the cone, he chooses a sequence of bounded domains R_j, each containing x, with $\overline{R_j} \subset R_{j+1}$ and whose union is the whole cone. He writes T_j for the exit time from R_j. If $T_\alpha^{1/2}$ was p^{th}-power integrable then so would $B^*_{T_\alpha}$. Moreover, for each j,

$$\left|u\left(B_{T_j}\right)\right| \leq |B_{T_j}|^p \leq B^*_{T_\alpha} \quad (\text{since } T_j \leq T_\alpha).$$

Since u is harmonic and bounded in R_{j+1}, $\{u(B_{T_j \wedge t})\}_{t \geq 0}$ is a martingale and so, by optional stopping,

$$u(x) = E_x u(B_{T_j}).$$

Hence, by dominated convergence,

$$0 < u(x) = \lim_{j\to\infty} E_x u(B_{T_j}) = E_x\left[\lim_{j\to\infty} u(B_{T_j})\right] = E_x u(B_{T_\alpha}) = 0,$$

a contradiction. Thus $T_\alpha^{1/2} \notin L^p$ if $p\alpha = \pi/2$. Hence

$$T_\alpha^{1/2} \in L^p \iff \alpha < \frac{\pi}{2p}.$$

The same method works for higher dimensional cones, with the role of $|x|^p \cos(p\theta)$ being played by $|x|^p h(\theta)$ where h is a certain hypergeometric function. This function has a smallest positive zero $\theta_{p,n}$ with $\theta_{p,n} < \pi$, and then

$$T_\alpha^{1/2} \in L^p \iff \alpha < \theta_{p,n}.$$

2.3. Some further developments

Burkholder's results, and the approach he took, gave rise to significant further research, for example that of Essén and Haliste [12]. Sakai [21] proved an interesting isoperimetric inequality in this area: if u is the least harmonic majorant of $|x|^p$ in a bounded domain D that contains 0, then

$$u(0)^{1/p} \leq c\, r(D),$$

for some finite c depending only on p and the dimension. Here $r(D)$ is the volume radius of D, the radius of a ball in $\mathbf{R}^n$ with the same volume as D. He furthermore proved several interesting results about the best constant $c(p,n)$, bringing into play, as did Burkholder, exit times of Brownian motion, Hardy spaces and estimates of solutions of Poisson equations.

Expressions for the distribution function of the exit time from a cone have been obtained by Spitzer [22] in the planar case, from which the integrability result $T_\alpha^{1/2} \in L^p$ if and only if $\alpha < \pi/(2p)$ follows, by DeBlassie [10] in higher dimensions, and also in Bañuelos and Smits [5].

§3. Paraboloids

3.1. Exit time

Relatively recently, Bañuelos, DeBlassie and Smits [4] set themselves the task of uncovering the tail distribution of the exit time of Brownian motion from another conic section, the parabola

$$\mathcal{P} = \left\{ z\colon \operatorname{Re} z > 0 \text{ and } |\operatorname{Im} z| < \sqrt{\operatorname{Re} z} \right\}$$

in the plane. The exit time from any bounded domain is exponentially integrable, while that from a cone is only power integrable. The authors' goal was to find a domain for which the integrability of its exit time was intermediate between these two extremes. The parabola can be fitted inside a cone whose aperture is as small as one may wish, simply by putting the vertex of the cone far out on the negative real axis and the axis of the cone in the direction of the positive real axis. The exit time from the parabola is less than that from the larger cone, and the latter will be in L^p for large p because the aperture of the cone is small. Consequently, the exit time $\tau_{\mathcal{P}}$ from the parabola is p^{th}-power integrable for every finite p. On the other hand, $E_x\left[\exp(c\tau_{\mathcal{P}})\right]$ cannot be finite for any positive c, since $\mathcal{P}$ contains disks of arbitrarily large radius. For a disk B of radius r and centre x contained in $\mathcal{P}$, one has that $E_x\left[\exp(c\tau_{\mathcal{P}})\right] \geq E_x\left[\exp(c\tau_B)\right]$, and the latter is infinite when $c > a/r^2$, for an absolute constant a.

The estimate obtained by Bañuelos, DeBlassie and Smits for the distribution function of the exit time is

$$A_1 \leq \liminf_{t\to\infty} t^{-1/3} \log \frac{1}{P_z(\tau_{\mathcal{P}} > t)} \leq \limsup_{t\to\infty} t^{-1/3} \log \frac{1}{P_z(\tau_{\mathcal{P}} > t)} \leq A_2$$

for some positive constants A_1 and A_2. This indicates that $P_z(\tau_\alpha > t)$ may be about the size of $\exp\left[-A\, t^{1/3}\right]$.

This estimate was extended to higher dimensions and to more general paraboloids by Wembo Li [17], but it was Lifshits and Shi [18] who solved the problem completely. They showed that, for the exit time τ_α from the parabola-shaped region

$$\mathcal{P}_\alpha = \{(x,Y) \in \mathbf{R} \times \mathbf{R}^{n-1} \colon x > 0,\ |Y| < Ax^\alpha\}$$

(where $A > 0$ and $0 < \alpha < 1$),

$$\lim_{t\to\infty} t^{\frac{\alpha-1}{\alpha+1}} \log \frac{1}{P_z(\tau_\alpha > t)}$$

exists, and they determined the finite, positive limit explicitly. In particular, they proved that

$$\lim_{t\to\infty} t^{-\frac{1}{3}} \log \frac{1}{P_z(\tau_{1/2} > t)} = \frac{3\pi^2}{8}.$$

The distribution function of the exit time from a related very general class of unbounded domains is investigated in [11]. Using the results of Lifshits and Shi, van den Berg [6] studied the behaviour of the heat kernel in parabola-shaped regions.

3.2. Exit place

In [3], Bañuelos and the author investigated the rate of decay of harmonic measure in the parabola-shaped regions $\mathcal{P}_\alpha$ in $\mathbf{R}^n$ or, equivalently, the distribution function of the exit position of Brownian motion from such domains. Setting E_ρ to be that part of the boundary of $\mathcal{P}_\alpha$ lying outside the ball of centre 0 and radius ρ, we wished to estimate

$$\omega(\rho) = \omega(z_0, E_\rho; \mathcal{P}_\alpha) = P_{z_0}\left(|B_{\tau_\alpha}| > \rho\right),$$

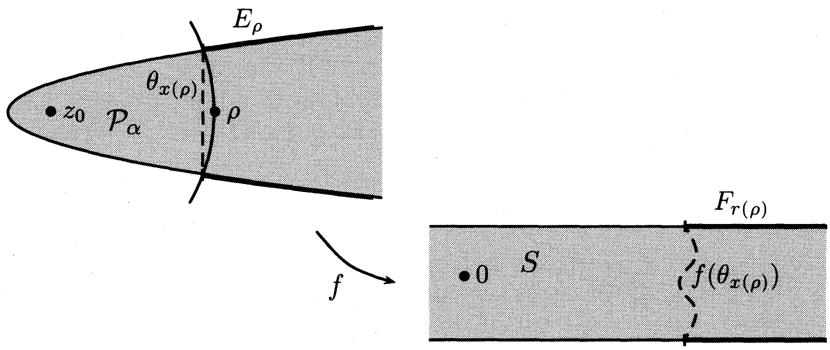
for some fixed z_0, say $z_0 = (1, 0, \ldots, 0)$, as accurately as possible.

This problem is easy to solve in the case of two dimensions using the techniques described earlier in this article. We map $\mathcal{P}_\alpha$ onto the strip S by a symmetric conformal mapping f with $f(z_0) = 0$. The part E_ρ of the boundary of $\mathcal{P}_\alpha$ starts from the cross cut at $x = x(\rho)$, where $\rho - \rho^{2\alpha-1} < x(\rho) < \rho$. The length of the cross cut of $\mathcal{P}_\alpha$ at x is $\theta(x) = 2x^\alpha$, which satisfies

$$\int^\infty \frac{\theta'(x)^2}{\theta(x)}\, dx < \infty.$$

Thus the Distortion Theorem for Certain Symmetric Domains of Section 1.3 is applicable and yields that the image of the cross cut $\theta_{x(\rho)}$ is within a bounded distance of

$$\frac{\pi}{2(1-\alpha)}\, x(\rho)^{1-\alpha}.$$



Writing F_r for that part of the boundary of the strip S where the real part is greater than r, the image of E_ρ under f is $F_{r(\rho)}$ where

$$r(\rho) = \frac{\pi}{2(1-\alpha)}\, x(\rho)^{1-\alpha} + \mathrm{O}(1).$$

At the end of Section 1.1, we worked out the rate of decay of harmonic measure in the strip, and found that

$$\omega(0, F_r; S) = \frac{2}{\pi}\, e^{-r} + \mathrm{o}(1) \quad \text{as } r \to \infty.$$

Thus

$$\omega(\rho) = \omega(1, E_\rho; \mathcal{P}_\alpha) = \omega(0, F_{r(\rho)}; S) \sim \exp\left[-\frac{\pi}{2(1-\alpha)}\, x(\rho)^{1-\alpha}\right].$$

A consequence of this is the following sharp integrability result,

$$E_1\left[\exp\left(b|B_{\tau_\alpha}|^{1-\alpha}\right)\right] < \infty \iff b < \frac{\pi}{2(1-\alpha)},$$

which is analogous to that of Burkholder for cones.

It does not seem to be so straightforward, however, to prove such precise results in higher dimensions. At the beginning of my talk at IWPT 2004, I asked the audience whether $P_{z_0}(|B_\tau| > \rho)$ is larger (for large ρ) for the exit time τ from

(i) the parabola $\{(x,y) : x > 0 \text{ and } |y| < \sqrt{x}\}$ in the plane

or from

(ii) the paraboloid $\{(x,Y) : x > 0 \text{ and } |Y| < \sqrt{x}\}$ in three dimensions.

The answer to this question follows from the explicit asymptotics that we derive in [3]. Chris Burdzy was kind enough to explain to me how he answered the question without knowing the exact asymptotics.

Brownian motion $\{B_t\}$ in the parabola-shaped region $\mathcal{P}_\alpha$ may be thought of as a one-dimensional Brownian motion $\{B_t^1\}$ in the x_1-direction and an independent Bessel process $\{X_t\}$ of order $n-1$ in the orthogonal direction. He showed that, for fixed α and ρ, $P_{z_0}\left(B_{\tau_\alpha}^{1,*} > \rho\right)$, where

$$B_{\tau_\alpha}^{1,*} = \sup_t B_{t\wedge\tau_\alpha}^1,$$

decreases as the dimension increases. [This probability is the harmonic measure at z_0 of the cross section θ_ρ of $\mathcal{P}_\alpha$ at $x_1 = \rho$ with respect to that part of $\mathcal{P}_\alpha$ to the left of this cross section.] The probability that $B_{\tau_\alpha}^{1,*} > \rho$ is the probability that the one-dimensional Brownian motion B_t^1 hits level ρ before the Bessel process X_s exceeds $(B_s^1)^\alpha$. The Bessel process of order $n-1$ satisfies the stochastic differential equation

$$dX_t = dZ_t + \frac{n-2}{2}\frac{1}{X_t}\,dt$$

while the Bessel process of order n corresponding to $\mathcal{P}_\alpha$, but one dimension higher, satisfies

$$dY_t = dZ_t + \frac{n-1}{2}\frac{1}{Y_t}\,dt,$$

where we may take $\{Z_t\}$ to be the *same* one dimensional Brownian motion in each case and to be independent of $\{B_t^1\}$. Since $X_0 = Y_0$ and since the drift coefficient $(n-1)/2$ for Y_t is always greater than that for X_t, which is $(n-2)/2$, it follows from a general comparison result that, for any time t, $Y_t \geq X_t$ a.s. Thus $Y_s - (B_s^1)^\alpha$ will become non negative before $X_s - (B_s^1)^\alpha$ becomes non negative.

In [3], it is shown that the rate of decay of harmonic measure in $\mathcal{P}_\alpha$ satisfies, for each positive ϵ,

$$\exp\left[-\frac{\sqrt{\lambda_1}}{1-\alpha}\,(1+\epsilon)\,\rho^{1-\alpha}\right] \leq \omega(\rho) \leq \exp\left[-\frac{\sqrt{\lambda_1}}{1-\alpha}\,(1-\epsilon)\,\rho^{1-\alpha}\right]$$

for all sufficiently large ρ, where λ_1 is the smallest eigenvalue for the Dirichlet Laplacian in the unit ball in $\mathbf{R}^{n-1}$. The upper bound comes from the Carleman estimate, which belongs to the family of upper bounds for harmonic measure described in Section 1.4. Lower bounds for harmonic measure, in situations such as that under consideration here, are harder to obtain. In his lower bound on harmonic measure mentioned in Section 1.3, Warschawski needed to take into account the oscillation of the width and the oscillation of the central line of the domain. The central lines of our parabola-shaped domains don't oscillate, while the width is increasing but not too quickly in that $\int \theta'(x)^2/\theta(x)\,dx$ is finite. It is natural to expect that the upper bound for harmonic measure given by the Carleman method would be achieved in this case. The problem, however, was how to prove this.

The harmonic measure $\omega(x, E_\rho; \mathcal{P}_\alpha)$ has symmetry that it inherits from the symmetry of the domain and that of its boundary values: at (x, Y) in $\mathcal{P}_\alpha$ (where $x \in \mathbf{R}$, $Y \in \mathbf{R}^{n-1}$),

$$\omega((x, Y), E_\rho; \mathcal{P}_\alpha) = u(x, |Y|)$$

for some function $u(x, y)$ that is defined on the upper half of the domain $\mathcal{P}_\alpha$ in the plane. Whereas the harmonic measure satisfies Laplace's equation, the function u satisfies

$$\Delta u + (n-2)\frac{u_y}{y} = 0.$$

We would like to transform from the parabola to the strip, as this worked so well in two dimensions. However, unlike Laplace's equation, this Bessel type operator is not conformally invariant. Transformation to the strip results in what is, at first sight, a messy expression that involves both the mapping g from the strip to the parabola and its derivative. The asymptotics of the mapping g and of g' can be deduced relatively easily from Warschawski's work. However, the asymptotic estimates for the derivative are restricted to sub strips $S_\rho = \{z\colon |\operatorname{Im} z| < \rho\}$ where $\rho < \pi/2$. To transform the partial differential equation $\Delta u + (n-2)u_y/y = 0$ successfully from the parabola $\mathcal{P}_\alpha$ to the strip S we needed *uniform* estimates on the derivative of the conformal mapping. These were obtained as part of the results in [8] and lead directly to the following result:

Suppose that g is a symmetric conformal mapping of the infinite strip S onto $\mathcal{P}_\alpha$, with $g(x) \to \infty$ as $x \to \infty$. There is a function $\epsilon(w)$ in the strip S with $\epsilon(w) \to 0$ as $\operatorname{Re} w \to \infty$, uniformly in the imaginary part of w. Moreover, whenever $u(x, y)$ satisfies the p.d.e. $\Delta u + (n-2)u_y/y = 0$ in $\mathcal{P}_\alpha^+ = \{(x, y) : 0 < y < x^\alpha\}$ then $v = u \circ g$ satisfies the p.d.e.

$$\Delta v + (n + \epsilon(w) - 2)\frac{v_y}{y} = 0$$

in the upper half of the strip.

The Bessel operator is, in some sense, asymptotically conformally invariant. In our situation, the function v arises from the harmonic measure of the exterior of a ball of radius ρ in $\mathcal{P}_\alpha$ in $\mathbf{R}^n$. We know the rate of growth of the mapping from the planar parabola-shaped domain onto the strip, from which we can deduce the boundary values for v.

With a little more careful analysis involving the maximum principle, the lower bound for harmonic measure follows.

These bounds on harmonic measure lead directly to integrability results for the exit position, valid in each finite dimension:

$$E_1\left[\exp\left(b|B_{\tau_\alpha}|^{1-\alpha}\right)\right]$$

is finite if $b < \sqrt{\lambda_1}/(1-\alpha)$ and is infinite if $b > \sqrt{\lambda_1}/(1-\alpha)$. Just as in Burkholder's work on cones, we proved that $B^*_{\tau_\alpha}$ has the same integrability properties.

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