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Neumann eigenfunctions and Brownian couplings

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Abstract.

This is a review of research on geometric properties of Neumann eigenfunctions related to the "hot spots" conjecture of Jeff Rauch. The paper also presents, in an informal way, some probabilistic techniques used in the proofs.

§1. Introduction

In 1974 Jeff Rauch stated a problem at a conference, since then referred to as the "hot spots conjecture" (the conjecture was not published in print until 1985, in a book by Kawohl [K]). Informally speaking, the conjecture says that the second Neumann eigenfunction for the Laplacian in a Euclidean domain attains its maximum and minimum on the boundary. There was hardly any progress on the conjecture for 25 years but a number of papers have been published in recent years, on the conjecture itself and on problems related to or inspired by the conjecture. This article will review some of this body of research and techniques used in it, with focus on author's own research and probabilistic methods used in proofs of analytic results.

The paper is organized as follows. First, we will state and explain the conjecture. Then we will review the main results on the conjecture and related problems. Finally, we will review some techniques used in the proofs.

In order to explain the intuitive contents of the hot spots conjecture we will start with the heat equation. Suppose that D is an open connected bounded subset of $\mathbb{R}^d$, $d \geq 1$. Let $u(t,x), t \geq 0, x \in D$, be the solution of the heat equation $\partial u/\partial t = \Delta_x u$ in D with the Neumann

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boundary conditions and the initial condition $u(0,x) = u_0(x)$. That is, $u(t,x)$ is a solution to the following initial-boundary value problem,

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta_x u(t,x), & x \in D,\, t > 0, \\ \dfrac{\partial u}{\partial \mathbf{n}}(t,x) = 0, & x \in \partial D,\, t > 0, \\ u(0,x) = u_0(x), & x \in D, \end{cases}$$

where $\mathbf{n}(x)$ denotes the inward normal vector at $x \in \partial D$. The long time behavior of a "generic" solution (i.e., the solution corresponding to a "typical" initial condition) can be derived from the properties of the second eigenfunction using the following eigenfunction expansion. Under suitable conditions on the domain, such as convexity or Lipschitz boundary, and for a "typical" initial condition $u_0(x)$, we have

$$(1.2) \qquad u(t,x) = c_1 + c_2\varphi_2(x)e^{-\mu_2 t} + R(t,x),$$

where $c_1 \in \mathbb{R}$ and $c_2 \neq 0$ are constants depending on the initial condition, $\mu_2 > 0$ is the second eigenvalue for the Neumann problem in D, $\varphi_2(x)$ is a corresponding eigenfunction, and $R(t,x)$ goes to 0 faster than $e^{-\mu_2 t}$, as $t \to \infty$. Note that the first eigenvalue is equal to 0 and the first eigenfunction is constant. Suppose that $\varphi_2(x)$ attains its maximum at the boundary of D. Under this assumption, for "most" initial conditions $u_0(x)$, if z_t is a point at which the function $x \to u(t,x)$ attains its maximum, then the distance from z_t to the boundary of D tends to zero as t tends to ∞. In other words, the "hot spots" move towards the boundary.

Hot Spots Conjecture (Rauch (1974)). *The second eigenfunction for the Laplacian with Neumann boundary conditions in a bounded Euclidean domain attains its maximum at the boundary.*

The above version of the hot spots conjecture is somewhat ambiguous as it does not specify whether the maximum has to be strict, i.e., whether the eigenfunction can attain the same maximal value somewhere in the interior of the domain; it does not address the question of what might happen when the second eigenvalue is not simple, i.e., whether all eigenfunctions corresponding to the second eigenvalue have to satisfy the conjecture (in some domains, for example, the square, there are infinitely many eigenfunctions corresponding to the second eigenvalue). As we will see, it turns out that a precise statement of the conjecture is not needed because the results do not depend in a subtle way on its formulation.

The hot spots conjecture can be justified by appealing to our physical intuition and by examples amenable to explicit analysis. Intuitively, the "heat" and "cold" are substances that annihilate each other so it is easy to believe that the hottest and coldest spots lie as far as possible from each other, hence on the boundary of the domain. One can find explicit formulas for the eigenfunctions in some simple domains, for example, in a rectangle $[0,a] \times [0,b]$ with $a > b > 0$, we have $\varphi_2(x_1, x_2) = \cos(\pi x_1/a)$. All such explicit examples support the hot spots conjecture, i.e., the second eigenfunction attains the maximum on ∂D in simple domains such as rectangles, discs and balls.

§2. Main theorems on the "hot spots" problem

For 25 years, from 1974 to 1999, almost nothing was known about the "hot spots" conjecture. A notable exception was a result by Kawohl that appeared in his book [K] in 1985. Kawohl proved that if a set $D \subset \mathbb{R}^d$ is a cylindrical domain, i.e., if $d > 1$, and D can be represented as $D = D_1 \times [0,1]$ for some $D_1 \subset \mathbb{R}^{d-1}$, then the hot spots conjecture holds for D. This result has a simple proof based on the factorization of eigenfunctions in cylindrical domains. Kawohl's most lasting contributions are the realization that one should restrict attention to some classes of domains, and the statement of the currently most significant open problem in the area—Kawohl suggested that the hot spot conjecture might not be true in general but it should be true for convex domains.

The next paper on the hot spots conjecture, [BB1], appeared in 1999. The paper contained the proof of the hot spots conjecture for two classes of planar domains: domains with a line of symmetry and "lip" domains, to be described shortly. The results were not complete, in the sense that the authors imposed some extra "technical" assumptions on domains in each family. Those extra assumptions were removed for symmetric domains by Pascu [P] and for "lip" domains in [AB2].

Recall that a function f is called Lipschitz with constant c if $|f(x) - f(y)| \leq c|x-y|$ for all x and y. A "lip" domain is a bounded planar domain such that its boundary consists of two graphs of Lipschitz functions with the Lipschitz constant equal to 1. For example, any obtuse triangle (i.e., a triangle with an angle greater than π) is a lip domain if it is properly oriented. In Fig. 2.1, D_1, D_2 and the interior of $\overline{D_1 \cup D_2}$ are lip domains.

Theorem 2.1. *The hot spots conjecture holds for $D \subset \mathbb{R}^2$ if*

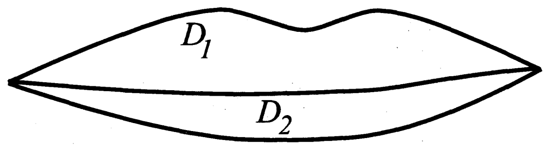

Figure 2.1.

(i) ([BB1], [P]) D is convex and has a line of symmetry, or
(ii) ([BB1], [AB2]) D is a lip domain.

The methods and techniques developed in [BB1] to prove the hot spots conjecture for some classes of domains turned out to be useful also in deriving negative results. The first of such results, [BW], appeared in 1999. The authors showed that there exists a planar domain where the second eigenvalue is simple and the eigenfunction corresponding to the second eigenvalue attains its maximum in the interior of the domain. This result was strengthened in [BB2], where it was shown that in some other planar domain, the second eigenvalue is simple and the second eigenfunction attains both its minimum and maximum in the interior of the domain. The domain constructed in [BB2] had many holes and the one constructed in [BW] had 2 holes. The intuitive idea behind the examples constructed in [BW] and [BB2] suggested that every counterexample to the hot spots conjecture in the plane must have at least two holes, and every counterexample in $\mathbb{R}^d$, $d \geq 3$, must have at least d handles. This turned out not to be true—a new counterexample ([B2]) shows that there exists a planar domain with one hole and simple second eigenvalue, and such that the second eigenfunction attains both its maximum and minimum in the interior of the domain. The domain is depicted in Fig. 2.2. Its shape is much simpler than that of examples in [BW] and [BB2]. The maximum and minimum of the second eigenfunction are attained at the points marked on the figure.

Theorem 2.2. ([BW], [BB2], [B2]) *The hot spots conjecture fails for some domains* $D \subset \mathbb{R}^2$.

Before we discuss results related to the hot spots conjecture in various ways, we will state the most intriguing open problems in this area. The first one was proposed by Kawohl in [K], and the second one is known among the researchers interested in the subject.

Open problems. (i) ([K]) *Does the hot spots conjecture hold for bounded convex domains* $D \subset \mathbb{R}^d$ *for all* $d \geq 1$*?*

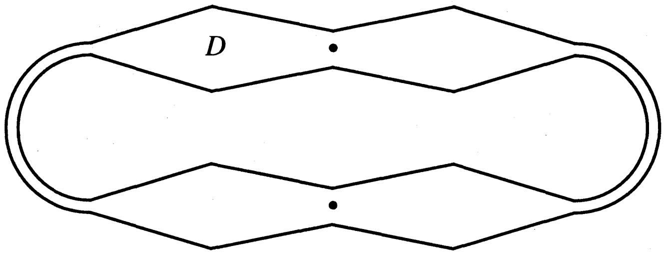

Figure 2.2.

(ii)*Does the hot spots conjecture hold for bounded simply connected planar domains?*

§3. Results related to the "hot spots" problem

The hot spots conjecture inspired a number of papers on the properties of Neumann eigenfunctions. We will review those that seem to be the closest in spirit to the original conjecture. For a review of research in related areas, see [NTJ].

First of all, we mention a paper by Hempel, Seco and Simon [HSS], which appeared in 1991, long time before the current interest in the hot spots conjecture. The authors studied the spectrum of the Neumann Laplacian in bounded Euclidean domains with non-smooth boundaries. Roughly speaking, their results show that the spectrum does not need to be discrete, and in a sense, it can be completely arbitrary. For this reason, the hot spots conjecture must be limited to domains where the spectrum is discrete, such as domains with Lipschitz boundaries.

Athreya [A2] showed that some monotonicity properties of Neumann eigenfunctions hold also for solutions of some semi-linear partial differential equations related to a class of stochastic processes known as "superprocesses." He adapted the probabilistic techniques used in the research on the hot spots conjecture to the new setting.

Jerison [J] found the location (in an asymptotic sense) of the nodal line (i.e., the line where the eigenfunction vanishes) of the second Neumann eigenfunction in long and thin domains. Strictly speaking, this result is not directly related to the hot spots conjecture. However, the information about the location of the nodal line can be effectively used in the research on the hot spots conjecture. This was first done in [BB1], where the nodal line was identified with the line of symmetry in domains possessing a line of symmetry. The knowledge of the nodal line can be used to transform the Neumann problem to a problem with mixed Neumann and Dirichlet conditions—a problem much easier than the original one. Jerison and Nadirashvili considered in [JN] convex planar domains with two perpendicular lines of symmetry, and showed that under these strong assumptions one can provide some accurate information about the second eigenfunction. The location of the nodal line for the second eigenfunction is treated as a problem of its own interest in [AB1], where probabilistic techniques are used to give some results in this direction.

Atar investigated in [A1] a class of multidimensional domains. Techniques used in other papers on the hot spots problem seem to work only in planar domains so [A1] is the only paper (except for an early result in [K]) that contains results on the multidimensional version of the problem.

It was known for a long time, as a "folk law" among the experts in the field, that the hot spots conjecture does not hold for manifolds, see, e.g., remarks to this effect in [BB1] or [BB2]. However, the first rigorous paper studying the hot spots problem for manifolds was published by Freitas [F].

Although a paper by Ishige and Mizoguchi [IM] is not devoted to the hot spots problem in the sense of this article, it is related because it studies geometric properties of the heat equation solutions.

Two recent papers by Bañuelos and Pang, one of them joint with Pascu ([BP] and [BPP]) are devoted to variations of the hot spots problem. The purpose of [BP] is to prove an inequality for the distribution of integrals of potentials in the unit disk composed with Brownian motion which, with the help of Lévy's conformal invariance, gives another proof of Pascu's result [P]. The paper [BPP] investigates the "hot spots" property for the survival time probability of Brownian motion with killing and reflection in planar convex domains whose boundary consists of two curves, one of which is an arc of a circle, intersecting at acute angles. This leads to the "hot spots" property for the mixed Dirichlet-Neumann eigenvalue problem in the domain with Neumann conditions on one of the curves and Dirichlet conditions on the other.

§4. Review of selected probabilistic techniques

The following review of techniques used in proofs of results related to the hot spots conjecture is highly subjective in its choices, dealing mostly with methods used by the author of this article in his own research. The review will mainly focus on "essential probabilistic techniques," i.e., those techniques that involve stochastic processes and cannot be easily translated into the language of analysis. A good way to illustrate this idea is to look at an example of a probabilistic concept that is *not* essential. The hitting distribution of Brownian motion on the boundary of a set can be identified with the harmonic measure—the two concepts are equivalent but knowing this equivalence does not immediately lead to any new results. We will focus on a probabilistic technique called "couplings." The technique was invented by Doeblin in 1930's and one can find a general review of this method in books by Lindvall [L] and Mu-Fa Chen [C]. The most frequent application of the coupling technique consists of a construction of two processes on the same probability space, run with the same clock. Often, the processes meet at a certain time, called the coupling time. Typically, the processes are *not* independent. One usually tries to find a coupling with as small coupling time as possible. A distinguishing feature of applications of couplings in the context of the hot spots conjecture is that the properties of the coupling time usually do not matter, and in a somewhat perverse way, the coupling time is infinite for some of the couplings. Couplings were used for the first time to study the hot spots conjecture in [BB1] but that paper owes a lot to an earlier project, [BK], devoted to a seemingly unrelated problem.

Many proofs of results on the hot spots conjecture are based on the eigenfunction expansion (1.2). First, a geometric property is proved for the heat equation and then it is translated into a statement about the second eigenfunction using (1.2), as $t \to \infty$.

For an introductory presentation of probabilistic concepts used below, such as Brownian motion, and their relationship to analysis, see a book by Bass [B1].

Let X_t and Y_t be reflected Brownian motions in D starting from $x \in D$ and $y \in D$, resp. Then we can represent the solution $u(t,x)$ of the heat equation (1.1) as $u(t,x) = Eu_0(X_t)$, and similarly $u(t,y) = Eu_0(Y_t)$. We have by (1.2),

$$\begin{aligned} \varphi_2(x) - \varphi_2(y) &= c_3 e^{\mu_2 t}(u(t,x) - u(t,y)) + R_1(t,x,y) \\ &= c_3 e^{\mu_2 t}(Eu_0(X_t) - Eu_0(Y_t)) + R_1(t,x,y), \end{aligned} \tag{4.1}$$

where $R_1(t,x,y)$ goes to 0 as $t \to \infty$. Without loss of generality we will assume that $c_3 > 0$. Suppose that we can prove for some initial condition u_0 that for all $t > 0$,

$$Eu_0(X_t) - Eu_0(Y_t) \le 0. \tag{4.2}$$

This and (4.1) will then show that $\varphi_2(x) \leq \varphi_2(y)$. If the last inequality can be proved for an appropriate family of pairs (x, y), the hot spots conjecture will follow. We will next present a technique of proving (4.2).

For $x, y \in \mathbb{R}^2$, write $x \preceq y$ if the angle between $y - x$ and the positive horizontal half-line is within $[-\pi/4, \pi/4]$. Suppose that D is a lip domain (defined in Section 2) and $x, y \in D$, $x \preceq y$. Suppose that X_t and Y_t are reflected Brownian motions in D, driven by the same Brownian motion, and starting from x and y, resp. In other words,

$$\begin{aligned} X_t &= x + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X, \\ Y_t &= y + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y, \end{aligned} \tag{4.3}$$

where $\mathbf{n}(z)$ is the unit inward normal vector at $z \in \partial D$ and L_s^X is the local time of X on the boundary of D, i.e., L^X is a non-decreasing process that does not increase when X is inside D. In other words,

$$\int_0^\infty \mathbf{1}_D(X_s) dL_s^X = 0.$$

Similar remarks apply to the formula for Y_t. For domains which are piecewise C^2-smooth, the existence of processes satisfying (4.3) follows from results of Lions and Sznitman [LS]. For lip domains, one can use a recent result from [BBC]. The existence of a strong unique solution to an equation analogous to (4.3) but in a multidimensional Lipschitz domain remains an open problem at this time.

We have assumed that the domain D is a lip domain so if the normal vector $\mathbf{n}(z)$ is well defined at $z \in \partial D$ (this is the case for almost all boundary points), it has to form an angle less than $\pi/4$ with the vertical. Then easy geometry shows that the "local time push" in (4.3), i.e., the term represented by the integral, is such that if $x \preceq y$ then

$$X_t \preceq Y_t \qquad \text{for all } t \geq 0. \tag{4.4}$$

Now consider a set $A \subset D$, such that both A and $D \setminus A$ have a non-empty interior and $\partial A \cap \partial(D \setminus A)$ is a vertical line segment. Suppose that A lies to the right of $D \setminus A$ and let the initial condition be $u_0(z) = \mathbf{1}_A(z)$. If (4.4) is satisfied, then for any fixed time $t \geq 0$, we may have $X_t, Y_t \in A$, or $X_t, Y_t \in D \setminus A$, or $X_t \in D \setminus A, Y_t \in A$, but we will never have $X_t \in A, Y_t \in D \setminus A$. This and the definition of u_0 imply (4.2). We combine this with (4.1) to conclude that $\varphi_2(x) \leq \varphi_2(y)$ for $x \preceq y$. Any lip domain has the "leftmost" and "rightmost" points in the sense of the partial order "$\leq$" (see Fig. 2.1) so our argument has shown that the maximum and the minimum of the second eigenfunction are attained at these two points. Hence, the hot spots conjecture holds in lip domains.

Planar domains with a line of symmetry have to be approached in a different manner. Suppose that $D \subset \mathbb{R}^2$ is symmetric with respect to a vertical line K and let D_1 be the part of D lying to the right of K. Under some extra assumptions, the second eigenfunction φ_2 in D with the Neumann boundary conditions is antisymmetric with respect to K (this follows from a simple symmetrization argument). Therefore, φ_2 must vanish on K and we see that φ_2 is the first eigenfunction for the Laplacian in D_1 with the Neumann boundary conditions on $\partial D_1 \setminus K$ and Dirichlet boundary conditions on K. Such boundary conditions correspond to the Brownian motion in D_1 that is reflected on $\partial D_1 \setminus K$ and killed on K.

We will choose the initial condition u_0 to be identically equal to 1 in D_1. Let T_K^X be the hitting time of K by X and let T_K^Y have the analogous meaning for Y. The strategy now is to construct Brownian motions X_t and Y_t in D_1, reflected on $\partial D_1 \setminus K$, killed on K, starting from x and y, and such that (4.2) holds not for a fixed time t but for an appropriate stopping time T. Let $T = T_K^X$. If we can show that X must hit K before Y does, then (4.2) follows and we have $\varphi_2(x) \leq \varphi_2(y)$ for this particular pair (x, y). We will not go into details of how it is best to choose x and y and what assumptions one must make about the geometry of D to carry out the argument outlined above. Instead, we will describe a coupling of reflected Brownian motions (the "mirror" coupling) that keeps the two Brownian particles in a relative position that ensures that $T_K^X \leq T_K^Y$.

Let us start by defining the mirror coupling for free Brownian motions in $\mathbb{R}^2$. Suppose that $x, y \in \mathbb{R}^2$, $x \neq y$, and that x and y are symmetric with respect to a line M. Let X_t be a Brownian motion starting from x and let τ be the first time t with $X_t \in M$. Then we let Y_t be the mirror image of X_t with respect to M for $t \leq \tau$, and we let $Y_t = X_t$ for $t > \tau$. The process Y_t is a Brownian motion starting from y. The pair (X_t, Y_t) is a "mirror coupling" of Brownian motions in $\mathbb{R}^2$.

Next we turn to the mirror coupling of reflected Brownian motions in a half-plane $\mathcal{H}$, starting from $x, y \in \mathcal{H}$. One can construct reflected Brownian motions X_t and Y_t in $\mathcal{H}$, starting from x and y, so that they have the following properties. The processes X_t and Y_t behave like free Brownian motions coupled by the mirror coupling as long as they are both strictly inside $\mathcal{H}$. When one of the processes hits the boundary, the two particles cannot behave as a "free" mirror coupling in the whole plane. We will describe their motion by specifying constraints on the particles—otherwise they can move in an arbitrary way. Let M be the line of symmetry for x and y and $H = M \cap \partial\mathcal{H}$. Then for every t, the distance from X_t to H is the same as for Y_t. Let M_t be the line of symmetry for X_t and Y_t. The "mirror" M_t may move, but only in a continuous way, while the point $M_t \cap \partial\mathcal{H} = H$ will never move. The absolute value of the angle between the mirror and the normal vector to $\partial\mathcal{H}$ at H can only decrease. These properties are illustrated in Fig. 4.1. The processes stay together after the first time they meet. The most important property of the mirror coupling is that the two processes X_t and Y_t remain at the same distance from a fixed point, the "hinge" H.

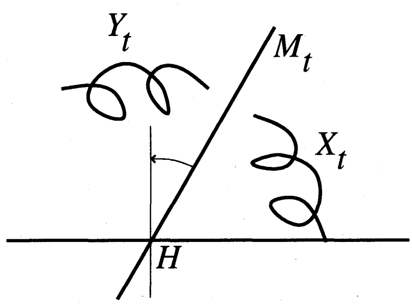

Figure 4.1.

When D is a polygonal domain, the processes X_t and Y_t will reflect on different sides of ∂D at different times. Since the reflecting particle cannot sense the global shape of the domain, the above description of the mirror coupling in a half-plane can be applied to describe the possible motions of the mirror (the line of symmetry between the processes) whenever only one of the processes is on the boundary. This simple recipe breaks down when the two processes hit the boundary at the same time. It is not obvious that two processes forming a mirror coupling can indeed hit the boundary at the same time but we conjecture that it is indeed true. The construction of the mirror coupling following the time when the two processes are simultaneously on the boundary has not been properly addressed in [BK] and [BB1]. In an earlier paper of Wang [W], mirror couplings were used without any proof of their existence. This unsatisfactory situation has been remedied recently as the full proof of the existence of mirror couplings in piecewise smooth domains has been given in [AB2], and the motion of the mirror following the time when both particles are on the boundary has been analyzed in [B2].

We will not present a detailed analysis of the motion of two particles related by a mirror coupling in a planar domain. The arguments involve no more than high school geometry.

The last coupling to be presented here is a "scaling coupling" introduced by Pascu [P]. This coupling is the most complex of the three couplings so we will only sketch the main ideas of this technique. The main objective of any coupling technique is to construct two processes whose relative motion is highly restricted, although each of the processes by itself is a reflected Brownian motion. This can lead to a condition such as (4.4) that can be in turn translated into an analytic statement using a formula such as (4.2).

Pascu's idea was to start with a planar Brownian motion X_t and let $Y_t = X_{at}/\sqrt{a}$, for some fixed $a > 0$. It is well known that Y is also a planar Brownian motion. The novelty of this coupling lies in the fact that although the shape of the trajectory of Y is a scaled image of the shape of the trajectory of X, the corresponding pieces of the trajectory occur at different times. In other words, the two processes run with different clocks. This rules out straightforward reasoning such as that in (4.1)-(4.4) but nevertheless Pascu managed to translate the information about possible geometric positions of the two processes into an analytic statement.

Two further technical aspects of scaling couplings should be mentioned here. The hot spots problem needs a construction of a pair of reflected Brownian motions in a domain D, not free Brownian motions in the whole plane. Hence, the simple scaling idea has to be modified in a way somewhat reminiscent of the way the mirror coupling in the plane is modified to handle reflected Brownian motions, because if X is a reflected Brownian motion in D then $Y_t = X_{at}/\sqrt{a}$ is not. Second, Pascu combined scaling couplings with conformal mappings in order to be able to handle arbitrary convex domains with a line of symmetry (the first step was to do the construction in a semi-disc). Conformal mappings preserve reflected Brownian motions but they require a time change. It was a very non-trivial observation of Pascu that the time change involved in his argument had the properties needed to finish the argument when the domain was convex.

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