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Thom polynomial computing strategies. A survey

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Abstract.

Thom polynomials compute the cohomology classes of degeneracy loci. In this paper we use a simple example to review the core ideas in different—mostly recently found—methods of computing Thom polynomials. Our goal is to show the underlying topology/geometry/algebra without involving combinatorics.

§1. Introduction

Global topology can force singularities to occur. That is, in a family of objects (where the 'object' can be a linear map, a map germ, a differential form, a diagram of maps, a variety, a stable bundle over a variety, etc) some has to be singular *because* of the topology of the family. This global aspect of singularities is encoded by their Thom polynomials.

Let G be a group acting on a vector space V, and let η be a G-invariant subvariety. Then the Poincaré dual of η in equivariant cohomology is called the Thom polynomial of η, denoted by $\mathrm{Tp}_\eta \in H^*_G(V) = H^*_G(\text{point}) = H^*(BG)$. Sometimes η is an open subset of a G-invariant subvariety. Then we define $\mathrm{Tp}_\eta := \mathrm{Tp}_{\bar\eta}$. Tracing back this definition one finds the following topological statement: *whenever a fiber bundle $E \to X$ with fiber V and structure group G is given, the cohomology class represented by the preimage S of the η-points under a generic section is equal to the Thom polynomial of the bundle.* That is, if V is the collection of 'objects', G is a natural equivalence on them, η is the collection of 'singular objects' then the mentioned sections are the 'families of objects' over the parameter space X, and S is the locus of points

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where the object is singular. Hence the knowledge of Thom polynomial tells us the (cohomology class of the) locus of the singular points.

The determination of concrete Thom polynomials is often difficult. What makes the case even worse is that Thom polynomial problems come in natural infinite series, and the combinatorial organization of calculating the infinitely many Thom polynomials at the same time often conceals the actual topological method used. The goal of this paper is to survey some Thom polynomial calculational methods without involving combinatorics. Hence we will deal with just one concrete (quite trivial) example, and show the calculation in five different ways.

Let $G = GL_3(\mathbf{C}) \times GL_3(\mathbf{C})$ act on the vector space of 3×3 matrices $V = \mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3)$, by $(A, B) \cdot M := BMA^{-1}$. Let Σ^2 denote the invariant set of matrices whose corank is 2, i.e. whose rank is 1. We will calculate the Thom polynomial of (the closure of) Σ^2. Hence $\mathrm{Tp} = \mathrm{Tp}_{\Sigma^2}$ is a degree 4 polynomial in $H^*(BGL_2 \times BGL_2) = \mathbf{Z}[A_1, A_2, A_3, B_1, B_2, B_3]$ (degree of the Chern class X_i is i), or what is the same, a degree 4 polynomial in $\mathbf{Z}[a_1, a_2, a_3, b_1, b_2, b_3]$ (degree of the Chern root x_i is 1), symmetric in a_1, a_2, a_3 and in b_1, b_2, b_3. Here $a_1 + a_2 + a_3 = A_1, a_1a_2 + a_1a_3 + a_2a_3 = A_2$, $a_1a_2a_3 = A_3$ and the same for the B's.

Theorem 1.1. *Tp_{Σ^2} is*

$$c_2^2 - c_1c_3, \tag{1}$$

where c_i is the i'th Taylor coefficient of

$$\frac{1 + B_1t + B_2t^2 + B_3t^3}{1 + A_1t + A_2t^2 + A_3t^3},$$

that is Tp_{Σ^2} =

$$B_2^2 - B_2A_1B_1 + B_2A_1^2 - 2B_2A_2 - A_1B_1A_2 + A_2^2 - B_1B_3+$$

$$+A_2B_1^2 + B_1A_3 + A_1B_3 - A_1A_3, \tag{2}$$

or in Chern roots, it is

$$(b_1b_2+b_1b_3+b_2b_3)^2-(b_1b_2+b_1b_3+b_2b_3)(a_1+a_2+a_3)(b_1+b_2+b_3)+\dots \tag{3}$$

In Sections 2-6 we will give 5 proofs. Before that we make two preliminary remarks. One is that the geometric counterpart of giving the Thom polynomial in Chern roots is that we restrict the group action to the maximal torus. Because of the splitting lemma, this does not mean any loss of information about Tp_η. The other remark is that when η happens to be smooth in V (e.g. it is a coordinate subspace), then Tp_η is the Euler class of the representation normal to η in 0. This follows from the definition. Some of the proofs below reduce the computation to this special case.

§2. The restriction equations

In this method, when computing the Thom polynomial of η, one needs to work with the *simpler* orbits (ones not contained in the closure of η). For such a ζ we pick a representative and find its stabilizer subgroup $G_\zeta \subset G$. This inclusion induces a map $BG_\zeta \to BG$ between the classifying spaces, and in turn a homomorphism $f_\zeta : H^*(BG) \to H^*(BG_\zeta)$.

Theorem 2.1. [14, Th. 2.4], [5, Th. 3.2] *Let ζ not be contained in the closure of η. Then the Thom polynomial of η vanishes at f_ζ. Moreover, if the representation satisfies a technical condition (see* [5, 3.4-3.5]*), then in the expected degree, only integer multiples of the Thom polynomial of η satisfy all these vanishing conditions.*

In our situation Σ^0 and Σ^1 play the role of ζ, with representatives the identity matrix and diag$(1,1,0)$, respectively. Now G_{Σ^0} and G_{Σ^1} could be determined explicitly, but we will only compute their maximal tori—this is enough, since $H^*(BG)$ injects into $H^*(BT)$ in general. Thus we will take

$$G_{\Sigma^0} = \{(\mathrm{diag}(x,y,z), \mathrm{diag}(x,y,z)) : x,y,z \in \mathbf{C}^*\},$$

$$G_{\Sigma^1} = \{(\mathrm{diag}(x,y,u), \mathrm{diag}(x,y,v)) : x,y,u,v \in \mathbf{C}^*\}.$$

From these the induced map can be read, as follows:

$$f_{\Sigma^0} : \mathbf{Z}[A_1,A_2,A_3,B_1,B_2,B_3] \to \mathbf{Z}[x,y,z]$$

maps both A_i and B_i to the i'th elementary symmetric polynomial of x,y,z. The map

$$f_{\Sigma^1} : \mathbf{Z}[A_1,A_2,A_3,B_1,B_2,B_3] \to \mathbf{Z}[x,y,u,v]$$

maps A_i to the i'th elementary symmetric polynomial of x,y,u, while maps B_i to the i'th elementary symmetric polynomial of x,y,v.

We need the intersection of the kernels of these two homomorphisms. In fact, one factors through the other, so we only need ker f_{Σ^1}, which turns out (Macaulay2) to be an ideal generated by polynomials in degrees

4, 5 and 6. The degree 4 generator, $A_2^2-A_1A_3-A_1A_2B_1+A_3B_1+\ldots$ thus has to be $\pm$ the Thom polynomial of Σ^2. The sign can be determined by the so-called *principal equation* of [5, Th. 3.5], which states that the f_η image of the Thom polynomial of η is the equivariant Euler class of η. In our case

$$G_{\Sigma^2} = \{(\mathrm{diag}(x,u,v), \mathrm{diag}(x,w,z)) : x,u,v,w,z \in \mathbf{C}^*\},$$

and f_{Σ^2} is analogous to the above. The normal slice to Σ^2 at diag$(1,0,0)$ is the space of matrices whose 1'st row and column is 0. Therefore the equivariant Euler class is $(w-u)(w-v)(z-u)(z-v)$. Computation shows that this is the image of the above polynomial at f_{Σ^2}, so the above polynomial is the sought Thom polynomial.

Remark 2.2. For a reference of this method as well as many applications see [5], [14], [10]. The restriction method is very effective if the representation has finitely many orbits. When dealing with natural infinite series, a connection with various resultant formulas can be established, see [3].

§3. Resolution and integral

In the following method it is assumed that η is a cone in V, and, instead of $\eta \subset V$, we consider the projectivization $\mathbf{P}\eta \subset \mathbf{P}V$. The starting point is looking for an equivariant resolution of $\mathbf{P}\eta$ considered as a map $\varphi : R \to \mathbf{P}V$.

Theorem 3.1. [6, Th. 3.1] *Let $\alpha_i \in H^*(BT)$ be the weights of the representation of G on V. Denote by q the polynomial*

$$\frac{\prod(x+\alpha_i)-\prod\alpha_i}{x}$$

in the equivariant cohomology ring

$$H_G^*(\mathbf{P}V) = \frac{H^*(BG)[x]}{\prod(x+\alpha_i)}.$$

Then the Thom polynomial of η is

$$\int_R \varphi^*(q).$$

In our case $\mathbf{P}\Sigma^2 = \mathbf{P}^2 \times \mathbf{P}^2$ is already smooth, hence $\varphi : R = \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^8$ is the Segre embedding. The ring $H_G^*(\mathbf{P}^2\times\mathbf{P}^2)$ is $H^*(BG)[y,z]$

modulo the two relations $r_1 := \prod_{i=1}^3 (y - a_i)$ and $r_2 := \prod_{i=1}^3 (z + b_i)$. Since $\varphi^*(x) = y + z$ (x, y, z are the classes of hyperplane sections of $\mathbf{P}^8$ and the two copies of $\mathbf{P}^2$'s, respectively) we have that the Thom polynomial of Σ^2 is

$$\int_{\mathbf{P}^2 \times \mathbf{P}^2} \frac{\prod_{i=1}^3 \prod_{j=1}^3 (y + z - a_i + b_j) - \prod_{i=1}^3 \prod_{j=1}^3 (-a_i + b_j)}{y + z}.$$

Integration means taking the top coefficient, i.e. the coefficient of yz. Hence the procedure is to consider the integrand above, use the relations r_1, r_2 to reduce its (y, z)-degree to $(1, 1)$, and take the coefficient of yz. Note that taking the minimal degree representative in a factor ring is *automatically* done in computer algebra packages, which makes this method very easy to code.

Remark 3.2. For a reference of this method, see [6]. It is most effective if we can find a resolution with simple cohomology ring. In these cases the integration part is often encoded as an interpolation problem, so the combinatorics of divided differences enters the calculations.

§4. Resolution and integral via localization

The method presented in this section is not really a new method, it's rather an improvement of that of Section 3. The novelty is that we compute the integral $\int_R \varphi^*(q)$, which is the Thom polynomial, by localization techniques. This is a vital help when R is more complicated than a projective space or Grassmannian.

We will use the Atiyah-Bott localization formula [1], as follows. Let a torus T act on a manifold with fixed point set the disjoint union of some F_i's. Then the integral of an equivariant cohomology class $\alpha \in H_T^*(M)$ can be 'localized':

$$\int_M \alpha = \sum_i \int_{F_i} \frac{j_i^* \alpha}{e(\nu_i)},$$

where $j_i : F_i \subset M$ is the embedding and ν_i is its normal bundle. When the fixed point set is discrete we can integrate by just "counting":

$$\int_M \alpha = \sum_i \frac{j_i^* \alpha}{e(T_{F_i} M)}. \tag{4}$$

In our case R is $\mathbf{P}^2 \times \mathbf{P}^2$, with 9 fixed points $P_{1,1} := ((1:0:0),(1:0:0))$, $P_{1,2} := ((1:0:0),(0:1:0))$, etc. It will be convenient to use a different form of $q \in H_G^*(\mathbf{P}^8)$, namely $q = \frac{-\prod\prod(-a_i+b_j)}{x}$ (recall that $H^*_G(\mathbf{P}V) = H^*(BG)[x]/\prod\prod(x - a_i + b_j))$. Then the term in (4) corresponding to e.g. $P_{1,1}$ is

$$L_{1,1} := \prod_{i=1}^{3}\prod_{j=1}^{3}(-a_i+b_j)\frac{1}{(-a_1+b_1)\cdot(a_2-a_1)(a_3-a_1)(b_2-b_1)(b_3-b_1)}.$$

The Thom polynomial is then the sum of 9 similar terms, or

$$\mathrm{Tp}_{\Sigma^2} = \frac{1}{4}\sum_{\sigma\in S_3\times S_3} L_{\sigma(1,1)}.$$

Remark 4.1. See [7] for a general reference. This method is most effective if we can simplify the resulting sum using algebra (e.g. Lagrange interpolation). In cases like ours, when the resolution is trivial, the localized integral formula coincides with the *Dusitermaat-Heckman* formula.

§5. Gröbner degeneration

The goal of this method is to "perturb" Σ^2 in $\mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3)$ without changing its Thom polynomial, and eventually degenerate it to another set, whose Thom polynomial is trivial to compute. The first obstacle is that Σ^2 can not be perturbed at all to another G-invariant subset. However, we can restrict the group action to the maximal torus T without losing any Thom polynomial information, and there are lots of T-invariant perturbations.

Let us consider the following example. The torus $GL_1(\mathbf{C})\times GL_1(\mathbf{C})$ acts on $\mathbf{C}^3 = \mathbf{C}\{x,y,z\}$ by $(\alpha,\beta).(x,y,z) = (\alpha^2x, \beta^2y, \alpha\beta z)$. Then the cone $xy - z^2$ is invariant. But so is $xy - t\cdot z^2$ for every $t\in\mathbf{R}$. In the $t=0$ limit case we get $xy=0$, which is the union of two planes: $x=0$ with Thom polynomial $(2b)(a+b)$, and $y=0$ with Thom polynomial $(2a)(a+b)$ (see the last paragraph of the Introduction). It is easy to believe that the perturbation did not change the Thom polynomial, hence the Thom polynomial of the cone is $(2a)(a+b)+(2b)(a+b)$.

What are the "legal" perturbation (where the Thom polynomial does not change), and how to imitate this process when the variety has higher codimension? The theory of Gröbner basis gives an answer (for a general reference for Gröbner basis theory see e.g. [4, Ch. 15]).

Let I be the ideal of the torus-invariant variety X. Fix a term-order, and consider $in(I)$, the ideal generated by the initial terms of polynomials in I. Then the variety (scheme) corresponding to $in(I)$ is a *flat* deformation of X, hence their Thom polynomials are the same. (Well, one has to be a little careful about the multiplicities of the irreducible components of $in(I)$.)

Note that if we have a Gröbner basis f_i of I, then the leading terms of the f_i's generate $in(I)$. In our case

$$I = I(\Sigma^2) = (a_{11}a_{22} - a_{12}a_{13}, a_{11}a_{23} - a_{13}a_{21}, \ldots)$$

(the 9 2×2 minors) is given by a Gröbner basis with respect to e.g. the "graded reverse lexicographic" term order generated by $a_{11} > a_{12} > a_{13} > a_{21} > \ldots$. Thus

$$in(I) = (a_{12}a_{13}, a_{13}a_{21}, \ldots)$$

(the 'antidiagonals' of the 9 2×2 minors). A computer algebra package (e.g. Macaulay2 "primaryDecomposition in(I)") can be used to find the primary decomposition of $in(I)$ which is:

$$(a_{12}, a_{13}, a_{22}, a_{23}), \qquad (a_{12}, a_{13}, a_{23}, a_{31}),$$

$$(a_{13}, a_{21}, a_{23}, a_{31}), \qquad (a_{12}, a_{13}, a_{31}, a_{32}),$$

$$(a_{13}, a_{21}, a_{31}, a_{32}), \qquad (a_{21}, a_{22}, a_{31}, a_{32}).$$

They all describe linear spaces, whose Thom polynomials are obtained by the last remark of the Introduction, hence the Thom polynomial of Σ^2 is the sum of the following polynomials

$$(b_1-a_2)(b_1-a_3)(b_2-a_2)(b_2-a_3), \qquad (b_1-a_2)(b_1-a_3)(b_2-a_3)(b_3-a_1),$$

$$(b_1-a_3)(b_2-a_1)(b_2-a_3)(b_3-a_1), \qquad (b_1-a_2)(b_1-a_3)(b_3-a_1)(b_3-a_2),$$

$$(b_1-a_3)(b_2-a_1)(b_3-a_1)(b_3-a_2), \qquad (b_2-a_1)(b_2-a_2)(b_3-a_1)(b_3-a_2),$$

which turns out to be (3).

Remark 5.1. The theory behind this method is worked out in [11], see also [12]. An advantage is that the Thom polynomial is obtained as a sum with *positive* coefficients, which is sometimes important in enumerative geometry. When working with natural infinite series one meets subtle combinatorics (e.g. the "pipe dreams" of [12]).

§6. Porteous' method of embedded resolution

As a preparation we study Gysin maps associated with Grassmann bundles. Let $E^3 \to X$ be a bundle of rank 3 and $\pi : Gr_2(E^3) \to X$ its associated Grassmann-2 bundle (ie. we replace the fiber over $e \in E$ from E_e to $Gr_2(E_e)$.) The goal is to understand the Gysin map $\pi_!$ on some naturally defined cohomology classes of $Gr_2(E^3)$—namely the Chern monomials of the tautological 2-bundle S on $Gr_2(E^3)$. We claim that $\pi_!(c_{\lambda_1,\lambda_2})(-S) = c_{\lambda_1-1,\lambda_2-1}(-E)$. Here $c_{u,v}$ is the determinant of the matrix $\begin{pmatrix} c_u & c_{u+1} \\ c_{v-1} & c_v \end{pmatrix}$. Moreover, if F is any other bundle on X, and we denote its pullback to $Gr_2(E^3)$ also by F, then $\pi_!(c_{\lambda_1,\lambda_2})(F-S) = c_{\lambda_1,\lambda_2}(F-E)$, for a recent reference see [8, p.43].

With this knowledge we can calculate Tp_{Σ^2} as follows. Consider two 3-bundles E and F over X, and a generic homomorphism h between them. We want to resolve the closure of $\Sigma^2(h) \subset X$. Let $\pi : Gr_2(E) \to X$ be as above and consider the bundles S, E, F over $Gr_2(E)$. Let $\bar{h} : S \to F$ be the composition of the natural map $S \to E$ with the pullback of h. The 0-points of $\bar{h}$ can also be considered as $\Sigma^2(\bar{h})$. Now one fact [Port] is that the genericity of h implies that $\bar{h}$ is transversal to the 0-section of $\mathrm{Hom}(S,F)$, so we know the cohomology class $[\Sigma^2(\bar{h})] = e(\mathrm{Hom}(S,F))$. The other fact [Port] is that π restricted to $\Sigma^2(\bar{h})$ is a resolution of $\Sigma^2(h)$, thus $\pi_![\Sigma^2(\bar{h})] = [\Sigma^2(h)]$, what we want to compute. In the light of the above description of $\pi_!$ we only need to write $e(\mathrm{Hom}(S,F))$ as a linear combinations of $c_{\lambda_1,\lambda_2}(F-S)$'s. The Euler class $e(\mathrm{Hom}(S,F))$ is the product of differences of Chern roots of F and S, which is the same as $c_{3,3}(F-S)$. Hence $\pi_!(e(\mathrm{Hom}(S,F)) = c_{2,2}(F-E)$, which is (1), what we wanted to prove.

Remark 6.1. This method was historically the first, applied in many different situations, see [13], [15], [9] (singularities), works of Pragacz, Fulton, Harris-Tu, Buch and others (algebraic geometry, see [8] for references and e.g. [2] for a recent application). The effective usage of this method requires the handling of the combinatorics of Gysin homomorphisms, Schur and Schubert polynomials, Young tableaux, etc.

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