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# Arc spaces, motivic integration and stringy invariants

**Willem Veys**

**Abstract.**

The concept of *motivic integration* was invented by Kontsevich to show that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. He constructed a certain measure on the *arc space* of an algebraic variety, the motivic measure, with the subtle and crucial property that it takes values not in $\mathbb{R}$, but in the Grothendieck ring of algebraic varieties. A whole theory on this subject was then developed by Denef and Loeser in various papers, with several applications.

Batyrev introduced with motivic integration techniques new singularity invariants, the *stringy invariants*, for algebraic varieties with mild singularities, more precisely log terminal singularities. He used them for instance to formulate a topological Mirror Symmetry test for pairs of singular Calabi-Yau varieties. We generalized these invariants to almost arbitrary singular varieties, assuming Mori's Minimal Model Program.

The aim of these notes is to provide a gentle introduction to these concepts. There exist already good surveys by Denef-Loeser [DL8] and Looijenga [Loo], and a nice elementary introduction by Craw [Cr]. Here we merely want to explain the basic concepts and first results, including the $p$-adic number theoretic pre-history of the theory, and to provide concrete examples.

The text is a slightly adapted version of the 'extended abstract' of the author's talks at the 12th MSJ-IRI "Singularity Theory and Its Applications" (2003) in Sapporo. At the end we included a list of various recent results.

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## 1 Pre-history

**1.1.** Let $f \in \mathbb{Z}[x_1, \cdots, x_m]$ and $r \in \mathbb{Z}_{>0}$. A very general problem in number theory is to compute the number of solutions of the congruence $f(x_1, \cdots, x_m) = 0 \bmod r$ (in $(\mathbb{Z}/r\mathbb{Z})^m$). Thanks to the Chinese remainder theorem it is enough to consider the case where $r$ is a power of a prime.

So we fix a prime number $p$ and we investigate congruences modulo varying powers of $p$. We denote by $F_n$ the number of solutions of $f(x_1, \cdots, x_m) = 0 \bmod p^{n+1}$.

**1.2.** *Examples.*

1. $f_1 = y - x^2$. It should be clear that $F_n = p^{n+1}$.

2. $f_2 = x \cdot y$. EXERCISE : $F_n = (n+2)p^{n+1} - (n+1)p^n$.

3. $f_3 = y^2 - x^3$. We list $F_n$ for small $n$ : $F_0 = p$,

$$
\begin{array}{lll}
F_1 = p(2p-1) & F_5 = p^5(p^2+p-1) & F_7 = p^7(2p^2-1) \\
F_2 = p^2(2p-1) & F_6 = p^6(p^2+p-1) & F_8 = p^8(2p^2-1) \\
F_3 = p^3(2p-1) & & F_9 = p^9(2p^2-1) \\
F_4 = p^4(2p-1) & & F_{10} = p^{10}(2p^2-1) \\
& F_{11} = p^{11}(p^3+p^2-1) & \\
& F_{12} = p^{12}(p^3+p^2-1). &
\end{array}
$$

Note that the plane curve $\{f_1 = 0\}$ is nonsingular, $\{f_2 = 0\}$ has the easiest curve singularity, an ordinary node, and $\{f_3 = 0\}$ has a slightly more complicated singularity, an ordinary cusp. It is in fact this cusp which is responsible for the at first sight not so nice behavior of the $F_n$ for $f_3$.

More generally, the problem of the behavior of the $F_n$ turns out to be non-obvious precisely when $\{f = 0\}$ has singularities.

**1.3.** We now know that, for any $f \in \mathbb{Z}[x_1, \cdots, x_m]$, the $F_n$ do satisfy the following 'regular' behavior.

**Conjecture** [Borewicz, Shafarevich] = **Theorem** [Igusa]. *The generating formal series $J_p(T) := J_p(f, T) = \sum_{n\geq 0} F_n T^n$ is a rational function in $T$. (In particular the $F_n$ are determined by a* finite *number of them.)*

Igusa showed this in 1975 [Ig1] using

(1) a 'translation' of $J_p(T)$ into a $p$-adic integral (more precisely into $\int_{\mathbb{Z}_p^m} |f|_p^s |dx|$, which is now called *Igusa's local zeta function*, and which is the ancestor of the motivic zeta function of section 6),

(2) an embedded resolution of singularities for $\{f = 0\}$,

(3) the change of variables formula for integrals.

(We will see later an analogue of this strategy in the theory of motivic integration.)

**1.4.** *Examples* (continuing 1.2).

1. $J_p(f_1; T) = \frac{p}{1-pT}$ (easy).
2. EXERCISE : $J_p(f_2; T) = \frac{2p-1-p^2T}{(1-pT)^2}$.
3. Claim : $J_p(f_3; T) = p\frac{1+(p-1)T+(p^6-p^5)T^5-p^7T^6}{(1-p^7T^6)(1-pT)}$.

**1.5.** We already want to mention another connection with singularity theory; the famous (still open) *monodromy conjecture* of Igusa relates the poles of $J_p(T)$ with eigenvalues of local monodromy of $f$ considered as a map $f : \mathbb{C}^n \to \mathbb{C}$, see (6.8).

**1.6.** Before introducing arc spaces and motivic integration in the next sections, we present a hopefully motivating analogy between this number theoretic setting and the geometric arc setting.

| $f \in \mathbb{Z}[x_1, \cdots, x_m]$ | $f \in \mathbb{C}[x_1, \cdots, x_m]$ |
|---|---|
| solution of $f = 0$ over the ring $\mathbb{Z}/p^{n+1}\mathbb{Z} \cong \mathbb{Z}_p/p^{n+1}\mathbb{Z}_p$, i.e. an $m$-tuple with coordinates of the form $a_0 + a_1 p + ... + a_n p^n$, $a_i \in \{0, 1, ..., p-1\}$ | solution of $f = 0$ over the ring $\mathbb{C}[t]/(t^{n+1}) \cong \mathbb{C}[[t]]/(t^{n+1})$, i.e. an $m$-tuple with coordinates of the form $a_0 + a_1 t + ... + a_n t^n$, $a_i \in \mathbb{C}$ ("n-jet" of $\{f = 0\}$) |
| solution of $f = 0$ over $\mathbb{Z}_p = \lim\limits_{\leftarrow} \mathbb{Z}/p^{n+1}\mathbb{Z}$, i.e. with coordinates of the form $\sum_{n=0}^{\infty} a_i p^i$ | solution of $f = 0$ over $\mathbb{C}[[t]] = \lim\limits_{\leftarrow} \mathbb{C}[t]/(t^{n+1})$, i.e. with coordinates of the form $\sum_{n=0}^{\infty} a_i t^i$ ("arc" of $\{f = 0\}$) |
| integrate over $\mathbb{Z}_p^m$ | integrate over $\mathcal{L}(\mathbb{C}^m) := \{$ arcs of $\mathbb{C}^m\}$ |

*Warning.* Here and further on we sometimes use other (better ?) normalizations than in the original papers.

## 2 Arc spaces

Let $X$ be an algebraic variety over $\mathbb{C}$. (The theory of arc spaces and motivic integration can be generalized to any field of characteristic zero, see e.g. [DL8].)

**2.1.** The *space of arcs modulo* $t^{n+1}$ or *space of n-jets on* $X$ is an algebraic variety $\mathcal{L}_n(X)$ over $\mathbb{C}$ such that

$$\begin{aligned}&\{\text{points of } \mathcal{L}_n(X) \text{ with coordinates in } \mathbb{C}\}\\ =&\{\text{points of } X \text{ with coordinates in } \frac{\mathbb{C}[t]}{(t^{n+1})}\}.\end{aligned}$$

For all $n$ there are obvious 'truncation maps' $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$, obtained by reducing $(n+1)$-jets modulo $t^{n+1}$, and more generally $\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$ for $m \geq n$. This description is somewhat informal, but is essentially what is needed. We now first provide examples and give the 'exact' definition later.

**2.2.** *Example.* Let $X = \mathbb{C}^d$. Then

$$\begin{aligned}\mathcal{L}_n(X) =&\{(a_0^{(1)} + a_1^{(1)}t + \cdots + a_n^{(1)}t^n, \cdots, a_0^{(d)} + a_1^{(d)}t + \cdots + a_n^{(d)}t^n),\\ &\quad \text{with all } a_i^{(j)} \in \mathbb{C}\}\\ \cong&\mathbb{C}^{(n+1)d}.\end{aligned}$$

**2.3.** *Example.* Let $X = \{y^2 - x^3 = 0\}$.

(0) $\mathcal{L}_0(X) = \{(a_0, b_0) \in \mathbb{C}^2 | b_0^2 = a_0^3\} = X$.

(1) $\mathcal{L}_1(X)$

$$\begin{aligned}&= \{(a_0 + a_1t, b_0 + b_1t) \in (\mathbb{C}[t]/(t^2))^2 \mid (b_0 + b_1t)^2 = (a_0 + a_1t)^3 \text{mod } t^2\}\\ &= \{(a_0 + a_1t, b_0 + b_1t) \in (\mathbb{C}[t]/(t^2))^2 \mid b_0^2 = a_0^3 \text{ and } 2b_0b_1 = 3a_0^2a_1\}.\end{aligned}$$

So we can consider $\mathcal{L}_1(X)$ as the (two-dimensional) algebraic variety in $\mathbb{C}^4$ with equations $b_0^2 = a_0^3$ and $2b_0b_1 = 3a_0^2a_1$ in the coordinates $a_0, a_1, b_0, b_1$. The map $\pi_0^1 : \mathcal{L}_1(X) \to \mathcal{L}_0(X) = X$ is induced by the projection $\mathbb{C}^4 \to \mathbb{C}^2 : (a_0, a_1, b_0, b_1) \mapsto (a_0, b_0)$.

The fibre of $\pi_0^1$ above $(0,0)$ is $\{(0, a_1, 0, b_1)\} \cong \mathbb{C}^2$; this corresponds to the fact that the tangent space to $X$ at $(0,0)$ is the whole $\mathbb{C}^2$. The fibre above $(a_0, b_0) \neq (0,0)$ is the line in the $(a_1, b_1)$-plane with equation $2b_0b_1 = 3a_0^2a_1$, which corresponds to the tangent line at $X$ in $(a_0, b_0)$. In other words : $\mathcal{L}_1(X)$ is the tangent bundle $TX$, and $\pi_0^1$ is the natural projection $TX \to X$.

(2) $\mathcal{L}_2(X) = \{(a_0 + a_1t + a_2t^2, b_0 + b_1t + b_2t^2) \in (\mathbb{C}[t]/(t^3))^2 \mid (b_0 + b_1t + b_2t^2)^2 = (a_0 + a_1t + a_2t^2)^3 \text{mod } t^3\}$ is given in $\mathbb{C}^6$ by the equations

$$\begin{cases} b_0^2 = a_0^3 \\ 2b_0b_1 = 3a_0^2a_1 \\ b_1^2 + 2b_0b_2 = 3a_0a_1^2 + 3a_0^2a_2. \end{cases}$$

Exercise. a) Verify the description of $\mathcal{L}_2(X)$ and note that the map $\pi_1^2 : \mathcal{L}_2(X) \to \mathcal{L}_1(X)$ is not surjective. More precisely, the fibre of $\pi_0^2$ above $(0,0)$ is $\{(0, a_1, a_2, 0, 0, b_2)\} \cong \mathbb{C}^3$, but its image by $\pi_1^2$ is not the whole $(a_1, b_1)$-plane; it is just the line $\{b_1 = 0\}$.

b) Compute $\mathcal{L}_3(X)$ and note that also $\pi_2^3 : \mathcal{L}_3(X) \to \mathcal{L}_2(X)$ is not surjective.

c) However, above the nonsingular part of $X = \mathcal{L}_0(X)$ all considered maps $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ are fibrations with fibre $\mathbb{C}$.

**2.4.** Some observations in the examples are easily seen to be satisfied in general.
(1) $\mathcal{L}_0(X) = X, \quad \mathcal{L}_1(X) = TX$.
(2) If $X$ is smooth of dimension $d$, then all $\pi_n^{n+1}$ are locally trivial fibrations (w.r.t. the Zariski topology) with fibre $\mathbb{C}^d$.

**2.5.** The *space of arcs on* $X$ is an 'algebraic variety of infinite dimension' $\mathcal{L}(X)$ over $\mathbb{C}$ such that

$$\begin{aligned} &\{\text{points of } \mathcal{L}(X) \text{ with coordinates in } \mathbb{C}\} \\ =&\{\text{points of } X \text{ with coordinates in } \mathbb{C}[[t]]\}. \end{aligned}$$

We provide the 'exact' definition after continuing the examples. Now we have for all $n$ truncation maps $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$, obtained by reducing arcs modulo $t^{n+1}$.

**2.6.** *Example.* Let $X = \mathbb{C}^d$. Then

$$\mathcal{L}(X) = \{(\sum_{n=0}^{\infty} a_n^{(1)} t^n, \cdots, \sum_{n=0}^{\infty} a_n^{(d)} t^n), \text{ with all } a_n^{(j)} \in \mathbb{C}\},$$

which can be considered as an infinite dimensional affine space.

**2.7.** *Example.* Let $X = \{y^2 - x^3 = 0\}$. Then $\mathcal{L}(X)$ is given in the infinite dimensional affine space with coordinates

$$\begin{cases} a_0, a_1, a_2, \cdots, a_n, \cdots \\ b_0, b_1, b_2, \cdots, b_n, \cdots \end{cases}$$

by the infinite number of equations

$$\begin{cases} b_0^2 = a_0^3 \\ 2b_0 b_1 = 3a_0^2 a_1 \\ b_1^2 + 2b_0 b_2 = 3a_0 a_1^2 + 3a_0^2 a_2 \\ \cdots \end{cases}$$

**2.8.** *More precise definitions.*

(i) The 'base extension operation' $Y \to Y \times_{\mathbb{C}} \mathbb{C}[t]/(t^{n+1})$ is a covariant functor on the category of complex algebraic varieties, and it has a right adjoint $X \to \mathcal{L}_n(X)$. (Even more precisely we should say that we consider the *reduced* scheme $\mathcal{L}_n(X)$ associated to this right adjoint scheme.) This says that, for any $\mathbb{C}$-algebra $R$, the set of $R$-valued points of $\mathcal{L}_n(X)$ is in natural bijection with the set of $R[t]/(t^{n+1})$-valued points of $X$. In particular, as we said in (2.1), the $\mathbb{C}$-valued points of $\mathcal{L}_n(X)$ can be naturally identified with the $\mathbb{C}[t]/(t^{n+1})$-valued points of $X$.

(ii) Then $\mathcal{L}(X)$ is the inverse limit $\varprojlim \mathcal{L}_n(X)$. (Technically, it is important here that the truncation morphisms $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ are affine.) The $K$-valued points of $\mathcal{L}(X)$, for any field $K \supset \mathbb{C}$, are in natural bijection with the $K[[t]]$-valued points of $X$. We mention the following result, attributed to Kolchin : if $X$ is irreducible, then $\mathcal{L}(X)$ is irreducible.

See [DL3] for more information.

**2.9.** When $X$ is an affine variety, i.e. given by a finite number of polynomial equations, one can describe equations for the $\mathcal{L}_n(X)$ and for $\mathcal{L}(X)$ as in Examples 2.3 and 2.7.

**2.10.** Some first natural and fundamental questions are how the $\mathcal{L}_n(X)$ and $\pi_n(\mathcal{L}(X))$ change with $n$. (For $\pi_n(\mathcal{L}(X))$ this was already considered by Nash [Na].) Note that $\mathcal{L}_n(X)$ describes by definition the $n$-jets on $X$, and $\pi_n(\mathcal{L}(X))$ those $n$-jets that can be lifted to arcs on $X$.

This can be compared with the number theoretical setting of the previous section : there the question was how the solutions over $\mathbb{Z}/p^{n+1}\mathbb{Z}$ changed with $n$, and we could consider the same question for those solutions over $\mathbb{Z}/p^{n+1}\mathbb{Z}$ that can be lifted to solutions over $\mathbb{Z}_p$.

**2.11.** We now introduce the Grothendieck ring of algebraic varieties, which is the 'best' framework to answer these questions, and which is moreover (essentially) the value ring for motivic integration, to be explained in the next section.

Recall first two fundamental properties of the topological Euler characteristic $\chi(\cdot) \in \mathbb{Z}$ on complex algebraic varieties :

(1) $\chi(V) = \chi(Z) + \chi(V \setminus Z)$ if $Z$ is (Zariski-)closed in $V$,

(2) $\chi(V \times W) = \chi(V) \cdot \chi(W)$.

A finer invariant satisfying these properties is the Hodge-Deligne polynomial $H(\cdot) = H(\cdot; u, v) \in \mathbb{Z}[u, v]$, given for an algebraic variety $V$ of dimension $d$ by

$$H(V;u,v) := \sum_{p,q=0}^{d} (\sum_{i=0}^{2d}(-1)^i h^{p,q}(H_c^i(V,\mathbb{C})))u^p v^q,$$

where $h^{p,q}(\cdot)$ denotes the dimension of the $(p,q)$-component of the mixed Hodge structure. (When we would work over an arbitrary field of characteristic zero, we use an embedding into $\mathbb{C}$ of the field of definition of $V$. The $u^p v^q$–coefficients of $H(V;u,v)$ do not depend on the chosen embedding, since for a smooth projective $V$ they are equal to $(-1)^{p+q}\dim H^q(V,\Omega_V^p)$.)

Note that $H(V;1,1) = \chi(V)$.

The Grothendieck ring is the value ring of the 'universal Euler characteristic' on algebraic varieties.

**Definition.** (i) The Grothendieck group of (complex) algebraic varieties is the abelian group $K_0(Var_{\mathbb{C}})$ generated by symbols $[V]$, where $V$ is an algebraic variety, with the relations $[V] = [W]$ if $V$ and $W$ are isomorphic, and $[V] = [Z] + [V \setminus Z]$ if $Z$ is (Zariski-) closed in $V$.

(ii) there is a natural ring structure on $K_0(Var_{\mathbb{C}})$ given by $[V] \cdot [W] := [V \times W]$.

— So by construction the map {Varieties over $\mathbb{C}$} $\to K_0(Var_{\mathbb{C}}) : V \mapsto [V]$ is indeed universal with respect to the two properties above. Of course we still loose some information by this operation. For example $X = \{y^2 - x^3 = 0\} \subset \mathbb{A}^2$ satisfies $[X] = [\mathbb{A}^1]$. Also, when $V \to B$ is a locally trivial fibration with fibre $F$, then $[V] = [B] \cdot [F]$. —

(iii) Let $C$ be a constructible subset of some variety $V$, i.e. a disjoint union of (finitely many) locally closed subvarieties $A_i$ of $V$, then $[C] \in K_0(Var_{\mathbb{C}})$ is well defined as $[C] := \sum_i [A_i]$.

(iv) We denote $1 :=$ [point], $\mathbb{L} := [\mathbb{A}^1]$ and $\mathcal{M}_{\mathbb{C}} := K_0(Var_{\mathbb{C}})_{\mathbb{L}}$ the ring obtained from $K_0(Var_{\mathbb{C}})$ by inverting $\mathbb{L}$.

The rings $K_0(Var_{\mathbb{C}})$ and $\mathcal{M}_{\mathbb{C}}$ are quite mysterious. For instance, it was shown only recently that $K_0(Var_{\mathbb{C}})$ is not a domain [Po], and it is still not known whether $\mathcal{M}_{\mathbb{C}}$ is a domain or not, or whether the natural map $K_0(Var_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}}$ is injective.

*Remark.* There is an interesting alternative description of $K_0(Var_{\mathbb{C}})$ as the abelian group, generated by isomorphism classes $[V]$ of *nonsingular* *projective* varieties $V$, with the relations $[\emptyset] = 0$ and $[\tilde{V}]-[E] = [V]-[Z]$, where $\tilde{V} \to V$ is the blowing-up with centre $Z$ and exceptional variety $E$ [Bi1].

**2.12.** We now answer the questions in (2.10). We will consider $[\mathcal{L}_n(X)]$ and $[\pi_n(\mathcal{L}(X))]$ in $\mathcal{M}_{\mathbb{C}}$. For the latter we use a theorem of Greenberg [Gr], saying that $\pi_n(\mathcal{L}(X))$ is a constructible subset of $\mathcal{L}_n(X)$.

**Theorem** [DL3][DL8]. *The generating formal series*

$$J(T) := \sum_{n\geq 0} [\mathcal{L}_n(X)]T^n \text{ and } P(T) := \sum_{n\geq 0} [\pi_n(\mathcal{L}(X))]T^n$$

*in $\mathcal{M}_{\mathbb{C}}[[T]]$ are rational, with moreover as denominators products of polynomials of the form $1 - \mathbb{L}^a T^b$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{>0}$.*

The proof uses motivic integration, which 'explains' why $\mathcal{M}_{\mathbb{C}}$ is needed instead of $K_0(Var_{\mathbb{C}})$; see section 3.

This result specializes to the analogous statement, replacing $[\cdot]$ by $\chi(\cdot)$ or $H(\cdot)$. Note for this that $\chi : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}$ and $H : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}[u,v]$ obviously extend to $\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}$ and $H : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[u,v][\frac{1}{uv}]$. When $X = \{f = 0\}$ for some polynomial $f$, the statement for $J(T)$ should be compared with Theorem 1.3 for $J_p(T)$ ! In this case, we will outline a proof for $J(T)$ later. We just mention that the proof for $P(T)$ uses techniques from logic, more precisely quantifier elimination.

**2.13.** *Example.* When $X$ is smooth of dimension $d$, all $\mathcal{L}_n(X) = \pi_n(\mathcal{L}(X))$ are locally trivial over $X$ with fibre $\mathbb{C}^{nd}$. Hence

$$J(T) = P(T) = \sum_{n\geq 0} [X]\mathbb{L}^{nd}T^n = \frac{[X]}{1-\mathbb{L}^d T}.$$

**2.14.** *Example.* Let $X = \{y^2 - x^3 = 0\}$. The descriptions in Example 2.3 yield $[\mathcal{L}_0(X)] = [X] = \mathbb{L}, [\mathcal{L}_1(X)] = \mathbb{L}^2 + (\mathbb{L}-1)\mathbb{L} = 2\mathbb{L}^2 - \mathbb{L}, [\mathcal{L}_2(X)] = \mathbb{L}^3 + (\mathbb{L}-1)\mathbb{L}^2 = 2\mathbb{L}^3 - \mathbb{L}^2$. We claim that

$$J(T) = \mathbb{L}\frac{1 + (\mathbb{L}-1)T + (\mathbb{L}^6 - \mathbb{L}^5)T^5 - \mathbb{L}^7T^6}{(1-\mathbb{L}^7T^6)(1-\mathbb{L}T)},$$

see section 6. (Compare with 1.4(3)!) The formula in [DL5, Proposition 10.2.1] yields

$$P(T) = \frac{\mathbb{L} + (1-\mathbb{L})T - \mathbb{L}T^2}{(1-\mathbb{L}T)(1-T^2)}.$$

**2.15.** *Example.* Let $X = \{xy = 0\}$. EXERCISE :
(i) $[\mathcal{L}_n(X)] = (n+2)\mathbb{L}^{n+1} - (n+1)\mathbb{L}^n$. Then

$$J(T) = \frac{2\mathbb{L} - 1 - \mathbb{L}^2 T}{(1-\mathbb{L}T)^2}.$$

(Compare again with Examples 1.2 and 1.4.)
(ii) $[\pi_n(\mathcal{L}(X))] = 2\mathbb{L}^{n+1} - 1$. Then

$$P(T) = \frac{2\mathbb{L} - 1 - \mathbb{L}T}{(1-\mathbb{L}T)(1-T)}.$$

**2.16.** [Mul] To conclude this section, we relate some properties of the spaces of $n$-jets on $X$ to properties of $X$. Let $X$ be irreducible of dimension $d$.

(i) The closure in $\mathcal{L}_n(X)$ of $(\pi_0^n)^{-1}(X_{\text{reg}})$ is an irreducible component of $\mathcal{L}_n(X)$ of dimension $d(n+1)$.

(ii) Suppose that $X$ is locally a complete intersection. Then
(1) $\mathcal{L}_n(X)$ is pure dimensional if and only if $\dim \mathcal{L}_n(X) \leq d(n+1)$.
(2) $\mathcal{L}_n(X)$ is irreducible if and only if $\dim(\pi_0^n)^{-1}(X_{\text{sing}}) < d(n+1)$.
(3) If $\mathcal{L}_{n+1}(X)$ is pure dimensional or irreducible, then so is $\mathcal{L}_n(X)$.
(4) If $\mathcal{L}_n(X)$ is irreducible for some $n > 0$, then $X$ is normal.
(5) $\mathcal{L}_n(X)$ is irreducible for all $n > 0$ if and only if $X$ has rational singularities.

(iii) When $d = 1$ we have for any $n > 0$ that $\mathcal{L}_n(X)$ is irreducible if and only if $X$ is nonsingular.

## 3 Motivic integration

This notion is due to Kontsevich [Ko] on nonsingular varieties. It has been further developed by Batyrev [Ba2][Ba3], and especially by Denef and Loeser [DL3][DL4][DL6][DL8], with some improvements by Looijenga [Loo]. Probably the best way to view and understand it, is as being an analogue of $p$-adic integration.

Let in this section $X$ be any algebraic variety of pure dimension $d$.

**3.1.** A subset $A$ of $\mathcal{L}(X)$ is called *constructible* or *cylindric* or *a cylinder* if $A = \pi_m^{-1}C$ for some $m$ and some constructible subset $C$ of $\mathcal{L}_m(X)$. These can be considered as 'reasonably nice' subsets of the arc space $\mathcal{L}(X)$, being precisely all arcs obtained by lifting a nice subset of a jet space.

**3.2.** Suppose that $X$ is nonsingular. Then such a constructible subset $A = \pi_m^{-1}C$ satisfies the property

$$[\pi_n(A)] = \mathbb{L}^{(n-m)d}[C] \qquad \text{for all } n \geq m,$$

since $\pi_m^n : \mathcal{L}_n(X) = \pi_n(\mathcal{L}(X)) \to \mathcal{L}_m(X) = \pi_m(\mathcal{L}(X))$ is a locally trivial fibration with fibre $\mathbb{C}^{(n-m)d}$. We have in particular that the

$$\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$$

are all equal in $\mathcal{M}_{\mathbb{C}}$ for $n \geq m$.

For general $X$, a constructible set $A \subset \mathcal{L}(X)$ which is disjoint with $\mathcal{L}(X_{sing})$ still satisfies the property that the $\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$ stabilize for $n$ big enough [DL3, Lemma 4.1]. More precisely we have the following.

**Definition.** We call a set $A \subset \mathcal{L}(X)$ *stable* if for some $m \in \mathbb{N}$ we have

(i) $\pi_m(A)$ is constructible and $A = \pi_m^{-1}(\pi_m(A))$, and

(ii) for all $n \geq m$ the projection $\pi_{n+1}(A) \to \pi_n(A)$ is a piecewise trivial fibration with fiber $\mathbb{C}^d$.

(So in particular $A$ is constructible.)

**Lemma** [DL3]. *If $A \subset \mathcal{L}(X)$ is constructible and $A \cap \mathcal{L}(X_{\text{sing}}) = \emptyset$, then $A$ is stable.*

Hence for such $A$ it makes sense to consider $\lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}} \in \mathcal{M}_{\mathbb{C}}$ as an invariant of $A$; it is called its *naive motivic measure*. Note that for nonsingular $X$ the measure of $\mathcal{L}(X)$ is just $[X]$.

**3.3.** For arbitrary constructible $A \subset \mathcal{L}(X)$ the sequence $\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$ will not stabilize.

*Example.* Let $X = \{xy = 0\}$. From Example 2.15 we see that

$$\frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}} = \frac{2\mathbb{L}^{n+1} - 1}{\mathbb{L}^n} = 2\mathbb{L} - \frac{1}{\mathbb{L}^n}.$$

This sequence 'almost' stabilizes (the singular point of $X$ of course causes the trouble), and it would be nice to be able to consider $2\mathbb{L}$ as the limit of this sequence.

This will indeed work in Kontsevich's *completed Grothendieck ring* $\hat{\mathcal{M}}_{\mathbb{C}}$. This is by definition the completion of $\mathcal{M}_{\mathbb{C}}$ with respect to the decreasing filtration $F^m, m \in \mathbb{Z}$, of $\mathcal{M}_{\mathbb{C}}$, where $F^m$ is the subgroup of $\mathcal{M}_{\mathbb{C}}$ generated by the elements $\frac{[S]}{\mathbb{L}^i}$ with $S$ an algebraic variety and $\dim S - i \leq -m$. Note that this is indeed a ring filtration : $F^m \cdot F^n \subset F^{m+n}$. So $\hat{\mathcal{M}}_{\mathbb{C}} = \varprojlim_m \frac{\mathcal{M}_{\mathbb{C}}}{F^m}$.

*Continuing the example.* Indeed in $\hat{\mathcal{M}}_{\mathbb{C}}$ we have

$$\lim_{n\to\infty} \frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}} = 2\mathbb{L} - \lim_{n\to\infty} \frac{1}{\mathbb{L}^n} = 2\mathbb{L}.$$

**Theorem** [DL3]. *Let $A$ be a constructible subset of $\mathcal{L}(X)$. Then the limit*

$$\mu(A) := \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$$

*exists in $\hat{\mathcal{M}}_{\mathbb{C}}$.*

We call $\mu(A)$ the *motivic measure* of $A$. This yields a $\sigma$-additive measure $\mu$ on the Boolean algebra of constructible subsets of $\mathcal{L}(X)$. Thus, given any sequence $A_i, i \in \mathbb{Z}_{>0}$, of disjoint constructible subsets in $\mathcal{L}(X)$ such that $\lim_{i\to\infty} \mu(A_i) = 0$, we have that $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.

*Note.* It is not known whether the natural map $\mathcal{M}_{\mathbb{C}} \to \hat{\mathcal{M}}_{\mathbb{C}}$ is injective; its kernel is $\cap_{m\in\mathbb{Z}} F^m$. However, e.g. the topological Euler characteristic $\chi(\cdot)$ and the Hodge-Deligne polynomial $H(\cdot)$ factor through the image of $\mathcal{M}_{\mathbb{C}}$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.

*Remark.* Let $S \subsetneq X$ be a closed subvariety; it is not difficult to see that $\mathcal{L}(S)$ is *not* a constructible subset of $\mathcal{L}(X)$. It is possible to introduce more generally *measurable* subsets of $\mathcal{L}(X)$, and to associate analogously a motivic measure (in $\hat{\mathcal{M}}_{\mathbb{C}}$) to those subsets [Ba2][DL6]; we then have that such $\mathcal{L}(S)$ are measurable of measure zero.

**3.4.** We briefly compare with the $p$-adic case. Let $M$ be a $d$-dimensional submanifold of $\mathbb{Z}_p^m$, defined algebraically. We denote by $|\pi_n(M)|$ the cardinality of the image of $M$ under the natural truncation map $\pi_n : (\mathbb{Z}_p)^m \to (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^m = (\mathbb{Z}/p^{n+1}\mathbb{Z})^m$. Then $\frac{|\pi_n(M)|}{p^{(n+1)d}} \in \mathbb{Z}[\frac{1}{p}]$ is constant for $n$ big enough and is called the volume $\mu_p(M)$ of $M$.

For a singular $d$-dimensional subvariety $Z$ of $\mathbb{Z}_p^m$ one defines its *volume* as $\mu_p(Z) := \lim_{\epsilon\to 0} \mu_p(Z \setminus T_\epsilon(Z_{\text{sing}})) \in \mathbb{R}$, where $T_\epsilon$ denotes a small tubular neighbourhood 'of radius $\epsilon$'. Then by a Theorem of Oesterlé [Oe] we have, with analogous notation $|\pi_n(Z)|$,

$$\mu_p(Z) = \lim_{n\to\infty} \frac{|\pi_n(Z)|}{p^{(n+1)d}} .$$

Note the analogy

| | $p$-adic | motivic |
|---|---|---|
| integrate over | $\mathbb{Z}_p^m$ | $(\mathbb{C}[[t]])^m$ |
| value rings | $\mathbb{Z}$ | $K_0(Var_{\mathbb{C}})$ |
| | $\mathbb{Z}[\frac{1}{p}]$ | $\mathcal{M}_{\mathbb{C}}$ |
| | $\mathbb{R}$ | $\hat{\mathcal{M}}_{\mathbb{C}}$ |

The brilliant idea of Kontsevich was to use $\hat{\mathcal{M}}_{\mathbb{C}}$ instead of $\mathbb{R}$ as a value ring for integration.

**3.5.** We can now consider in a natural way *motivic integration.* We do not treat the most general setting; the following suffices in practice. Let $A \subset \mathcal{L}(X)$ be constructible and $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ a function with constructible fibres $\alpha^{-1}\{n\}, n \in \mathbb{Z}$. Then

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n\in\mathbb{Z}} \mu(\alpha^{-1}\{n\})\mathbb{L}^{-n}$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$, whenever the right hand side converges in $\hat{\mathcal{M}}_{\mathbb{C}}$. Then we say that $\mathbb{L}^{-\alpha}$ is *integrable* on $A$. (This will always be the case if $\alpha$ is bounded from below.)

**3.6.** An important example of an integrable function is induced by an effective Cartier divisor $D$ on $X$, i.e. $D$ is an (eventually non-reduced) subvariety of $X$ which is locally given by one equation. Define $ord_t D : \mathcal{L}(X) \to \mathbb{N} \cup \{+\infty\} : \gamma \mapsto ord_t f_D(\gamma)$, where $f_D$ is a local equation of $D$ in a neighbourhood of the origin $\pi_0(\gamma)$ of $\gamma$. Note e.g. that $(ord_t D)(\gamma) = +\infty$ if and only if $\gamma \in \mathcal{L}(D_{\text{red}})$ and $(ord_t D)(\gamma) = 0$ if and only if $\pi_0(\gamma) \notin D_{\text{red}}$. One easily verifies that $\mathbb{L}^{-ord_t D}$ is integrable on $\mathcal{L}(X)$.

We note that $(ord_t D)^{-1}(+\infty) = \mathcal{L}(D_{\text{red}})$ is *not* constructible; it is however measurable with measure zero.

*Example.* Take $X = \mathbb{A}^1$ and $D$ the divisor associated to the function $x^N$, i.e. the 'origin with multiplicity $N$'.

EXERCISE. (i) $N|(ord_t D)(\gamma)$ for all $\gamma \in \mathcal{L}(\mathbb{A}^1)$ and

$$\mu(\{\gamma \in \mathcal{L}(\mathbb{A}^1) \mid (ord_t D)(\gamma) = iN\}) = \frac{\mathbb{L}-1}{\mathbb{L}^i} \text{ for all } i \in \mathbb{N}.$$

(ii) $\int_{\mathcal{L}(\mathbb{A}^1)} \mathbb{L}^{-ord_t D} d\mu = \frac{(\mathbb{L}-1)\mathbb{L}^{N+1}}{\mathbb{L}^{N+1}-1} = (\mathbb{L}-1) + \frac{\mathbb{L}-1}{\mathbb{L}^{1+N}-1}$.

This example is the easiest case of the following very useful formula.

**Proposition** [Ba3][Cr]. *Let $X$ be nonsingular and take a normal crossings divisor $D = \sum_{i \in S} N_i D_i$ on $X$, i.e. all $D_i$ are nonsingular hypersurfaces intersecting transversely (and occurring with multiplicity $N_i$). Denote $D_I^\circ := (\cap_{i \in I} D_i) \setminus (\cup_{\ell \notin I} D_\ell)$ for $I \subset S$; the $D_I^\circ, I \subset S$, form a natural locally closed stratification of $X$ (note that $D_\emptyset^\circ = X \setminus (\cup_{\ell \in S} D_\ell)$). Then*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t D} d\mu = \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{1+N_i}-1}.$$

**3.7.** The construction in (3.6) can be generalized as follows. Let $\mathcal{I}$ be a sheaf of ideals on $X$. Then we define

$$ord_t \mathcal{I} : \mathcal{L}(X) \to \mathbb{N} \cup \{+\infty\} : \gamma \mapsto \min_g ord_t g(\gamma),$$

where the minimum is taken over $g \in \mathcal{I}$ in a neighbourhood of $\pi_0(\gamma)$. Of course, when $\mathcal{I}$ is the ideal sheaf of an effective Cartier divisor $D$, then $ord_t\mathcal{I} = ord_t D$.

**3.8.** The most crucial ingredient in the theory of motivic integration is the *change of variables formula* or *transformation rule* for motivic integrals under a birational morphism.

**Theorem** [DL3]. *(i) Let $h : Y \to X$ be a proper birational morphism between algebraic varieties $X$ and $Y$, where $Y$ is nonsingular. Let $A \subset \mathcal{L}(X)$ be constructible and $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ such that $\mathbb{L}^{-\alpha}$ is integrable on $A$. Then*

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}A} \mathbb{L}^{-(\alpha \circ h) - ord_t(Jac_h)} d\mu.$$

*Here the ideal sheaf $Jac_h$ is defined as follows. When also $X$ is nonsingular, it is locally generated by the 'ordinary' Jacobian determinant with respect to local coordinates on $X$ and $Y$. For general $X$, the sheaf of regular differential d-forms $h^*(\Omega^d_X)$ is still a submodule of $\Omega^d_Y$; but now $h^*(\Omega^d_X)$ is not necessarily locally generated by one element. Taking (locally) a generator $\omega_Y$ of $\Omega^d_Y$, each $h^*(\omega)$ for $\omega \in \Omega^d_X$ can be written as $h^*(\omega) = g_\omega \omega_Y$, and $Jac_h$ is defined as the ideal sheaf which is (locally) generated by these $g_\omega$.*

*(ii) When also $X$ is nonsingular and $\alpha = ord_t D$ for some effective divisor $D$ on $X$, we can rewrite the formula as follows :*

$$\int_A \mathbb{L}^{-ord_t D} d\mu = \int_{h^{-1}A} \mathbb{L}^{-ord_t(h^*D + K_{Y|X})} d\mu.$$

*Here $h^*D$ is the pullback of $D$, i.e. locally given by the equation $f \circ h$, if $D$ is given by the equation $f$. And $K_{Y|X}$ is the relative canonical divisor, which is precisely the effective divisor with equation the Jacobian determinant. Alternatively, $K_{Y|X} = K_Y - h^*K_X$ where $K_.$ denotes the (ordinary) canonical divisor, i.e. the divisor of zeros and poles of a differential d-form.*

*Note.* The birational morphism $h$ above must be proper in order to induce a bijection from $\mathcal{L}(Y)$ to $\mathcal{L}(X)$ outside subsets of measure zero. More precisely, denoting by $Exc$ the exceptional locus of $h$, we have a bijection from $\mathcal{L}(Y) \setminus \mathcal{L}(Exc)$ to $\mathcal{L}(X) \setminus \mathcal{L}(h(Exc))$. This is an easy consequence of the valuative criterion of properness [Har, Theorem II.4.7].

EXERCISE. Check the change of variables formula in the following special case : $h$ is the blowing-up of a nonsingular $X$ in a nonsingular centre, $A = \mathcal{L}(X)$ and $\alpha$ is the zero function.

## 4 First applications

**4.1.** Here we mean by a *Calabi-Yau manifold* $M$ of dimension $d$ a nonsingular complete (=compact) algebraic variety, which admits a nowhere vanishing regular differential $d$-form $\omega_M$. Alternative formulations of this last condition are that the first Chern class of the tangent bundle of $M$ is zero, or that the canonical divisor $K_M$ of $M$ is zero.

**Theorem** [Ko]. *Let $X$ and $Y$ be birationally equivalent Calabi-Yau manifolds. Then $[X] = [Y]$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.*

*Proof.* Since $X$ and $Y$ are birationally equivalent there exist a nonsingular complete algebraic variety $Z$ and birational morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$. By the definition of the motivic measure and the change of variables formula we have in $\hat{\mathcal{M}}_{\mathbb{C}}$ :

$$[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(X)} 1 d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t K_{Z|X}} d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t K_Z} d\mu$$

and of course $[Y]$ is given by the same right hand side. Q.E.D.

This implies that birationally equivalent Calabi-Yau manifolds have the same Hodge-Deligne polynomial, meaning that they have the same Hodge numbers. This result was Kontsevich's motivation to invent motivic integration !

The same proof gives the following more general result. Two nonsingular complete algebraic varieties are called *K-equivalent* if there exists a nonsingular complete algebraic variety $Z$ and birational morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$ such that $h_X^* K_X = h_Y^* K_Y$. This is an important notion in birational geometry.

**Theorem.** *Let $X$ and $Y$ be $K$-equivalent varieties. Then $[X] = [Y]$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.*

**4.2.** Let $h : Y \to X$ be a proper birational morphism between nonsingular algebraic varieties. We assume that the exceptional locus *Exc* of $h$, i.e. the subvariety of $Y$ where $h$ is not an isomorphism, is a normal crossings divisor. Let $E_i, i \in S$, be the irreducible components of *Exc*. The relative canonical divisor $K_{Y|X}$ is supported on *Exc*; let $\nu_i - 1$ be the multiplicity of $E_i$ in this divisor, so $K_{Y|X} = \sum_{i \in S}(\nu_i - 1)E_i$. Denoting $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$, we have

$$[X] = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i} - 1} = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{1}{[\mathbb{P}^{\nu_i - 1}]}$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$. Indeed, by the change of variables formula we have again that

$$[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu,$$

and then Proposition 3.6 yields the stated formula. Specializing to the topological Euler characteristic yields the remarkable formula

$$\chi(X) = \sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i},$$

which was first surprisingly obtained in [DL1], using $p$-adic integration and the Grothendieck-Lefschetz trace formula.

## 5 Motivic volume

Here $X$ is again any algebraic variety of pure dimension $d$.

**5.1. Definition.** The *motivic volume* of $X$ is $\mu(\mathcal{L}(X)) \in \hat{\mathcal{M}}_{\mathbb{C}}$, thus the motivic measure of the whole arc space of $X$. Recall that $\mu(\mathcal{L}(X)) = \lim_{n \to \infty} \frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}}$, and that it equals $[X]$ when $X$ is nonsingular.

We computed in (3.3) the motivic volume of $X = \{xy = 0\}$ as $\mu(\mathcal{L}(X)) = 2\mathbb{L}$ by the defining limit procedure. For more complicated

$X$, the following formula in terms of a suitable resolution of singularities is very useful.

**5.2. Theorem** [DL3]. *Let $h : Y \to X$ be a log resolution of $X$; i.e. $h$ is a proper birational morphism from a nonsingular $Y$ such that the exceptional locus Exc of $h$ is a normal crossings divisor. Assume also that the image of $h^*(\Omega^d_X)$ in $\Omega^d_Y$ is locally principal, i.e. locally generated by one element.*

*Denote by $E_i, i \in S$, the irreducible components of Exc, and let $\rho_i - 1$ be the multiplicity along $E_i$ of the divisor associated to $h^*(\Omega^d_X)$, i.e. the (effective) divisor locally given by the zeroes of a generator of $h^*(\Omega^d_X)$. Finally, set $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$. Then*

$$\mu(\mathcal{L}(X)) = \sum_{I\subset S} [E_I^\circ] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\rho_i}-1} = \sum_{I\subset S} [E_I^\circ] \prod_{i\in I} \frac{1}{[\mathbb{P}^{\rho_i - 1}]}$$

*in $\hat{\mathcal{M}}_{\mathbb{C}}$; in particular $\mu(\mathcal{L}(X))$ belongs to the subring of $\hat{\mathcal{M}}_{\mathbb{C}}$, obtained from (the image of) $\mathcal{M}_{\mathbb{C}}$ by inverting the elements $1+\mathbb{L}+\cdots+\mathbb{L}^j = [\mathbb{P}^j]$.*

We will denote this subring by $\mathcal{M}_{loc}$.

**5.3.** *Example.* Let $X = \{y^2 - x^3 = 0\}$ in $\mathbb{A}^2$. We take $\mathbb{A}^1 \to X : u \mapsto (u^2, u^3)$ as a log resolution. Since $\Omega^1_X$ is generated by $dx$ and $dy$ (subject to the relation $2ydy = 3x^2dx$), one easily verifies that $h^*\Omega^1_X$ is generated by $udu$. Hence the image of $h^*\Omega^1_X$ in $\Omega^1_Y$ is principal and we can apply Theorem 5.2.

Note that $Exc = E_1 = \{0\}$, occurring with multiplicity 1 in the divisor of $udu$. So $\rho_1 = 2$ and

$$\mu(\mathcal{L}(X)) = \mathbb{L} - 1 + \frac{1}{[\mathbb{P}^1]} = \frac{\mathbb{L}^2}{\mathbb{L}+1}.$$

(Recall that $[X] = \mathbb{L}$.)

**5.4.** *Example.* Let $X = \{z^2 = xy\}$ in $\mathbb{A}^3$.

EXERCISE. (i) Verify that $\mu(\mathcal{L}(X)) = \mathbb{L}^2$. (The 'obvious' log resolution satisfies the assumption of Theorem 5.2, and the unique component $E_1$ of the exceptional locus has $\rho_1 = 2$.)

(ii) Note that also $[X] = \mathbb{L}^2$; this could be interpreted as the singularity of $X$ being 'very mild'.

**5.5.** EXERCISE. Compute again that the motivic volume of $X = \{xy = 0\}$ is $2\mathbb{L}$; now using Theorem 5.2. (Note here that $[X] = 2\mathbb{L} - 1$; one could say that the motivic volume counts the double point twice.)

**5.6.** Recall that for nonsingular $X$ its universal Euler characteristic $[X] \in K_0(Var_{\mathbb{C}})$ specializes to its Hodge-Deligne polynomial $H(X) \in \mathbb{Z}[u,v]$ and further to $\chi(X) \in \mathbb{Z}$.

Since $\chi(\cdot)$ and $H(\cdot)$ factor through the image of $\mathcal{M}_{\mathbb{C}}$ in $\hat{\mathcal{M}}_{\mathbb{C}}$, they induce natural maps $\chi : \mathcal{M}_{loc} \to \mathbb{Q}$ and $H : \mathcal{M}_{loc} \to \mathbb{Z}[[u,v]]$. Applying these specialization maps to the motivic measure of $X$ yields new (numerical) singularity invariants, which generalize the usual $\chi(X)$ and $H(X)$ for nonsingular $X$. Denef and Loeser call $\chi(\mu(\mathcal{L}(X)))$ the *arc-Euler characteristic* of $X$.

For example the arc-Euler characteristic of $\{y^2 - x^3 = 0\}$ is $\frac{1}{2}$ and the one of $\{xy = 0\}$ is 2.

## 6 Motivic zeta functions

In this section $M$ is a nonsingular irreducible algebraic variety of dimension $m$, and $f : M \to \mathbb{C}$ is a non-constant regular function.

**6.1.** For each $n \in \mathbb{N}$ the morphism $f : M \to \mathbb{A}^1 = \mathbb{C}$ induces a morphism $f_n : \mathcal{L}_n(M) \to \mathcal{L}_n(\mathbb{A}^1)$. A point $\alpha \in \mathcal{L}_n(\mathbb{A}^1)$ corresponds to an element $\alpha(t) \in \mathbb{C}[t]/(t^{n+1})$; we denote as usual the largest $e$ such that $t^e$ divides $\alpha(t)$ by $ord_t\alpha \in \{0, 1, \cdots, n, +\infty\}$. We set

$$\mathcal{X}_n := \{\gamma \in \mathcal{L}_n(M) \mid ord_t f_n(\gamma) = n\} \qquad \text{for } n \in \mathbb{N};$$

it is a locally closed subvariety of $\mathcal{L}_n(M)$.

EXERCISE. Denote $X := \{f = 0\}$. Then $[\mathcal{X}_n] = \mathbb{L}^m[\mathcal{L}_{n-1}(X)] - [\mathcal{L}_n(X)]$ for $n \geq 1$, and $[\mathcal{X}_0] = [M] - [X]$.

**Definition.** The *motivic zeta function* of $f : M \to \mathbb{C}$ is the formal power series

$$Z(T) := \sum_{n \geq 0} [\mathcal{X}_n](\mathbb{L}^{-m}T)^n$$

in $\mathcal{M}_{\mathbb{C}}[[T]]$.

**6.2.** Considering the exercise above, it is not a surprise that for $X := \{f = 0\}$ the series $J(T) = \sum_{n\geq 0}[\mathcal{L}_n(X)]T^n$ and $Z(T)$ determine each other. Indeed, one easily verifies that

$$J(T) = \frac{Z(\mathbb{L}^m T) - [M]}{\mathbb{L}^m T - 1}.$$

**6.3.** The definition of $Z(T)$ is inspired by the $p$-adic Igusa zeta function, associated to a polynomial $f \in \mathbb{Z}_p[x_1, \cdots, x_m]$, which is defined as

$$Z_p(s) := \int_{\mathbb{Z}_p^m} |f(x)|_p^s |dx|$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$. Recall that each $z \in \mathbb{Z}_p \setminus \{0\}$ can be expressed as $z = p^\ell u$ with $\ell \in \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{Z}_p^\times$. One denotes $\mathrm{ord}_p(z) := \ell$ and $|z|_p := p^{-\mathrm{ord}_p z} = p^{-\ell}$. To compare with 6.1, note that $Z_p(s)$ can be rewritten as

$$\begin{aligned} Z_p(s) &= \sum_{n\geq 0} \mathit{volume}\{x \in \mathbb{Z}_p^m \mid \mathit{ord}_p f(x) = n\} p^{-ns} \\ &= \frac{1}{p^m} \sum_{n\geq 0} \#\{x \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^m \mid \mathit{ord}_p f(x) = n\}(p^{-m}p^{-s})^n. \end{aligned}$$

**6.4.** Exercise. Write $D$ for the (effective) divisor of zeros of $f$, i.e. $D$ is "$\{f = 0\}$ with multiplicities". Then

$$\int_{\mathcal{L}(M)} \mathbb{L}^{-\mathit{ord}_t D} d\mu = Z(\mathbb{L}^{-1})$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$, meaning in particular that the substitution in the right hand side yields a well-defined element of $\hat{\mathcal{M}}_{\mathbb{C}}$.

**6.5.** As for the motivic volume, there is an important (similar) formula for $Z(T)$ in terms of a resolution.

**Theorem** [DL2]. *Let $h : Y \to M$ be an embedded resolution of $\{f = 0\}$; i.e. $h$ is a proper birational morphism from a nonsingular $Y$ such that $h$ is an isomorphism on $Y \setminus h^{-1}\{f = 0\}$ and $h^{-1}\{f = 0\}$ is a normal crossings divisor. Let $E_i, i \in S$, be the irreducible components* *of $h^{-1}\{f = 0\}$. For $i \in S$ we denote by $N_i$ the multiplicity of $E_i$ in the divisor of $f \circ h$ on $Y$, and by $\nu_i - 1$ the multiplicity of $E_i$ in the divisor of $h^*\omega$, where $\omega$ is a local generator of $\Omega^m_M$. (Equivalently: $div(f \circ h) = \sum_{i \in S} N_i E_i$ and $K_{Y|M} = \sum_{i \in S}(\nu_i - 1)E_i$.) Set finally $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$. Then*

$$Z(T) = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{(\mathbb{L} - 1)T^{N_i}}{\mathbb{L}^{\nu_i} - T^{N_i}};$$

*in particular $Z(T)$ is rational and belongs more precisely to the subring of $\mathcal{M}_{\mathbb{C}}[[T]]$ generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $\frac{T^N}{L^\nu - T^N}$, where $\nu, N \in \mathbb{Z}_{>0}$.*

**6.6.** Corollaries.

(i) In the special case that $X = \{f = 0\}$ is a hypersurface this yields the stated rationality of $J(T)$ in (2.12).

(ii) Let $M = \mathbb{A}^m$ and $f \in \mathbb{Z}[x_1, \cdots, x_m]$. Then by a similar formula of Denef [De2] for the $p$-adic Igusa zeta functions $Z_p(s)$, Theorem 6.5 yields that $Z(T)$ specializes to the $Z_p(s)$ for all $p$ except a finite number. See [DL2] for a precise statement. Similarly $J(T)$ specializes to $J_p(T)$ for all $p$ except a finite number [DL8, Theorem 6.1].

(iii) For any $f : M \to \mathbb{C}$ we now explain how $Z(T)$ specializes to the *topological zeta function* of $f$. Using Theorem 6.5 and the notation there, we evaluate $Z(T)$ at $T = \mathbb{L}^{-s}$ for any $s \in \mathbb{N}$; this yields the well-defined elements

$$\sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i + sN_i} - 1} = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{1}{[\mathbb{P}^{\nu_i + sN_i - 1}]}$$

in (the image in $\hat{\mathcal{M}}_{\mathbb{C}}$ of) the localization of $\mathcal{M}_{\mathbb{C}}$ with respect to the elements $[\mathbb{P}^j]$. Applying the Euler characteristic specialization map $\chi(\cdot)$ yields the rational numbers

$$\sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i + sN_i}$$

for $s \in \mathbb{N}$. The *topological zeta function* $Z_{top}(s)$ of $f$ is the unique rational function in one variable $s$ admitting the values above for $s \in \mathbb{N}$.

Without the specialization argument above it is not at all clear that $Z_{top}(s)$ does not depend on the chosen resolution $h : Y \to M$. In fact $Z_{top}(s)$ was first introduced in [DL1], in terms of a resolution, and $p$-adic

Igusa zeta functions and the Grothendieck-Lefschetz trace formula were needed to prove independence of the chosen resolution.

**6.7.** We just mention that there is an important generalization of the motivic zeta function, working over a relative and equivariant Grothendieck ring; it specializes by a limit procedure to objects in (an equivariant version of) $\mathcal{M}_{\mathbb{C}}$, which are shown to be a good *virtual motivic incarnation* of the Milnor fibres of $f$ at the points of $\{f = 0\}$. It is quite remarkable that a definitely non-algebraic notion as the Milnor fibre has such an algebraic incarnation. See [DL2][DL7].

Moreover these objects satisfy a *motivic Thom-Sebastiani Theorem*, generalizing the known results of Varchenko and Saito. See [DL4].

**6.8.** *Monodromy Conjecture.*

There is an intriguing conjectural relation between the poles of the topological zeta function and the eigenvalues of the local monodromy of $f$.

**Monodromy conjecture.** *If $s_0$ is a pole of $Z_{top}(s)$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy action on the cohomology of the Milnor fibre of $f$ at some point of $\{f = 0\}$.*

One can also state the analogous conjecture for the motivic zeta function, but then one has to be careful with the notion of pole, see [RV2]. Alternatively, we can formulate this monodromy conjecture for $Z(T)$ as follows, without mentioning poles [DL2] :

*$Z(T)$ belongs to the ring generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $\frac{T^N}{L^\nu - T^N}$, where $\nu, N \in \mathbb{Z}_{>0}$ and $e^{2\pi i \frac{\nu}{N}}$ is an eigenvalue of the local monodromy as above.*

Actually, it was originally stated for the $p$-adic Igusa zeta function, being even more remarkable, for then it relates number theoretical invariants of $f \in \mathbb{Z}[x_1, \cdots, x_m]$ to differential topological invariants of $f$, considered as function $\mathbb{C}^n \to \mathbb{C}$.

The conjecture was shown by Loeser for $M = \mathbb{A}^2$ [Loe1]; a shorter proof in dimension 2 is in [Ro]. In dimension 3 there is a lot of 'experimental evidence' [Ve1], and by now various special cases are proved [ACLM1][ACLM2][Loe2][RV1].

*Example.* Let $M = \mathbb{A}^2$ and $f = y^2 - x^3$.
EXERCISE. Compute, using Theorem 6.5,

$$Z(T) = \mathbb{L}^2(\mathbb{L}-1)\frac{\mathbb{L}^5 - \mathbb{L}^3 T + \mathbb{L}^3 T^2 - T^5}{(\mathbb{L}^5 - T^6)(\mathbb{L} - T)}$$

and

$$Z_{top}(s) = \frac{5+4s}{(5+6s)(1+s)}.$$

(This is how we computed $J(T)$ in Example 2.14.) In particular, the poles of $Z_{top}(s)$ are $-1$ and $-5/6$. On the other hand, it is well known that the monodromy eigenvalues of $f$ are $1, e^{\frac{\pi i}{3}}$, and $e^{-\frac{\pi i}{3}}$. Hence the monodromy conjecture is indeed satisfied here.

*Note.* The previous example was too simple to exhibit the 'typical' situation. Each irreducible component $E_i$ in Theorem 6.5 induces a candidate-pole $-\frac{\nu_i}{N_i}$, and quite miraculously, for a generic example with a lot of components $E_i$, 'most' of these candidates cancel. This experimental fact is compatible with the monodromy conjecture, see [Ve1].

## 7 Batyrev's stringy invariants

Using motivic integration, Batyrev [Ba1][Ba2] introduced new singularity invariants for algebraic varieties with 'mild' singularities, more precisely with at worst log terminal singularities. He used them for instance to formulate a topological mirror symmetry test for singular Calabi-Yau varieties, to give a conjectural definition for stringy Hodge numbers, and to prove a version of the McKay correspondence.

We first explain log terminal and related singularities; for this we need the Gorenstein notion.

**7.1.** Let $X$ be a *normal* algebraic variety of dimension $d$. In particular $X$ is irreducible, $X_{\text{sing}}$ has codimension at least 2 in $X$, and $X$ has a well defined canonical divisor $K_X$ (up to linear equivalence). One can view (a representative of) $K_X$ as the divisor of zeroes and poles of a *rational* differential $d$-form on $X$; it is also the Zariski-closure of the usual canonical divisor on $X_{\text{reg}}$.

When $X$ is nonsingular, $K_X$ is a Cartier divisor, i.e. locally given by one equation. This is not true in general.

**Definition.** A normal variety $X$ is *Gorenstein* if $K_X$ is a Cartier divisor. Alternatively : $X$ is Gorenstein if the rational differential $d$-forms on $X$, which are regular on $X_{\text{reg}}$, are locally generated by one element.

*Example.* Let $X = \{z^2 = xy\}$; then those differential 2-forms are generated by $\frac{dx\wedge dy}{2z} = \frac{dx\wedge dz}{x} = -\frac{dy\wedge dz}{y}$ (which is indeed regular on $X_{\text{reg}}$).

This notion is quite general; for instance all (normal) hypersurfaces and even complete intersections are Gorenstein.

*Note.* In the literature one often uses the term *Gorenstein* alternatively for varieties $X$ for which all local rings $\mathcal{O}_{X,x} (x \in X)$ are Gorenstein rings, and then the property that $K_X$ is a Cartier divisor is called *1-Gorenstein.*

**7.2.** We now introduce a certain 'badness' for singularities, in terms of numerical invariants of a resolution.

Let $X$ be Gorenstein of dimension $d$. Take a log resolution $\pi : Y \to X$ of $X$ and denote by $E_i, i \in S$, the irreducible components of the exceptional locus $Exc$ of $h$. We associate as follows an integer $a_i$ to each $E_i$.

(1) *Description with divisors.* Since $K_X$ is Cartier, the pullback $\pi^* K_X$ makes sense and one can consider the relative canonical divisor $K_{Y|X} = K_Y - \pi^* K_X$, which is supported on $Exc$. Then $a_i - 1$ is the multiplicity of $E_i$ in $K_{Y|X}$, i.e. $K_{Y|X} = \sum_{i\in S}(a_i - 1)E_i$.

(2) *Description with differential forms.* Take a general point $Q_i$ of $E_i$ and local coordinates $y_1, y_2, \cdots, y_d$ around $Q_i$ such that the local equation of $E_i$ is $y_1 = 0$. Let $\omega_i$ be a local generator around $\pi(Q_i)$ of the $d$-forms on $X$, which are regular on $X_{\text{reg}}$. (Such an $\omega_i$ exists by the Gorenstein property.) Then around $Q_i$ one can write $\pi^*\omega_i$ as

$$\pi^*\omega_i = u y_1^{a_i - 1} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_d,$$

where $u$ is regular and nonzero around $Q_i$.

In general the $a_i \in \mathbb{Z}$, and when $X$ is nonsingular they satisfy $a_i \geq 2$.

*Terminology.* One calls $a_i$ the *log discrepancy* of $E_i$ with respect to $X$ (and $a_i - 1$ the *discrepancy*).

*Example.* The standard log resolution of $X = \{z^2 = xy\}$ has one exceptional curve $E \cong \mathbb{P}^1$ with log discrepancy $a = 1$.

**7.3.** We also have to consider a technical generalization : the normal variety $X$ is called $\mathbb{Q}$-*Gorenstein* if $rK_X$ is Cartier for some $r \in \mathbb{Z}_{>0}$. In this case the log discrepancies are defined analogously by $K_{Y|X} = \sum_{i \in S}(a_i - 1)E_i$, which should be considered as an abbreviation of $rK_{Y|X} = rK_Y - rK_X = \sum_{i \in S} r(a_i - 1)E_i$. Now the $r(a_i - 1) \in \mathbb{Z}$, and hence $a_i \in \frac{1}{r}\mathbb{Z}$.

*Example.* Let $X$ be the quotient of $\mathbb{A}^2$ by the action of $\mu_3 = \{z \in \mathbb{C} \mid z^3 = 1\}$ given by $(x, y) \mapsto (\epsilon x, \epsilon y)$ for $\epsilon \in \mu_3$. Concretely, $X$ is given in $\mathbb{A}^4$ by the equations

$$\{u_1 u_3 - u_2^2 = u_2 u_4 - u_3^2 = u_1 u_4 - u_2 u_3 = 0\},$$

in particular it is *not* a complete intersection. Here $K_X$ is not Cartier; a representative of $K_X$ is for example $\{u_1 = u_2 = u_3 = 0\}$. However, $3K_X$ is Cartier; a representative is $\{u_1 = 0\}$.

The standard log resolution of $X$ has one exceptional curve $E \cong \mathbb{P}^1$ with log discrepancy $a = \frac{2}{3}$.

A nice introduction to these notions is in [Re1].

**7.4. Definition.** (i) Let $X$ be a $\mathbb{Q}$-Gorenstein variety. Take a log resolution $\pi : Y \to X$ of $X$; let $E_i, i \in S$, be the irreducible components of the exceptional locus of $\pi$ with log discrepancies $a_i$. Then $X$ is called *terminal, canonical, log terminal* and *log canonical* if $a_i > 1, a_i \geq 1$, $a_i > 0$ and $a_i \geq 0$, respectively, for all $i \in S$.

One can show that these conditions do not depend on the chosen resolution.

(ii) We say that $X$ is *strictly log canonical* if it is log canonical but not log terminal.

We should note that 0 is indeed the relevant 'border value' here; if some $a_i < 0$ on some log resolution, then one can easily construct log resolutions with arbitrarily negative $a_i$.

The log terminal singularities should be considered 'mild', the singularities which are not log canonical 'general', and the strictly log canonical ones as a special 'border' class.

**7.5.** *Example.* (1) When $X$ is a surface ($d=2$) terminal is equivalent to non-singular, the canonical singularities are precisely the so-called ADE singularities or rational double points, and the log terminal singularities are precisely the Hirzebruch-Jung or quotient singularities.

(2) Let $X = \{x_1^k + x_2^k + \cdots + x_{d+1}^k = 0\}$ in $\mathbb{A}^{d+1}$. The origin is the only singular point of $X$, and the blowing-up with the origin as centre yields a log resolution $\pi : Y \to X$ of $X$ with exceptional locus consisting of one irreducible component $E$, which is isomorphic to $\{x_1^k + x_2^k + \cdots + x_{d+1}^k = 0\} \subset \mathbb{P}^d$.

EXERCISE. (i) The log discrepancy of $E$ with respect to $X$ is $d+1-k$.

(ii) $X$ is log terminal, strictly log canonical, and not log canonical when $k < d+1, k = d+1$, and $k > d+1$, respectively.

**7.6.** There are nice results of Ein, Mustaţă and Yasuda, relating the previous notions with jet spaces.

**Theorem** [Mu1][EMY][EM]. *Let $X$ be a normal variety, which is locally a complete intersection. Then $X$ is terminal, canonical, and log canonical if and only if $\mathcal{L}_n(X)$ is normal, irreducible, and equidimensional, respectively, for every $n$.*

**7.7. Definition.** Let $X$ be a log terminal algebraic variety. Take a log resolution $\pi : Y \to X$ of $X$. Let $E_i, i \in S$, be the irreducible components of the exceptional locus of $\pi$ with log discrepancies $a_i$ ($\in \mathbb{Q}_{>0}$). Denote also $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$.

(i) The *stringy Euler number* of $X$ is

$$e_{st}(X) := \sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{a_i}.$$

(ii) The *stringy E-function* of $X$ is

$$E_{st}(X) := \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{uv - 1}{(uv)^{a_i} - 1}.$$

(iii) The *stringy $\mathcal{E}$-invariant* of $X$ is

$$\mathcal{E}_{st}(X) := \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1}.$$

*Remarks.* (1) Clearly $e_{st}(X) \in \mathbb{Q}$; $E_{st}(X)$ is a rational function in $u, v$ (with 'fractional powers'), and $\mathcal{E}_{st}(X)$ lives in a finite extension of $\hat{\mathcal{M}}_{\mathbb{C}}$. We have specialization maps $\mathcal{E}_{st}(X) \mapsto E_{st}(X) \mapsto e_{st}(X)$.

(2) Strictly speaking, Batyrev defined and used only the levels (i) and (ii) [Ba2][Ba3].

When $X$ is nonsingular, $\mathcal{E}_{st}(X) = [X]$ (this is 4.2), and of course $E_{st}(X) = H(X)$ and $e_{st}(X) = \chi(X)$. So also these invariants are new singularity invariants, generalizing $[\cdot], H(\cdot)$ and $\chi(\cdot)$, respectively, for nonsingular $X$. (Just as the motivic volume and its specializations. We give a comparing example in 7.11.)

**7.8.** The crucial point is that the defining expressions above do not depend on the chosen resolution. We indicate three different arguments, supposing for simplicity that $X$ is Gorenstein, i.e. the $a_i \in \mathbb{Z}_{>0}$.

(1) Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be two log resolutions of $X$. By the formula of Proposition 3.6 we have in fact

$$\sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu.$$

So we must show that $\int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu = \int_{\mathcal{L}(Y')} \mathbb{L}^{-ord_t K_{Y'|X}} d\mu$. To this end we take a log resolution $\rho : Z \to X$, dominating $\pi$ and $\pi'$; i.e. we have $\rho : Z \xrightarrow{\sigma} Y \xrightarrow{\pi} X$ and $\rho : Z \xrightarrow{\sigma'} Y' \xrightarrow{\pi'} X$. By the change of variables formula in (3.8) we have

$$\begin{aligned}\int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t (\sigma^* K_{Y|X} + K_{Z|Y})} d\mu \\ &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t (K_{Z|X})} d\mu ,\end{aligned}$$

and of course the same is true for the integral over $\mathcal{L}(Y')$.

This is essentially Batyrev's proof.

(2) We can define $\mathcal{E}_{st}(X)$ intrinsically, using motivic integration on $X$ [Ya1][DL6]. There is an ideal sheaf $\mathcal{I}_X$ on $X$ such that

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{ord_t \mathcal{I}_X} d\mu,$$

using the setting of (3.5) and (3.7). More precisely, denoting by $\omega_X$ the sheaf of differential $d$-forms on $X$ which are regular on $X_{\mathrm{reg}}$, we have a natural map $\Omega_X^d \to \omega_X$ whose image is $\mathcal{I}_X\omega_X$. See [Ya1, Lemma 1.16].

(3) Using the Weak Factorization Theorem, see below, one essentially has to show that the defining expressions in (7.7) do not change after blowing-up $Y$ in a nonsingular centre which intersects $\cup_{i\in S}E_i$ transversely. This is straightforward.

**7.9. Weak Factorization Theorem** [AKMW][Wł].

*(1) Let $\phi : Y \dashrightarrow Y'$ be a proper birational map between nonsingular irreducible varieties, and let $U \subset Y$ be an open set where $\phi$ is an isomorphism. Then $\phi$ can be factored as follows into a sequence of blow-ups and blow-downs with smooth centres disjoint from $U$.*

*There exist nonsingular irreducible varieties $Y_1, \dots, Y_{\ell-1}$ and a sequence of birational maps*

$$Y = Y_0 \overset{\phi_1}{\dashrightarrow} Y_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} Y_{i-1} \overset{\phi_i}{\dashrightarrow} Y_i$$
$$\overset{\phi_{i+1}}{\dashrightarrow} \cdots \overset{\phi_{\ell-1}}{\dashrightarrow} Y_{\ell-1} \overset{\phi_\ell}{\dashrightarrow} Y_\ell = Y'$$

*where $\phi = \phi_\ell \circ \phi_{\ell-1} \circ \cdots \circ \phi_2 \circ \phi_1$, such that each $\phi_i$ is an isomorphism over $U$ (we identify $U$ with an open in the $Y_i$), and for $i = 1, \dots, \ell$ either $\phi_i : Y_{i-1} \dashrightarrow Y_i$ or $\phi_i^{-1} : Y_i \dashrightarrow Y_{i-1}$ is the blowing-up at a nonsingular centre disjoint from $U$, and is thus a morphism.*

*(1′) There is an index $i_0$ such that for all $i \leq i_0$ the map $Y_i \to Y$ is a morphism, and for $i \geq i_0$ the map $Y_i \to Y'$ is a morphism.*

*(2) If $Y \setminus U$ and $Y' \setminus U$ are normal crossings divisors, then the factorization above can be chosen such that the inverse images of these divisors under $Y_i \to Y$ or $Y_i \to Y'$ are also normal crossings divisors, and such that the centres of blowing-up of the $\phi_i$ or $\phi_i^{-1}$ intersect these divisors transversely.*

*Remark.* (i) In [AKMW] and [Wł] the theorem is stated for a birational map $\phi$ between complete $Y$ and $Y'$; the generalization to proper birational maps between not necessarily complete $Y$ and $Y'$ is mentioned by Bonavero [Bo].

(ii) In [AKMW, Theorem 0.3.1] the first claim of (2) is not explicitly stated, but can be read off from the proof (see [AKMW, 5.9 and 5.10]).

**7.10.** *Important Intermezzo.* Using weak factorization instead of motivic integration, we can define $\mathcal{E}_{st}(X)$ in a localization of (a finite extension of) $\mathcal{M}_{\mathbb{C}}$, which is a priori finer than in (a finite extension of) $\hat{\mathcal{M}}_{\mathbb{C}}$, since we do not know whether the natural map $\mathcal{M}_{\mathbb{C}} \to \hat{\mathcal{M}}_{\mathbb{C}}$ is injective.

This remark also applies e.g. to (4.1), yielding $[X] = [Y]$ in the localization of $\mathcal{M}_{\mathbb{C}}$ with respect to the $[\mathbb{P}^j]$ instead of merely in $\hat{\mathcal{M}}_{\mathbb{C}}$.

**7.11.** *Example.* Let $X = \{x_1^k + x_2^k + \cdots + x_{d+1}^k = 0\} \subset \mathbb{A}^{d+1}$.
EXERCISE. We use the notation $E$ of Example 7.5.
(i) $\mathcal{E}_{st}(X) = (\mathbb{L}-1)[E] + [E]\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1}$,
(ii) $\mu(\mathcal{L}(X)) = (\mathbb{L}-1)[E] + [E]\frac{\mathbb{L}-1}{\mathbb{L}^{d}-1}$,
(iii) $[X] = (\mathbb{L}-1)[E] + 1$.

(Note also that (ii) and (iii) are consistent with Example 5.4.)

**7.12.** *Applications.*
(i) Topological mirror symmetry test for singular Calabi-Yau mirror pairs [Ba2].
(ii) A conjectural definition of stringy Hodge numbers for certain canonical Gorenstein varieties [Ba2].
(iii) A proof of a version of the McKay correspondence [Ba3][DL6][Ya1].
(iv) A new birational invariant for varieties of nonnegative Kodaira dimension, assuming the Minimal Model Program [Ve2, (2.8)].

## 8 Stringy invariants for general singularities

In this section $X$ is a $\mathbb{Q}$-Gorenstein variety.

**8.1.** For a log resolution $\pi : Y \to X$ of $X$, we use the notation $E_i$ and $a_i, i \in S$, and $E_I^\circ, I \subset S$, as before. There are (at least) two natural questions concerning a possible generalization of Batyrev's stringy invariants beyond the log terminal case.

QUESTION I. Suppose there exists at least one log resolution $\pi : Y \to X$ of $X$ for which *all* log discrepancies $a_i \neq 0$. Is (e.g.)

$$\sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{a_i}$$

independent of a chosen such resolution ?

This question is still open (a positive answer would yield a generalized stringy invariant for those $X$ admitting such a log resolution). Note that, when using the weak factorization theorem to connect two such log resolutions by chains of blowing-ups, log discrepancies on 'intermediate varieties' could be zero, obstructing an obvious attempt of proof.

QUESTION II. Do there exist any kind of invariants, associated to all or 'most' $\mathbb{Q}$-Gorenstein varieties, which coincide with Batyrev's stringy invariants if the variety is log terminal ?

Concerning this question, we obtained the following result [Ve4]. We associated invariants to 'almost all' $\mathbb{Q}$-Gorenstein varieties, more precisely to all $\mathbb{Q}$-Gorenstein varieties without strictly log canonical singularities, which do generalize Batyrev's invariants for log terminal varieties. (Note that in particular log discrepancies can be zero in a log resolution of a non log canonical variety !)

- To construct these invariants we have to assume Mori's Minimal Model Program (in fact the relative and log version).
- As in the previous section, we can work on any level : $\chi(\cdot), H(\cdot)$, and $[\cdot]$. For simplicity we treat here just the roughest level $\chi(\cdot)$; the other levels are analogous.

**8.2.** We associate to any $\mathbb{Q}$-Gorenstein $X$ without strictly log canonical singularities a rational function $z_{st}(X;s)$ in one variable $s$, the *stringy zeta function* of $X$. It will turn out that for log terminal $X$, this rational function is in fact a constant and equal to $e_{st}(X)$.

We just present the main idea of our construction. The 'pragmatic' idea is to split the log discrepancies $a_i$ of a log resolution $\pi : Y \to X$ as $a_i = \nu_i + N_i$ such that $(\nu_i, N_i) \neq (0,0)$ for all $i$, and to define $z_{st}(X;s)$ as

$$\sum_{I\subset S} \chi(E_I^\circ) \prod_{i\in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s).$$

This is done in a geometrically meaningful way via factoring $\pi$ through a certain 'partial resolution' $p : X^m \to X$ of $X$, which is called a *relative log minimal model of* $X$. This is a natural object in the (relative, log) Minimal Model Program; important here is that it is not unique and that $X^m$ can have certain mild singularities. (Its existence is the key point in this Program and this is for the moment proved only in dimensions 2 and 3.)

For the specialists : $p$ is a proper birational morphism, $X^m$ is $\mathbb{Q}$-factorial, the pair $(X^m, E^m)$ is divisorial log terminal, and $K_{X^m} + E^m$ is $p$-nef, where $E^m$ denotes the reduced exceptional divisor of $p$. References for these notions are e.g. in [KM][KMM][Ma].

We consider the factorization $\pi : Y \xrightarrow{h} X^m \xrightarrow{p} X$. In general $h$ is only a birational map (maybe not everywhere defined), but we suppose for the moment that it is a morphism. We justify this later. Denoting as usual by $E_i, i \in S$, the irreducible components of the exceptional divisor of $\pi$, we let $E_i^m, i \in S^m$, be the images in $X^m$ of those $E_i$ which 'survive' in $X^m$, i.e. which are not contracted by $h$ to varieties of smaller dimension. Then

$$\begin{aligned}\sum_{i\in S} a_i E_i &= K_Y + \sum_{i\in S} E_i - \pi^* K_X \\ &= \underbrace{K_Y + \sum_{i\in S} E_i - h^*(K_{X^m} + \sum_{i\in S^m} E_i^m)}_{(1)} \\ &\quad \underbrace{+\, h^*(K_{X^m} + \sum_{i\in S^m} E_i^m) - h^* p^* K_X}_{(2)} .\end{aligned}$$

Both (1) and (2) are divisors on $Y$, supported on $\cup_{i\in S} E_i$. We write (1) as $\sum_{i\in S} \nu_i E_i$; all $\nu_i \geq 0$ because the pair $(X^m, \sum_{i\in S^m} E_i^m)$ has only mild singularities (more precisely, because it is divisorial log terminal). We can rewrite (2) as

$$h^*(K_{X^m} + \sum_{i\in S^m} E_i^m - p^* K_X) = h^*(\sum_{i\in S^m} a_i E_i^m);$$

and it is well known that all $a_i, i \in S^m$, are non-positive (more precisely, this follows since $K_{X^m} + \sum_{i\in S^m} E_i^m$ is $p$-nef). So we can write (2) as $\sum_{i\in S} N_i E_i$ where all $N_i \leq 0$.

With these definitions of $\nu_i$ and $N_i$ we indeed have $a_i = \nu_i + N_i$ for $i \in S$, with moreover $\nu_i \geq 0$ and $N_i \leq 0$. One can show that, if $X$ has no strictly log canonical singularities, the situation $\nu_i = N_i = 0$ *cannot* occur.

When $X$ is log terminal, the morphism $p : X^m \to X$ has *no* exceptional divisors, so $S^m = \emptyset$, all $N_i = 0$ and $\nu_i = a_i$, and as promised $z_{st}(X; s) = e_{st}(X)$.

In fact we FIRST choose a relative log minimal model $p : X^m \to X$ of $X$, we secondly choose a log resolution $h : Y \to X^m$ of the pair $(X^m, E^m)$, where $E^m$ is the reduced exceptional divisor of $p$, and then we put $\pi := p \circ h$.

The point is again that $z_{st}(X; s)$ is independent of both choices, for which a crucial ingredient is the Weak Factorization Theorem.

**8.3. Theorem** [Ve4]. *Let $X$ be any* surface *without strictly log canonical singularities. Then*

$$\lim_{s \to 1} z_{st}(X; s) \in \mathbb{Q}.$$

(Recall that this is non-obvious since some $a_i$ can be zero. The clue is that if $a_i = 0$, then $E_i$ must be rational and must intersect exactly once or twice other components; this then easily implies the cancelling of $\nu_i + sN_i$ in the denominator of $z_{st}(X; s)$.) So we can define in dimension 2 a generalized stringy Euler *number* $e_{st}(X)$ as the limit above for any such surface $X$. In fact we constructed this generalized $e_{st}(X)$ in [Ve3] by a 'direct' approach.

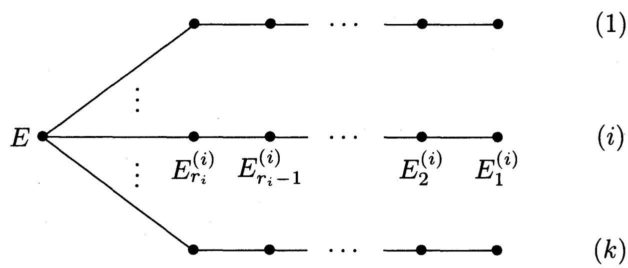


FIGURE 1

**8.4.** *Example* [Ve3]. Let $P \in X$ be a normal surface singularity with dual graph of its minimal log resolution $\pi : X \to S$ as in Figure 1. There is a central curve $E$ with genus $g$ and self-intersection number $-\kappa$, and all other curves are rational. Each attached chain $E_1^{(i)} - \cdots - E_{r_i}^{(i)}$ is determined by two co-prime numbers $n_i$ and $q_i$, which are the absolute value of the determinant of the intersection matrix of $E_1^{(i)}, \ldots, E_{r_i}^{(i)}$ and $E_1^{(i)}, \ldots, E_{r_i-1}^{(i)}$, respectively. Finally, we denote by $d$ the absolute value of the determinant of the total intersection matrix of $\pi^{-1}P$. This is a quite large class of singularities; it includes all weighted homogeneous isolated complete intersection singularities, for which the numbers $\{g;\kappa;(n_1,q_1),\cdots,(n_k,q_k)\}$ are called the *Seifert invariants* of the singularity.

If $P\in X$ is not strictly log canonical, then

$$e_{st}(X)=\lim_{s\to 1} z_{st}(X;s)=\frac{1}{a}(2-2g-k+\sum_{i=1}^{k} n_i)+\chi(X\setminus\{P\}),$$

where

$$a=\frac{2-2g-k+\sum_{i=1}^{k}\frac{1}{n_i}}{\kappa-\sum_{i=1}^{k}\frac{q_i}{n_i}}=\frac{\prod_{i=1}^{k} n_i}{d}(2-2g-k+\sum_{i=1}^{k}\frac{1}{n_i})$$

is the log discrepancy of $E$.

We note that some other log discrepancies might be zero. A particular example is the so-called *triangle singularity*, given by $g=0,\kappa=1$, $k=3$ and $r_1=r_2=r_3=1$. So, concretely, there is a central rational curve with self-intersection $-1$ to which three other rational curves are attached. Then $a=-1$ and the three other log discrepancies are zero, and $e_{st}(X)=1-(n_1+n_2+n_3)+\chi(X\setminus\{P\})$.

When such $P\in X$ is a weighted homogeneous isolated *hypersurface* singularity, this generalized stringy Euler number appears in some Taylor expansion associated to it, studied by Némethi and Nicolaescu [NN].

**8.5.** *Example.* [Ve4] Here we mention a concrete example of a threefold singularity $P\in X$, which has an exceptional surface with log discrepancy zero in a log resolution, and such that nevertheless $\lim_{s\to 1} z_{st}(X;s)\in\mathbb{Q}$, i.e. such that the evaluation $z_{st}(X;1)$ makes sense.

Let $X$ be the hypersurface $\{x^4+y^4+z^4+t^5=0\}$ in $\mathbb{A}^4$; its only singular point is $P=(0,0,0,0)$. We sketch the following constructions in Figure 2; we denote varieties and their strict transforms by the same symbol.

The blowing-up $\pi_1:Y_1\to X$ with centre $P$ is already a resolution of $X$ ($Y_1$ is smooth). Its exceptional surface $E_1$ is the affine cone over the smooth projective plane curve $C=\{x^4+y^4+z^4=0\}$. Let $\pi_2:Y_2\to Y_1$ be the blowing-up with centre the vertex $Q$ of this cone, and exceptional surface $E_2\cong\mathbb{P}^2$. Then $E_1\subset Y_2$ is a ruled surface over $C$ which intersects

$E_2$ in a curve isomorphic to $C$. The composition $\pi = \pi_1 \circ \pi_2$ is a log resolution of $P \in X$, and one easily verifies that the log discrepancies are $a_1 = 0$ and $a_2 = -1$; in particular $P \in X$ is not log canonical.

Now $E_1 \subset Y_2$ can be contracted (more precisely one can check that the numerical equivalence class of the fibre of the ruled surface $E_1$ is an extremal ray). Let $h : Y_2 \to X^m$ denote this contraction, and let $\pi = p \circ h$. As the notation suggests, one can verify that $K_{X^m} + E_2$ is $p$-nef, implying that $(X^m, E_2)$ is a relative log minimal model of $P \in X$.

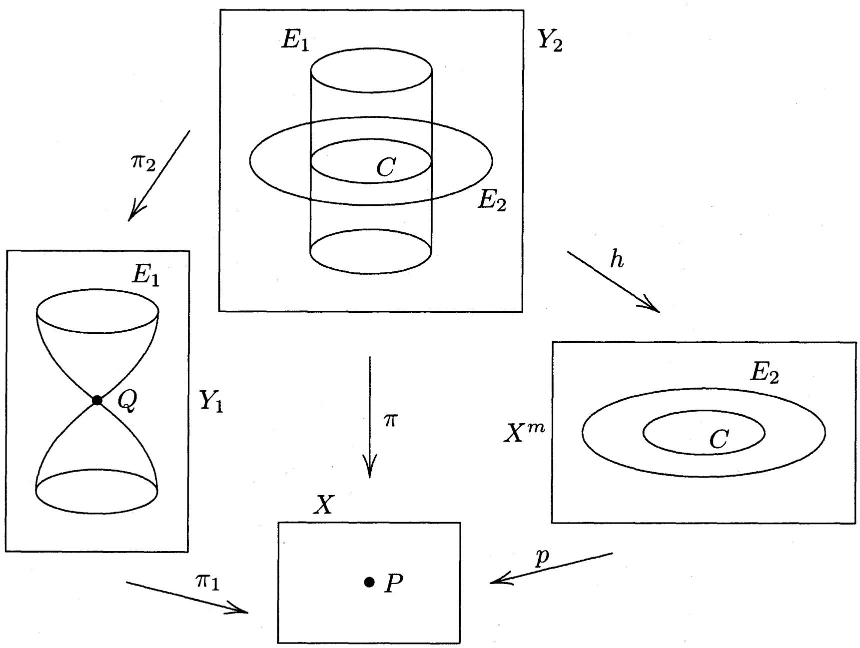


FIGURE 2

Denoting as usual

$$K_{Y_2} = h^*(K_{X^m} + E_2) + (\nu_1 - 1)E_1 + (\nu_2 - 1)E_2$$

and

$$h^*(a_2 E_2) = N_1 E_1 + N_2 E_2$$

we have clearly that $\nu_2 = 0$ and $N_2 = -1$, and one computes that $\nu_1 = \frac{1}{5}$ and $N_1 = -\frac{1}{5}$. So

$$\begin{aligned}
&z_{st}(X;s)\\
&= \frac{\chi(C)}{(\nu_1 + sN_1)(\nu_2 + sN_2)} + \frac{\chi(E_1 \setminus C)}{\nu_1 + sN_1} + \frac{\chi(E_2 \setminus C)}{\nu_2 + sN_2} + \chi(X \setminus \{P\})\\
&= \frac{-4}{(\frac{1}{5} - \frac{1}{5}s)(-s)} + \frac{-4}{\frac{1}{5} - \frac{1}{5}s} + \frac{7}{-s} + \chi(X \setminus \{P\}) = \frac{13}{s} + \chi(X \setminus \{P\}),
\end{aligned}$$

yielding $\lim_{s\to 1} z_{st}(X;s) = z_{st}(X;1) = 13 + \chi(X \setminus \{P\})$.

**8.6.** *Question.* Let $X$ be a $\mathbb{Q}$-Gorenstein variety of arbitrary dimension without strictly log canonical singularities. When is

$$\lim_{s\to 1} z_{st}(X;s) \in \mathbb{Q}\ ?$$

## 9 Miscellaneous recent results

Here we gather a collection of various results, which were obtained after the redaction of the survey paper [DL8].

• Aluffi noticed in [Al1] that the Euler characteristic formula in (4.2) implies interesting similar statements about Chern-Schwartz-MacPherson classes. Then in [Al2] he studies the birational behavior of Chern classes with respect to the 'motivic integration philosophy'. There he also introduces stringy Chern classes of log terminal varieties, which was done simultaneously by de Fernex, Lupercio, Nevins and Uribe in [dFLNU].

• Bittner [Bi2] calculated the relative dual of the motivic nearby fibre and constructed a nearby cycle morphism on the level of the Grothendieck group of varieties.

• More exotic motivic measures are introduced by Bondal, Larsen and Lunts [BLL] and Drinfeld [Dr].

• Using arc spaces and motivic integration, Budur [Bu] relates the Hodge spectrum of a hypersurface singularity to its *jumping numbers* (which come from multiplier ideals).

• Campillo, Delgado and Gusein-Zade [CDG1][CDG2][CDG3], and Ebeling and Gusein-Zade [EG1][EG2] studied filtrations on the ring of germs of functions on a germ of a complex variety, defined by arcs on the singularity. An important technique is integration with respect to the Euler characteristic over the projectivization of the space of function germs; this notion is similar to (and inspired by) motivic integration.

• Cluckers and Loeser [CL1][CL2][CL3] built a more general theory for relative motivic integrals, avoiding moreover the completion of Grothendieck rings. These integrals specialize to both 'classical' and arithmetic motivic integrals.

More 'relative theory' is in [Ni3].

• Dais and Roczen obtained formulas for the stringy Euler number and stringy $E$-function for some special classes of singularities [Da][DR].

• Now available are the ICM 2002 survey [DL9] and the recent expository paper of Hales [Hal3] on the theory of arithmetic motivic measure of Denef and Loeser [DL5]. Related work is in [DL10] and [Ni3].

• In [dSL] du Sautoy and Loeser associate motivic zeta functions to a large class of infinite dimensional Lie algebras.

• Ein, Lazarsfeld, Mustaţă and Yasuda have various other papers about spaces of jets, relating them for instance to singularities of pairs, in particular to the log canonical threshold, and to multiplier ideals [ELM][Mu2][Ya2].

• Koike and Parusiński [KP] associated motivic zeta functions to real analytic function germs and showed that these are invariants of blow-analytic equivalence. Fichou [Fi] obtained similar results in the context of Nash funcion germs. Both constructions are useful for classification issues.

• Gordon [Go] introduced a motivic analogue of the Haar measure for the (non locally compact) groups $G(k((t)))$, where $G$ is a reductive algebraic groups, defined over an algebraically closed field $k$ of characteristic zero.

• Guibert [Gui] computed the motivic zeta function associated to irreducible plane curve germs, yielding a new proof of the formula expressing the spectrum in terms of the Puiseux data. Here he studied also a motivic zeta function for a family of functions and related it with the Alexander invariants of the family; this is used to obtain a formula for the Alexander polynomial of a plane curve.

• Guibert, Loeser and Merle [GLM1] introduced iterated motivic vanishing cycles and proved a motivic version of a conjecture of Steenbrink concerning the spectrum of hypersurface singularities.

• Gusein-Zade, Luengo and Melle Hernández [GLM2] treat integration over spaces of non-parametrized arcs and introduce motivic versions of the classical monodromy zeta function. They indicate a formula connecting the motivic zeta function with this monodromy zeta function.

• Arithmetic motivic integration in the context of $p$-adic orbital integrals and transfer factors is considered by Gordon and Hales in [GH] and [Hal2]. An introduction to this theory is [Hal1].

• Ishii and Kollár [IK] found counter examples in dimensions at least 4 to the Nash problem, which relates irreducible components of the space of arcs through a singularity to exceptional components of a resolution. (And they proved it in general for toric singularities.) Reguera [Reg] showed in any dimension that the Nash problem is equivalent to the so-called wedge problem.

For a toric variety, Ishii [Is] described precisely the relation between arc families and valuations, and obtained the answer to the embedded version of the Nash problem.

• Ito produced an alternative proof that birational smooth minimal models have equal Hodge numbers [It1], and that Batyrev's stringy $E$-function is well defined [It2], using $p$-adic Hodge theory.

• Kapranov [Ka] introduced another motivic zeta function as the generating series for motivic measures of varying $n$-fold symmetric products of a fixed variety. Larsen and Lunts [LL1][LL2] determined for which surfaces this is a rational function over $K_0(Var_{\mathbb{C}})$. It is not known whether it is always a rational function over $\mathcal{M}_{\mathbb{C}}$. See also [DL10, §7] and [BDN].

• For toric surfaces, Lejeune-Jalabert and Reguera [LR] and Nicaise [Ni1] computed an explicit formula for the series $P(T)$ and $J(T)$, respectively. This last paper also contains a sufficient condition for the equality of $P(T)$ and the arithmetic Poincaré series of a toric singularity, which is always satisfied in the surface case. A counter example for this equality in dimension 3 is given.

In [Ni2] Nicaise provides a concrete formula for $P(T)$ if the variety has an embedded resolution of a simple form; this yields a short proof of the formula for toric surfaces.

• Loeser [Loe3] studied the behavior of motivic zeta functions of prehomogeneous vector spaces under castling transformations; he deduced in particular how the motivic Milnor fibre and the Hodge spectrum at the origin behave under such transformations.

• In [NS] Nicaise and Sebag establish the motivic zeta function as a Weil zeta function of the rigid Milnor fibre.

• Sebag [Se1][Se2] studied motivic integration and motivic zeta functions in the context of formal schemes. Loeser and Sebag [LS] developed a theory of motivic integration for smooth rigid varieties, obtained a motivic Serre invariant, and provided new geometric birational invariants of degenerations of algebraic varieties.

• The author introduces motivic principal value integrals and investigates their birational behavior in [Ve5].

• Vojta provides in [Vo] a general reference for jet spaces and jet differentials (at the level of EGA), using Hasse-Schmidt higher differentials.

• Yasuda [Ya1][Ya3] introduced so-called *twisted* jets and arcs over Deligne-Mumford stacks and studied then motivic integration over them. As applications he obtained a McKay correspondence for general orbifolds (see also [LP]), and a common generalization of the stringy $E$-function and the orbifold cohomology.

• Yokura [Yo] constructs Chern-Schwartz-MacPherson classes on pro-algebraic varieties and relates this to the motivic measure.

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*K. U. Leuven, Departement Wiskunde, Celestijnenlaan 200B,*
*B–3001 Leuven, Belgium,* `wim.veys@wis.kuleuven.be`
`http://www.wis.kuleuven.be/algebra/veys.htm`