Advanced Studies in Pure Mathematics 43, 2006
Singularity Theory and Its Applications
pp. 421–436

An infinitesimal criterion for topological triviality of families of sections of analytic varieties

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Abstract.

We present sufficient conditions for the topological triviality of families of germs of functions defined on an analytic variety V. The main result is an infinitesimal criterion using the integral closure of a convenient ideal as the tangent space to a subset of the set of topologically trivial deformations of a given germ. Applications to the problem of equisingularity of families of sections of V are also discussed.

§1. Introduction

Let $V, 0$ be the germ of an analytic subvariety of k^n ($k = \mathbb{R}$ or $\mathbb{C}$) and let $\mathcal{R}_V$ (respectively C^0-$\mathcal{R}_V$) be the group of germs of diffeomorphisms (respectively homeomorphisms) preserving $V, 0$. In this paper we introduce a sufficient condition for the C^0-$\mathcal{R}_V$- triviality of families of map germs $h : k^n \times k, 0 \to k^p, 0$, based on the integral closure of $T\mathcal{R}_V(h)$, the tangent space to the orbit of h under the action of the group $\mathcal{R}_V$. Our main result establishes that if $\frac{\partial h}{\partial t} \in \overline{T\mathcal{R}_V(h)}$, then h is topologically $\mathcal{R}_V$-trivial.

We are specially concerned with the case $p = 1$, that is, with families $h : k^n \times k, 0 \to k, 0$. In this case $h^{-1}(0)$ defines a family of sections of the analytic variety $V, 0$.

As a corollary of the method, we obtain sharp results when the analytic variety is weighted homogeneous and the family of sections is a deformation of a weighted homogeneous map germ h_0 (consistent with V) by terms of filtration higher than or equal to the filtration of h_0.

Received June 6, 2004.
Revised December 16, 2004.
2000 *Mathematics Subject Classification.* 32S15, 58K40, 58K15.
The first author was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, grant # 300066/88-0 and by FAPESP, grant # 03/03107-9.

This result was previously proved by Damon in [8]. In the final section, we introduce a notion of V-equisingularity of the family of sections and we show that the hypothesis of the main theorem implies this geometric condition. A weighted approach for the topological triviality of families of sections of analytic varieties was presented in [16]. For other results related to the subject discussed in this paper, see for instance [2], [8], [19].

§2. Basic results

Let $\mathcal{O}_n$ be the ring of germs of analytic functions $h : k^n, 0 \to k$, $k = \mathbb{R}$ or $\mathbb{C}$.

A germ of a subset $V, 0 \subset k^n, 0$ is the germ of an analytic variety if there exist germs of analytic functions $f_1, ..., f_r$ such that $V = \{x : f_1(x) = \cdots = f_r(x) = 0\}$.

Our aim is to study map germs $h : k^n, 0 \to k^p, 0$ under the equivalence relation that preserves the analytic variety $V, 0$. We say that two germs h_1 and $h_2 : k^n, 0 \to k^p, 0$ are $\mathcal{R}_V$-equivalent (respectively C^0-$\mathcal{R}_V$-equivalent) if there exists a germ of a diffeomorphism (respectively homeomorphism) $\phi : k^n, 0 \to k^n, 0$ with $\phi(V) = V$ and $h_1 \circ \phi = h_2$. That is,

$$\mathcal{R}_V = \{\phi \in \mathcal{R} : \phi(V) = V\},$$

where $\mathcal{R}$ is the group of germs of diffeomorphisms of $k^n, 0$.

A one parameter deformation $h : k^n \times k, 0 \to k^p, 0$ of $h_0 : k^n, 0 \to k^p, 0$ is topologically $\mathcal{R}_V$-trivial (or C^0-$\mathcal{R}_V$-trivial) if there exists homeomorphism $H : k^n \times k, 0 \to k^n \times k, 0$, $H(x,t) = (\bar{h}(x,t), t)$, such that $h \circ H(x,t) = h_0(x)$ and $H(V \times k) = V \times k$.

We denote by θ_n the set of germs of tangent vector fields in $k^n, 0$; θ_n is a free $\mathcal{O}_n$ module of rank n. Let $I(V)$ be the ideal in $\mathcal{O}_n$ consisting of germs of analytic functions vanishing on V. We denote by $\Theta_V = \{\eta \in \theta_n : \eta(I(V)) \subseteq I(V)\}$, the submodule of germs of vector fields tangent to V (see [2] for more details).

The tangent space to the action of the group $\mathcal{R}_V$ is $T\mathcal{R}_V(h) = dh(\Theta_V^0)$, where Θ_V^0 is the submodule of Θ_V given by the vector fields that are zero at zero.

The group $\mathcal{R}_V$ is a geometric subgroup of the contact group, as defined by J.Damon [5], [6], hence the infinitesimal criterion for $\mathcal{R}_V$-determinacy holds (see [2] for a proof).

Theorem 2.1. *The germ h is $\mathcal{R}_V$-finitely determined if and only if there exists a positive integer k such that $T\mathcal{R}_V(h) \supset \mathcal{M}_n^k$.*

The following theorem is the geometric criterion for the $\mathcal{R}_V$-finite determinacy.

Theorem 2.2. ([2]) *Let $V, 0 \subseteq \mathbb{C}^n, 0$ be the germ of an analytic variety and let $h : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ be the germ of an analytic function. Let*

$$V(h) = \{x \in \mathbb{C}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}.$$

Then h is $\mathcal{R}_V$-finitely determined if and only if $V(h) = \{0\}$ or $\emptyset$.

As a consequence of this result, it follows that if h is $\mathcal{R}_V$-finitely determined, then $h^{-1}(c)$ is transverse to V away from 0, for sufficiently small values of c.

In the real case, the necessary condition remains true, that is, if h is $\mathcal{R}_V$-finitely determined then the set $\{x \in \mathbb{R}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}$ is $\{0\}$ or $\emptyset$.

§3. Basic facts on integral closure of ideals

Let I be an ideal in a ring A. An element $h \in A$ is said to be integral over I if it satisfies an integral dependence relation $h^n + a_1 h^{n-1} + ... + a_n = 0$ with $a_i \in I^i$. The set of such elements form an ideal in A, called the integral closure of I.

When $A = \mathcal{O}_{X,x_0}$, the local ring of a complex analytic set, Teissier gives in [18] various notions equivalent to the above concept.

Theorem 3.1. ([11], Proposition 1.2) *Let I be an ideal in $\mathcal{O}_{X,x_0}$ and $\overline{I}$ its integral closure, where X is a complex analytic space. The following statements are equivalent:*
(a) $h \in \overline{I}$.
(b) *For each choice of generators $\{g_i\}$ of I there exist a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:*

$$|h(x)| \leq C \sup_i |g_i(x)|.$$

(c) *For each analytic curve $\varphi : \mathbb{C}, 0 \to X, x_0$, $h \circ \varphi$ lies in $(\varphi^*(I))\mathcal{O}_1$.*
(d) *There exists a faithful $\mathcal{O}_{X,x_0}$ module L of finite type such that $h.L \subset I.L$.*

In the real case, the above algebraic definition of integral closure is not appropriate. But, one can use condition (c) above as a definition. More precisely,

Definition 3.2. *Let I be an ideal of the ring $\mathcal{O}_{X,x_0}$, where X is a real analytic set. The real integral closure $\overline{I}$ of I is the set of h such that for all analytic $\varphi : \mathbb{R}, 0 \to X, x_0$, we have $h \circ \varphi \in (\varphi^*(I))\mathcal{O}_1$.*

Gaffney ([11], p. 30) shows that $h \in \overline{I}$ if and only if for each choice of generators $\{g_i\}$ of I there exists a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:

$$|h(x)| \leq C\text{sup}_i |g_i(x)|.$$

§4. The main result

Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ of an analytic function and let $h : k^n \times k, 0 \to k, 0$ be an analytic deformation of h_0. In the sequel, we shall assume $h(0,t) = 0$. The property of being $\mathcal{R}_V$-finitely determined is open in the sense that the germ $\{x \in k^n : dh_t\xi(x) = 0, \forall\, \xi \in \Theta_V\}$ at 0 is $\{0\}$ or empty for sufficiently small values of the parameters ([2]). However, this does not guarantee the existence of a neighbourhood U of 0 in $k^n, 0$ and an open ε-ball, B_ε, centered at the origin in k such that the above condition holds $\forall\, x \in U$ and $\forall\, t \in B_\varepsilon$. We then need the following definition:

Definition 4.1. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ. We say that a deformation $h : k^n \times k, 0 \to k, 0$ of h_0 is a* good deformation *if $V(h) \subseteq \{0\} \times k, 0$, where $V(h) = \{(x,t) \in k^n \times k, 0\,; dh_t(x)\xi(x) = 0\, \forall\, \xi \in \Theta_V\}$.*

Example 4.2. *Let V be the x-axis in k^2; Θ_V is generated by $(1,0)$ and $(0,y)$. The germ $h_0(x,y) = x^2 + y^3$ is $\mathcal{R}_V$-finitely determined. The deformation $h_t(x,y) = x^2 + y^3 + ty^2$ of h_0 has the property that h_t is $\mathcal{R}_V$-finitely determined for each fixed t, but we cannot find $\varepsilon > 0$ such that the above condition holds for all $t \in B_\varepsilon$.*

Our main result is the following theorem:

Theorem 4.3. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ and let $h : k^n \times k, 0 \to k, 0$ be a good deformation of h_0. If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ for all $t \in k$ sufficiently near 0, then h is C^0-$\mathcal{R}_V$-trivial.*

The proof of the theorem is a consequence of the following results.

In what follows we can assume that $dh_t\xi(0) = 0, \forall\, \xi \in \Theta_V$. In fact, if $\xi \in \Theta_V$, then $dh_t\xi . \frac{\partial h}{\partial t} = dh_t(\frac{\partial h}{\partial t} . \xi)$. If $dh_t\xi_0(0) \neq 0$ for some ξ_0, then

$$\frac{\partial h}{\partial t} = dh_t\left(\frac{\frac{\partial h}{\partial t} . \xi_0}{dh_t\xi_0}\right)$$

and hence the deformation is C^ω-$\mathcal{R}_V$-trivial (i.e. analytically trivial). Observe that $\frac{\frac{\partial h}{\partial t} . \xi_0}{dh_t\xi_0} \in \Theta_V^0$.

Lemma 4.4. *Let I and J be ideals in $\mathcal{O}_n$ with $\mathcal{M}_n I \subseteq J \subseteq I$ and $V(I) = \{0\}$, where $V(I)$ is the variety of the ideal I. Then $V(J) = \{0\}$.*

Proof. From the hypothesis, $V(\mathcal{M}_n I) \supseteq V(J) \supseteq V(I)$. Since $V(M_n I) = V(\mathcal{M}_n) \cup V(I) = \{0\} \cup \{0\}$, we get $V(J) = \{0\}$. Q.E.D.

Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ and let $h : k^n \times k, 0 \to k, 0$ be a good deformation of h_0. Let $\{\xi_1, ..., \xi_r\}$ be generators of Θ_V and $I = \langle dh_t\xi_1, ..., dh_t\xi_r\rangle$ the ideal in $\mathcal{O}_{n+1}$ then $V(I) \subseteq \{0\} \times k$, since h is a good deformation of h_0. Let $\{\alpha_1, ..., \alpha_m\}$ be the generators of Θ_V^0, $dh_t\alpha_i = \rho_i$ and $J = \langle \rho_1, ..., \rho_m\rangle$. Since the α_i and hence the ρ_i vanish on $\{0\} \times k$, it follows that $V(J) \supseteq \{0\} \times k$. On the other hand, $M_n I \subset J \subset I$, and it follows from Lemma 4.4, that $V(J) \subseteq \{0\} \times k$.

Let $\rho(x,t) = \sum_{i=1}^m |\rho_i|^2$. The condition $V(J) = \{0\} \times k$ implies that $\rho \geq 0$, and $\rho_t(x) = 0$ is equivalent to $x = 0$. Then, the following result holds.

Lemma 4.5. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ and let $h : k^n \times k, 0 \to k, 0$ be a good deformation of h_0. If $\rho(x,t) = \sum_{i=1}^m |dh_t\alpha_i|^2$, then $V(\rho(x,t)) = \{0\} \times k$.*

Lemma 4.6. *Let $h : k^n \times k, 0 \to k, 0$ be a deformation of h_0. Suppose there is a continuous vector field $(W, 1) \in \Theta_{V\times k}$ such that:*
(i) $\rho\frac{\partial h}{\partial t} = dh_t(W)$, where ρ is a control function, that is, $\rho : k^n \times k, 0 \to \mathbb{R}$ with $\rho(x,t) \geq 0$ and $\rho(x,t) = 0$ if and only if $x = 0$.
(ii) $(-\frac{W}{\rho}, 1)$ is locally integrable.
Then h is topologically $\mathcal{R}_V$-trivial.

Proof. Let $\phi(x,t,\tau)$ be the flow of the on $k^n \times k, 0$ defined by $(-\frac{W}{\rho}, 1)$, so $\frac{\partial \phi}{\partial \tau} = (-\frac{W}{\rho}, 1) \circ \phi$, $\phi(x,t,0) = (x,t)$. When $k = \mathbb{R}$, we define

$$\varphi(x,t) = \phi(x,0,t) = (\overline{\varphi}(x,t), t).$$

Taking the derivative of $h(\varphi(x,t)) = h(\overline{\varphi}(x,t), t)$ with respect to t, we get

$$\begin{aligned}
\frac{\partial}{\partial t}(h(\varphi(x,t))) &= \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\overline{\varphi}(x,t),t)\frac{\partial \overline{\varphi_i}}{\partial t}(x,t) + \frac{\partial h}{\partial t}(\overline{\varphi}(x,t),t) \\
&= -\sum_{i=1}^n \frac{\partial h}{\partial x_i}(\overline{\varphi}(x,t),t)\frac{W_i}{\rho}(\overline{\varphi}(x,t),t) + \frac{\partial h}{\partial t}(\overline{\varphi}(x,t),t) \\
&= \left(\frac{\partial h}{\partial t} - \sum_{i=1}^n \frac{W_i}{\rho}\frac{\partial h}{\partial x_i}\right)(\overline{\varphi}(x,t),t) = 0
\end{aligned}$$

where W_i are the components of W. Hence, fixing x, it follows that $h(\varphi(x,t))$ is constant, that is, $h(\varphi(x,t)) = h(\varphi(x,0)) = h(x,0) = h_0(x)$

for all t and x. Therefore h is topologically $\mathcal{R}_V$-trivial. When $k=\mathbb{C}$, we consider the restriction

$$h^1 = h|\mathbb{C}^n \times \mathbb{R} \times \{0\} \to \mathbb{C}.$$

It is sufficient to show that h is a $\mathcal{R}_V$-topologically trivial deformation of h^1, which in turn is a $\mathcal{R}_V$-topologically trivial deformation of h_0.

Let $\phi(x,t,\tau)$ be such that $\frac{\partial\phi}{\partial\tau} = (-\frac{W}{\rho},1)\circ\phi$ and $\phi(x,t,0)=(x,t)$. We consider $\phi_1(x,u+iv) = \phi(x,u,v)$ and $\phi_2(x,u) = \phi(x,0,u)$. It follows that $h\circ\phi_1$ is constant with respect to v and hence $h(\phi_1(x,u+iv)) = h(\phi_1(x,u)) = h(\phi(x,u,0)) = h(x,u) = h^1(x,u)$. One can also show that $h^1\circ\phi_2$ is constant with respect to u, therefore $h^1(\phi_2(x,u)) = h^1(\phi_2(x,0)) = h^1(x,0) = h_0$ and the result follows. Q.E.D.

Proof of the Theorem 4.3. With the above notations, it follows that

$$|\rho_i|^2\frac{\partial h}{\partial t} = dh_t(\overline{\rho_i}\frac{\partial h}{\partial t}\alpha_i).$$

Since $\rho = \sum_{i=1}^m |\rho_i|^2$, it follows that

$$\rho\frac{\partial h}{\partial t} = dh_t\left(\frac{\partial h}{\partial t}(\overline{\rho_1}\alpha_1 + \ldots + \overline{\rho_m}\alpha_m)\right)$$

hence

$$\frac{\partial h}{\partial t} = dh_t\left(\frac{\partial h}{\partial t}\frac{1}{\rho}(\overline{\rho_1}\alpha_1 + \ldots + \overline{\rho_m}\alpha_m)\right).$$

From Lemma 4.5, $V(\rho(x,t)) = \{0\}\times k$. We define the vector field X in $k^n\times k, 0$,

$$X(x,t) = \begin{cases} \left(-\frac{\partial h}{\partial t}\frac{1}{\rho}(\overline{\rho_1}\alpha_1 + \ldots + \overline{\rho_m}\alpha_m), 1\right) & \text{if } x \neq 0 \\ (0,1) & \text{if } x = 0 \end{cases}$$

The vector field $X(x,t)$ is real analytic away from $\{0\}\times k$.

From the hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ and hence by item (b) of Theorem 3.1

$$|\frac{\partial h}{\partial t}| \leq c\sup\{|\rho_i|\}.$$

Then

$$\begin{aligned} |X(x,t)-X(0,t)| &= |\tfrac{\partial h}{\partial t}\tfrac{1}{\rho}(\overline{\rho_1}\alpha_1+...+\overline{\rho_m}\alpha_m)| \\ &\leq |\tfrac{\partial h}{\partial t}|\tfrac{1}{\rho}(|\overline{\rho_1}||\alpha_1|+...+|\overline{\rho_m}||\alpha_m|) \\ &\leq c\sup\{|\rho_i|\}\tfrac{1}{\rho}(|\rho_1||\alpha_1|+...+|\rho_m||\alpha_m|) \\ &\leq c(|\alpha_1|+...+|\alpha_m|)\leq C|x|. \end{aligned}$$

Thus, X satisfies the Lipschitz condition around the solution $(0,t)$, and it follows from [4] or [13] that $X(x,t)$ is locally integrable in a neighbourhood of $(0,0)\in k^n\times k$. Then, there exists a family of homeomorphisms $\phi(x,t,\tau)$, $\phi: k^n\times k\times\mathbb{R},0\to k^n\times k,0$ such that $\frac{\partial\phi}{\partial\tau}=-X\circ\phi$ and $\phi(x,t,0)=(x,t)$. The proof follows now from Lemma 4.6 (see Lemma 6.2, in [9]). Q.E.D.

§5. Weighted homogeneous germs and varieties

Definition 5.1. *(a) Given $(w_1,...,w_n:d_1,...,d_p)$, w_i, $d_j\in\mathbb{Q}^+$, a map germ $f:k^n,0\to k^p,0$ is weighted homogeneous of type $(w_1,...,w_n:d_1,...,d_p)$ if for all $\lambda\in k-\{0\}$:*

$$f(\lambda^{w_1}x_1,\lambda^{w_2}x_2,...,\lambda^{w_n}x_n)=(\lambda^{d_1}f_1(x),\lambda^{d_2}f_2(x),...,\lambda^{d_p}f_p(x)).$$

In this case, the value w_i is called weight of the variable x_i and the value d_i, is the filtration of f_i with respect to the weights $(w_1,...,w_n)$. We write: weight$(x_i)=w(x_i)=w_i$ and filtration$(f)=$ fil$(f)=(d_1,...,d_p)$.

(b) Given $(w_1,...,w_n)$, and any monomial $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$, we define fil$(x^\alpha)=\sum_{i=1}^n\alpha_iw_i$.

(c) We define a filtration in the ring $\mathcal{O}_n$ via the function defined by fil$(f)=\inf_{|\alpha|}\{$fil$(x^\alpha):\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(0)\neq 0\}$, $|\alpha|=\alpha_1+\ldots+\alpha_n$.

Definition 5.2. *A germ of an analytic variety $V,0\subseteq k^n,0$ is weighted homogeneous if it is defined by a weighted homogeneous map germ $f:k^n,0\to k^p,0$.*

Definition 5.3. *Let $V,0\subseteq k^n,0$ be the germ of a weighted homogeneous analytic variety. We say that a set $\{\alpha_1,...,\alpha_r\}$ of generators of Θ_V is weighted homogeneous of type $(w_1,...,w_n:d_1,...,d_r)$ if α_{ij} are weighted homogeneous polynomials of type $(w_1,...,w_n:d_i+w_j)$ whenever $\alpha_{ij}\neq 0$, where $\alpha_i=\sum_{j=1}^n\alpha_{ij}\frac{\partial}{\partial x_j}$, $i=1...r$.*

When V is a weighted homogeneous variety, we always can choose weighted homogeneous generators for Θ_V. A proof can be found in [10].

Following [7], we define:

Definition 5.4. *Let V be defined by weighted homogeneous polynomials. We say that h is weighted homogeneous consistent with V if h is weighted homogeneous with respect to the same set of weights assigned to V.*

Example 5.5. *Let $V = \phi^{-1}(0) \subset k^3$ where $\phi(x, y, z) = z^2 - x^2y$. We have that ϕ is weighted homogeneous of type $(1, 2, 2 : 4)$. Let $h(x, y, z) = x^3 + xy + xz$ and $f(x, y, z) = x^3 + xy + z^2$. Then h is consistent with V, f is weighted homogeneous but not consistent with V.*

The following result does not follow as a corollary of the Theorem 4.3, but the proof is similar. It was previously proved by J. Damon in [8], but we include it here for completeness. In [16], we discuss a weighted approach for the topological triviality of families of sections of analytic varieties, which also gives Theorem 5.6 as a corollary.

Theorem 5.6. *Let V be a weighted homogeneous subvariety of $k^n, 0$ and let $h_0 : k^n, 0 \to k, 0$ be weighted homogeneous consistent with V and $\mathcal{R}_V$-finitely determined. Then any deformation h of h_0 by terms of filtration greater than or equal to the filtration of h_0, is C^0-$\mathcal{R}_V$-trivial.*

Proof. Under the above conditions, any such h is a good deformation of h_0 (see [15]).

We have $dh_0(\alpha_i)$ is weighted homogeneous, where $\{\alpha_1, ..., \alpha_m\}$ is a set of weighted homogeneous generators of Θ_V. Let r_i be the filtration of $dh_0(\alpha_i)$, $i = 1, ..., m$ and

$$\omega_0(x) = |dh_0(\alpha_1)(x)|^{2s_1} + ... + |dh_0(\alpha_m)(x)|^{2s_r}$$

with $s_i = k/r_i$, and $k = \text{l.c.m.}\{r_i\}$. Let $\rho_i = dh_t(\alpha_i)$ and $\omega = \sum_{i=1}^m |\rho_i|^{2s_i}$. Since

$$|\rho_i|^2 \frac{\partial h}{\partial t} = dh_t(\overline{\rho_i} \frac{\partial h}{\partial t} \alpha_i),$$

it follows that

$$\omega \frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} (\overline{\rho_1} |\rho_1|^{2s_1 - 2} \alpha_1 + ... + \overline{\rho_m} |\rho_m|^{2s_m - 2} \alpha_m) \right).$$

Then

$$\frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} \frac{1}{\omega} (\overline{\rho_1} |\rho_1|^{2s_1 - 2} \alpha_1 + ... + \overline{\rho_m} |\rho_m|^{2s_m - 2} \alpha_m) \right).$$

The proof now follows analogously to the proof of Theorem 4.3.
Q.E.D.

Example 5.7. *Let $V, 0 \subset \mathbb{R}^3, 0$ (or $\mathbb{C}^3, 0$) be defined by $\varphi(x,y,z) = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y = 0$. This is the implicit equation for the S_k-singularities classified by D. Mond* [14]. *The function-germ φ is weighted homogeneous of weights 2, $2k+2$ and $3k+3$ for x, y and z respectively. We have that $h(x,y,z) = y + a_{k+1}x^{k+1}$ is $\mathcal{R}_V$-finitely determined for $a_{k+1} \neq 0, 1$ and consistent with V. Therefore deformations of h by terms of order higher than or equal to fil(h) are C^0-$\mathcal{R}_V$-trivial. For k odd, $h_1(x,y,z) = z + ax^{3(k+1)/2}$ and $h_2(x,y,z) = z + bx^{(k+1)/2}y$ are consistent with V and $\mathcal{R}_V$-finite for all $a^2 \neq -4/27$ and $b \neq \pm 2$. Thus deformations of h_1 and h_2, respectively by terms of order higher than or equal to fil(h_1) and fil(h_2) are C^0-$\mathcal{R}_V$-trivial.*

§6. V-Equisingularity

Bernard Teissier developed in [18] an infinitesimal theory and a theory of geometrical invariants to study the equisingularity of families of complex analytic hypersurfaces X_t^d with isolated singularities. The integral closure of an ideal I is the right object to the infinitesimal part of that theory. T. Gaffney in [11] extended Teissier results, using the integral closure of a convenient module to obtain necessary and sufficient conditions for the equisingularity of families of complete intersections with isolated singularities.

Definition 6.1. *Suppose (X,x) is a complex analytic germ, $\mathcal{O}_{X,x}$ its local ring and M a submodule of $\mathcal{O}^p_{X,x}$. Then an element $h \in \mathcal{O}^p_{X,x}$ is in $\overline{M}$ if and only if for all $\phi : \mathbb{C}, 0 \to X, x$, $h \circ \phi$ is in $(\phi^*(M))\mathcal{O}_1$.*

Theorem 6.2. ([11], Theorem 2.5) *Let $F : \mathbb{C}^t \times \mathbb{C}^N \to \mathbb{C}^p, 0$, defining $X = F^{-1}(0)$ with reduced structure, $Y = \mathbb{C}^t \times 0$ and X_0 the smooth part of X. Then $\frac{\partial F}{\partial s} \in \overline{\left\langle z_i \frac{\partial F}{\partial z_j} \right\rangle}_{\mathcal{O}_X}$ for all tangent vectors $\frac{\partial}{\partial s}$ to $\mathbb{C}^t \times 0$ iff (X_0, Y) are Whitney regular.*

Our purpose in this section is to show that the infinitesimal condition in Theorem 4.3 gives a sufficient condition for equisingularity of families of sections of analytic varieties. We also show with an example that it is not a necessary condition.

Let $V \subset \mathbb{C}^n$ be an analytic variety. The family of sections of V is defined by $h(x,t) = 0$, where $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$, $h(0,t) = 0$, is a good deformation of a $\mathcal{R}_V$-finitely determined map germ $h_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$.

In order to define the notion of V-equisingularity, we will construct a stratified diagram of mappings which satisfies Thom's second isotopy lemma.

From now on, we assume that V admits a Whitney stratification $\mathcal{S}_V$ in a neighbourhood U of the origin, for which $\{0\}$ is a stratum. We can also extend this stratification to the neighbourhood U of the origin in a natural way, that is, the strata are the strata of $\mathcal{S}_V$ and the complement of V in U. We denote by $\tilde{V}$ the subvariety of $\mathbb{C}^n \times \mathbb{C}, 0$ defined by $\tilde{V} = V \times \mathbb{C}$. The product stratification is clearly Whitney regular. Since the germ $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$ is a good deformation, we can choose a representative, which we also denote by h, given by $h : U \times B_r, 0 \to \mathbb{C}, 0$, where B_r is an open ball in $\mathbb{C}$ centered at the origin with the property that $h^{-1}(0)$ is transversal to the strata of $\tilde{V}$ away from $0 \times B_r$.

We refine the stratification $\tilde{\mathcal{S}}$ of $U \times B_r$ as follows. Given a stratum S of $\mathcal{S}$, we define the new strata $\tilde{S}$ of $\tilde{\mathcal{S}}$ as one of the following types: $(S \times B_r) - h^{-1}(0)$ and $(S \times B_r) \cap h^{-1}(0)$. This refinement defines a new stratification $U \times B_r$, since h is transversal to $\tilde{V}$ away from zero. We denote this new stratification by the same notation $\tilde{\mathcal{S}}$.

Definition 6.3. *With the above notation, h is V-equisingular if there exists $\varepsilon > 0$ such that:*
(1) $(B_\varepsilon \times B_r, \tilde{\mathcal{S}})$ is Whitney regular;
(2) $B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$ satisfies the second isotopy lemma, where B_ε is the closed ball in $\mathbb{C}^n$ with radius ε, B_r is the closed ball in $\mathbb{C}$ of radius r, and $F : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C} \times \mathbb{C}, 0$ is given by $F(x,t) = (h(x,t), t)$.

In the following theorem we show that $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ is a sufficient condition for V-equisingularity.

Theorem 6.4. *Let $V = \phi^{-1}(0)$, $\phi : \mathbb{C}^n, 0 \to \mathbb{C}, 0$, $h_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ $\mathcal{R}_V$-finitely determined and $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$ a good deformation of h_0. Let $h^{-1}(0) \cap \Sigma_\phi = \{0\} \times \mathbb{C}$, where Σ_ϕ is the singular set of ϕ. If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$, then h is V-equisingular.*

J.W. Bruce in [1] considers an analogous question. He describes the topological type of generic families of sections of a semialgebraic stratification $\mathcal{T}$ of a neighbourhood of the origin in $\mathbb{R}^n$, with 0 being a stratum. Such families are *generalised transverse* (G.T) with respect to the stratification, that is, for every pair of strata S_1 and S_2, and a sequence of points $(x_i) \in S_1$ such that $\lim_{i\to\infty} x_i = x \in S_2$ and the limit of the tangent spaces $\lim_{i\to\infty} T_{x_i}S_1 = T$ then $dh(x) : T \to \mathbb{R}$ has maximal rank, that is, $h^{-1}(h(x))$ is transversal to T.

The following theorem is proved in [1]:

Theorem 6.5. ([1], Proposition 1.4) *Let $\mathcal{T}$ a Whitney stratification of an open neighbourhood U of the origin in $\mathbb{R}^n$, with 0 being a stratum. Let $h : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be a family of submersions, with $h(0,t) = 0$ and $h_t(x) = h(x,t)$. If the family h is generalised transverse with respect to $\mathcal{T}$, for all $t \in [0,1]$, then there exists a germ of homeomorphism $G : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ preserving the strata of $\mathcal{T}$ such that $h_0 \circ G = h_1$.*

Good examples of families satisfying the G.T. condition are the families of sections of an analytic variety defined by generic families of hyperplanes in $\mathbb{R}^n$. In this work, we substitute the G.T. condition by the finite determinacy of h_0 and the integral closure condition. Under these hypothesis we are able to obtain the topological triviality of families that do not satisfy the G.T. condition.

Example 6.6. *Let $V, 0 \subseteq \mathbb{C}^3, 0$ be the swallowtail parametrized by $(x, -4y^3 - 2xy, -3y^4 - xy^2)$. The module Θ_V is generated by $\eta_1 = (2x, 3y, 4z)$, $\eta_2 = (6y, -2x^2 - 8z, xy)$ and $\eta_3 = (-4x^2 - 16z, -8xy, y^2)$. The $\mathcal{R}_V$ classification of germs $h : \mathbb{C}^3, 0 \to \mathbb{C}, 0$ given by Theorem 4.10 in [3], gives the normal form $z + ax^n + tx^{n+1}$, $n \geq 2$ which is finitely determined for $a \neq 0, n \neq 2$, and $a \neq 0, a \neq 1/12, n = 2$. Let $h_0(x,y,z) = z + ax^n$ from Theorem 4.3 we have that the family $h_t(x,y,z) = z + ax^n + tx^{n+1}$ is topologically $\mathcal{R}_V$-trivial. However this family h_t is not G.T. at 0, since $dh_t(0,0,0) = (0,0,1)$ and the limit of tangent planes to the smooth part of V is the xy-plane.*

To prove Theorem 6.4, we first prove the following Lemma.

Lemma 6.7. *Let $\phi : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ and $V = \phi^{-1}(0)$. Given $h : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, define $G : \mathbb{C}^{n+1}, 0 \to \mathbb{C}^2, 0$, by $G(x,t) = (h(x,t), \phi(x))$. If $g \in \overline{dh_t(\Theta_V^0)}_{\mathcal{O}_{n+1}}$ then $(g, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$.*

Proof. By hypothesis, for any analytic curve $\varphi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$, it follows that $g \circ \varphi \in \langle dh_t(\alpha_i) \circ \varphi \rangle$ where α_i are generators of Θ_V^0. Then for all $\varphi : \mathbb{C}, 0 \to V \times \mathbb{C}, 0$, we also have $(g \circ \varphi, 0) \in \langle dh_t(\alpha_i) \circ \varphi, d\phi(\alpha_i) \circ \varphi \rangle$, since $d\phi(\alpha_i) \in \langle \phi \rangle$ and $\phi(V) = 0$. Therefore $(g \circ \varphi, 0) \in \left\langle (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \circ \varphi \right\rangle$. Thus, $(g, 0) \in \overline{\left\langle (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \right\rangle}_{\mathcal{O}_{V \times \mathbb{C}}} = \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{V \times \mathbb{C}}}$. In particular, $(g, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$. Q.E.D.

Remark 6.8. *The above result remains true under the weaker hypothesis $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V \times \mathbb{C}}}$.*

We now proceed to prove Theorem 6.4; our proof is analogous to the proof of Theorem 6.5 in [1]. As in [1], we divide the proof in steps:

Step 1. The stratification $\tilde{S}$ is Whitney regular.

Proof. The Whitney regularity of a pair of strata (S_1, S_2) follows easily, with exception of the regularity condition of the strata over $\{0\} \times \mathbb{C}$. Clearly the strata of type $(S \times \mathbb{C}) - h^{-1}(0)$ are regular with respect to $\{0\} \times B_r$, since the original stratification satisfies the Whitney conditions. Then we only have to verify that $\overline{(S \times B_r) \cap h^{-1}(0)}$ is regular over $\{0\} \times B_r$. From hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t \Theta_V^0}$ and from Lemma 6.7 it follows that $(\frac{\partial h}{\partial t}, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$. Now, from Theorem 6.2, $(G^{-1}(0) - \Sigma_{G^{-1}(0)}, \{0\} \times B_r) = (h^{-1}(0) \cap \tilde{V} - \{0\} \times B_r, \{0\} \times B_r)$ is Whitney regular. Q.E.D.

Step 2. For some $\varepsilon' > 0$ and all $0 < \varepsilon \leq \varepsilon'$ the product of the boundary of the ε-ball, ∂B_ε, by B_r meets the strata of $\tilde{S}$ transversaly.

Proof. The argument is the same as in Theorem 6.5 in [1]. Let us suppose that the statement is false. Then we can find a sequence of points (x_i, t_i) in some stratum $\tilde{S}$ with $x_i \to 0$ and $T_{(x_i,t_i)}\tilde{S} \subset T_{(x_i,t_i)}(\partial B_{\varepsilon_i} \times B_r)$ where $\varepsilon_i = ||x_i||$. Then $(x_i, 0)$ is perpendicular to $T_{(x_i,t_i)}\tilde{S}$. This contradicts the Whitney condition B. Q.E.D.

We then have the first approximation to our stratified diagram, that is,

$$B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$$

where B_ε is the closed ball in $\mathbb{C}^n$ of radius ε, $\varepsilon \leq \varepsilon'$, $F(x,t) = (h(x,t), t)$ and π is the projection to the second factor. We stratify $\mathbb{C} \times B_r$ by $(\mathbb{C} - \{0\}) \times B_r \cup \{0\} \times B_r$ and we refine the stratification of $B_\varepsilon \times B_r$, taking the intersection of the strata in $\tilde{S}$ with $\partial B_\varepsilon \times B_r$ and $int B_\varepsilon \times B_r$. We would like to show that this stratification satisfies Thom's condition, but h_t might have critical points on ∂B_ε. To get around this difficulty we need the following.

Step 3. For some $\delta > 0$, $B_\delta - \{0\}$ in $\mathbb{C}$ consists only of regular values of h_t for every $t \in B_r$.

Proof. This follows from the fact that h is a good deformation of h_0. Q.E.D.

In the above diagram we change $\mathbb{C}$ by B_δ, where B_δ is the ball with radius δ, with the stratification $\partial B_\delta \cup \{0\} \cup int B_\delta - \{0\}$, and satisfying the conditions in Step 3. We then get a new stratification of $F^{-1}(B_\delta \times B_r)$ pulling back the strata. We consider now

$$F^{-1}(B_\delta \times B_r) \xrightarrow{F} B_\delta \times B_r \xrightarrow{\pi} B_r$$

Step 4. The above diagram is Thom stratified.

Proof. We have to show that the diagram satisfies the condition A_{h_t}. Given two strata $\tilde{S}_1$, $\tilde{S}_2$ with $(x_i, t_i) \in \tilde{S}_1$, and $(x_i, t_i) \to (x, t) \in \tilde{S}_2$, the restriction of the kernel of $dF(x_i, t_i)$ to $T_{(x_i,t_i)}\tilde{S}_1$, say K_i, is $T_{(x_i,t_i)}\tilde{S}_1 \cap (\ker dh_{t_i}(x_i) \times \{0\})$. The limit of this sequence of spaces is contained in $T \cap (\ker dh_t(x) \times \{0\})$ where $T = \lim_{i\to\infty} T_{(x_i,t_i)}\tilde{S}_1$. If $x \neq 0$ then $\ker dh_t(x) \times \{0\}$ is transversal to T, hence $\lim_{i\to\infty} K_i = T \cap (\ker dh_t(x) \times \{0\})$. Since $T \supset T_{(x,t)}\tilde{S}_2$ (Whitney condition A), then $\lim_{i\to\infty} K_i$ contains the restriction of the kernel of $dF(x,t)$ to $T_{(x,t)}\tilde{S}_2$. If $x = 0$ then $\tilde{S}_2 = \{0\} \times B_r$, and Thom condition follows trivially. Q.E.D.

Remark 6.9. *The V-equisingularity of a family h as above implies that h is topologically $\mathcal{R}_V$-trivial.*

In fact, from Thom's second isotopy lemma ([12], p.62), there exist homeomorphisms

$$H : F^{-1}(B_\delta \times \{0\}) \times B_r \to F^{-1}(B_\delta \times B_r)$$

$$H' : B_\delta \times \{0\} \times B_r \to B_\delta \times B_r,$$

preserving the stratifications, such that the following diagram commutes:

$$\begin{array}{ccccc} F^{-1}(B_\delta \times \{0\}) \times B_r & \xrightarrow{F \times id} & B_\delta \times \{0\} \times B_r & \xrightarrow{\pi_3} & B_r \\ \downarrow H & & \downarrow H' & & \downarrow id \\ F^{-1}(B_\delta \times B_r) & \xrightarrow{F} & B_\delta \times B_r & \xrightarrow{\pi_2} & B_r \end{array}$$

Then $H(x,0,t) = (\overline{h}(x,t),t)$, $F(\overline{h}(x,t),t) = (h(x,0),t)$ and it follows that $h(\overline{h}(x,t),t) = h_0(x)$ for all t and x. Therefore h is topologically $\mathcal{R}_V$-trivial.

The example below shows that the condition $g \in \overline{dh(\Theta_V^0)_{\mathcal{O}_{V\times\mathbb{C}}}}$ is stronger than the condition $(g,0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle_{\mathcal{O}_{G^{-1}(0)}}}$.

Example 6.10. *Let $V, 0 \subset k^3, 0$ be defined by $\phi(x,y,z) = 2x^2y^2 + y^3 - z^2 + x^4y = 0$ and $h : \mathbb{C}^4, 0 \to \mathbb{C}, 0$, $h(x,y,z,t) = y + (a+t)x^2$ and $G : \mathbb{C}^4, 0 \to \mathbb{C}^2, 0$ given by $G(x,y,z,t) = (y+(a+t)x^2, 2x^2y^2 + y^3 - z^2 + x^4y)$. The module Θ_V is generated by $\eta_1 = (2x, 4y, 6z)$, $\eta_2 = (0, 2z, x^4 + 4x^2y + 3y^2)$, $\eta_3 = (x^2 + 3y, -4xy, 0)$ and $\eta_4 = (z, 0, 2x^3y + 2xy^2)$. The element $\frac{\partial h}{\partial t} = x^2$ is not in the integral closure of the ideal $dh_t(\Theta_V^0)$ (it also follows that $x^2 \notin \overline{\langle dh(\eta_i)\rangle_{\mathcal{O}_{V\times\mathbb{C}}}}$). In fact, given $\phi : k, 0 \to k^4, 0$, $\phi(s) = (s, -as^2, 0, 0)$, it follows that $\frac{\partial h}{\partial t} \circ \phi$ is not in $(\phi^*(dh_t(\Theta_V^0)))\mathcal{O}_1$, then by Theorem 3.1, $\frac{\partial h}{\partial t} = x^2 \notin \overline{dh_t(\Theta_V^0)}$. We can verify that $(x^2, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$. In fact, we will show that $(x^2, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j}, e_iG_j \right\rangle}_{\mathcal{O}_4}$ and the result will follow from this. We have*

$$
\begin{aligned}
&(a)\ zG_z = (0, -2z^2)\\
&(b)\ e_1G_1 = (y + (a+t)x^2, 0)\\
&(c)\ x^2G_y = (x^2, 4x^4y + 3x^2y^2 + x^6)\\
&(d)\ e_2G_2 + \tfrac{1}{2}zG_z = (0, 2x^2y^2 + y^3 + x^4y)
\end{aligned}
$$

Let $\varphi : \mathbb{C}, 0 \to \mathbb{C}^4, 0$ be given by $\varphi(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u), \varphi_4(u))$. We shall see that $(\varphi_1^2, 0) \in \left\langle (x_i \frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi \right\rangle_{\mathcal{O}_1}$. Let $r = ord(\varphi_1)$ and $s = ord(\varphi_2)$, if $s \leq r$ or $2s = r$ then it follows from (b) that $(\varphi_1^2, 0) \in \left\langle (x_i \frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi \right\rangle_{\mathcal{O}_1}$.

If $s > r$ then it follows from (c) that

$$x^2G_y \circ \varphi = (\varphi_1^2, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6) = (\varphi_1^2, 0) + (0, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6)$$

and from (d) we get that $(0, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6) \in \left\langle (x_i \frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi \right\rangle_{\mathcal{O}_1}$, hence,
$(\varphi_1^2, 0) \in \left\langle (x_i \frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi \right\rangle_{\mathcal{O}_1}$ or $(x^2, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j}, e_iG_j \right\rangle}_{\mathcal{O}_4}$.

Remark 6.11. *When the variety V reduces to 0, Tessier in [17] proved that the set*

$$\{t \in \mathbb{C}, 0 \, : \, \frac{\partial h}{\partial t} \in \overline{\left\langle x_i \frac{\partial h}{\partial x_j} \right\rangle}\}$$

is open and dense. In the relative case, we can obtain a similar result as a consequence of Gaffney in [11], that is :

$$\{t \in \mathbb{C}, 0 \, : \, (\frac{\partial h}{\partial t}, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}\}$$

is open and dense. However, the corresponding statement does not hold for $\overline{dh_t(\Theta_V^0)}$. In fact, with a slight modification of the arguments in the above example, we see that the set:

$$\{t \in \mathbb{C}, 0 : \frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}\}$$

is empty.

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