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On time averaged optimization of dynamic inequalities on a circle

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Abstract.

We analyze the maximum averaged profit for one parameter families of dynamic inequalities and profit densities on a circle. Generic singularities of the profit for stationary strategies are classified. They are shown to be stable.

§1. Introduction

A smooth function F on the tangent bundle TM of a smooth manifold M defines a *dynamic inequality*: a tangent vector $v \in TM$ is an *admissible* velocity of the inequality if $F(v) \leq 0$. We consider only inequalities (called inequalities with *locally bounded derivatives*) such that the set of admissible velocities over any base point of M is compact. We identify the space of inequalities with the space of functions F. In particular, a *family* of inequalities is a family of functions.

An *admissible motion* is an absolutely continuous mapping $t \mapsto x(t)$ of the time axis segment to the manifold M with the derivative $\dot{x}(t)$ belonging to the convex hull of the admissible velocities in the fiber over $x(t)$ (whenever the derivative exists).

Given a continuous *profit density* function $f : M \mapsto R$, an admissible motion $x, x = x(t)$, on the interval $[0, T]$, $T > 0$, provides the *profit*

$$P(T) = \int_0^T f(x(t))dt \quad \text{and the } \textit{averaged profit} \quad A(T) = P(T)/T.$$

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An important well known control problem is to find an admissible motion providing the maximum averaged profit on the *infinite horizon*, that is when $T \to \infty$, [1], [7], [14]. Such a motion is called *optimal*.

V.I.Arnold suggested a new approach to the problem based on the singularity theory methods. He proved that a constant map (=*stationary strategy*): $x(t) = x_0 \in M$ for any t, or periodic motions can be optimal [2] (see also [1], [3]). The case studied by Arnold is a reasonable model for cyclic process with a prescribed trajectory in multidimensional phase space. An example of this process is a motion along the closed route with the velocity depending on a chosen control.

In the present paper we follow this approach and analyze an analog of Arnold's model [2] defining admissible velocities by a dynamic inequality. We classify generic singularities of the maximum averaged profit provided by stationary strategies in one parameter families of dynamic inequalities and profit densities on the circle. We use Γ-*equivalence*: two germs of functions have the same singularity if their graphs are diffeomorphic via a *parameter diffeomorphism*, that is via a difeomorphism, which respects the natural projection to the parameter sending any fiber to a fiber. We prove also the stability of these singularities with respect to small perturbations: an object has stable singularity, if any sufficiently close object has equivalent singularity and the corresponding equivalence diffeomorphism can be taken close to the identity.

For multidimensional parameter or phase space the classification problem remains open.

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§2. Classification of singularities

In this section the main results are stated. We consider only one parameter families of inequalities and densities on the circle. The phase variable and the parameter are denoted by x and p, respectively. A *generic* or *typical* object is an object from an open dense subset of the space of objects endowed with smooth or sufficiently smooth fine topology.

2.1. Stationary domain and shadows

Clearly, a point of the phase space is a stationary strategy if and only if the zero level set of the dynamic inequality over this point contains both non-positive and non-negative velocities. Denote by P_- and P_+ the subsets of this level set which consists of non-positive and non-negative velocities, respectively. Hence, the set of all stationary strategies (=*stationary domain*) of an inequality is the intersection of the images (=*shadows*)$\pi(P_-), \pi(P_+)$ of these subsets parts the natural projection $\pi : (x, \dot{x}, p) \mapsto (x, p)$ along the velocity axis.

We denote stationary domain by S and its intersection with the fiber $p = p_0$ by S_{p_0}.

Using the results of [4], [10], [11], [12], [13] on generic singularities of restrictions of projections to submanifolds and submanifolds with boundary and taking into account that the sets P_- and P_+ have the same boundary, we prove the following

Theorem 2.1. *The germ of the stationary domain of a generic dynamic inequality at any boundary point is fiber diffeomorphic to the germ at the origin of one of the following (eight) sets*

$$1)\, x \geq 0;\ 2_\pm)\, p \geq \pm x^2;\ 3_\pm)\, p \geq \pm|x|;\ 4_\pm)\, x \geq \pm|p|;\ 5)\, x \geq p|p| \tag{1}$$

Moreover the stationary domain of generic family is stable.

Remark 1. The theorem holds for a subset in the space of inequality families which is open in fine C^3-topology and dense in fine C^∞-topology.

Theorem 2.1 is proved in Subsection 3.1

2.2. Maximum profit for stationary strategies

The maximum averaged profit A_s for stationary strategies is a solution of the extremal problem

$$A_s(p) = \max_{x \in S_p} f(x, p) \tag{2}$$

over the set of all stationary strategies for parameter value p.

Theorem 2.2. *Any germ of the profit A_s for a generic pair of families of inequalities and profit densities is Γ - equivalent to the germ at the origin of one of the eight functions listed in the second column of Table 1.*

Remark 2. The third column of Table 1 contains more precise information on the equivalence used. Singularities 1–5 can be reduced

Table 1

No	Sing.	Eq.	Conditions
1	0	R^+	unique optimal strategy either of type 1) with $f_x \neq 0$ or an interior one of S
2	$\lvert p\rvert$	R^+	unique optimal strategy either of type 4_-) or the competition of two strategies with singularity 1
2_-	$-\lvert p\rvert$	R^+	unique optimal strategy of type 4_+)
3	$p\lvert p\rvert$	R^+	unique optimal strategy of type 1) with $f_x = 0$
4	$\sqrt{p}$	R^+	unique optimal strategy of type 2_+)
5	$0,\ p \geq 0$	R^+	unique optimal strategy of type 3_+)
6	$\begin{cases} 0, & p<0 \\ 1+\sqrt{p}, & p \geq 0 \end{cases}$	Γ	competition of two strategies with singularity 1 and 4
7	$\begin{cases} 0, & p<0 \\ 1, & p \geq 0 \end{cases}$	Γ	competition of two strategies with singularity 1 and 5

to normal form by a R^+-equivalence, which is a particular case of Γ-equivalence: the diffeomorphisms acts on each fiber just by a shift depending on a parameter [5].

The fourth column contains description of the type of strategy, and the type of singularity of the stationary domain from Theorem 2.1.

Theorem 2.2 is proved in Subsection 3.2.

§3. Proofs

Here Theorem 2.1 and Theorem 2.2 are proved sequentially.

3.1. Singularities of stationary domain

If a family of inequalities has no stationary strategies then this is also true for any family of inequalities sufficiently close to the given one in the fine C^0-topology.

Consider the case when the stationary domain is not empty. The zero level of a generic family of inequalities is non-critical. It is a smooth (hyper)surface. The restriction τ of the natural projection π along the velocity axis to this level is a proper map due to the imposed "locally bounded derivatives" condition.

J. Mather's theory for restrictions of projections [11] imply that for a generic family of inequalities the map τ is LR-stable . Moreover, it can have only Whitney fold and pleat singularities (as a map between two dimensional manifolds [10]). Hence, in a generic case the set C of critical values of the map τ is either empty or a smooth curve with cusps and transversal self-intersections.

LR-stability and transversality theorems imply that in a generic case the set C and its singularities take typical position with respect to the natural fibering over the parameter space. Consequently, sets C for a generic inequality and an inequality sufficiently close to it are parameter diffeomorphic (via a diffeomorphism close to the identity). For all the detailes which we omit here see [8], where a similar result is proven.

The stationary domain is the intersection of shadows of the sets P_- and P_+. Clearly, for a generic family of inequalities, the boundary of shadows belongs to the union of the set C and the intersection I of the zero section $\dot{x} = 0$ with zero level of the family of inequalities. Similar arguments show that in a generic case the union $C \cup I$ is also stable (with respect to parameter difeomorphisms). So, generic stationary domain and its local singularities are stable. Now we classify these singularities.

According to [10], a generic shadow of two dimensional manifold with boundary near any its boundary point in appropriate smooth local coordinates u, v takes the form of one of the three sets

$$a)\, u \geq 0 \quad \text{or} \quad b)\, v \leq |u|, \quad \text{or else} \quad c)\, v \geq u|u|$$

near the origin.

The first singularity occurs either at a Whitney fold critical point of the map τ outside the zero section $\dot{x} = 0$, or at a regular point of the map τ which belongs to the zero section.

For a generic family of inequalities, the second singularity is a transversal superposition of two singularities of the first type. Finally, the third singularity occurs at a Whitney fold critical point of the map τ which belongs to the zero section $\dot{x} = 0$.

Hence, the first and the third singularities are local (completely defined by the germ at a single point of the zero level of the family of dynamic inequalities), and the second singularity is defined by two germs.

To classify generic singularities of stationary domain one needs to study the singularities of the intersection of shadows of the subsets P_+ and P_-.

These intersections yield singularities a) - c)at the point which belongs to the interior of one of these shadows and to the boundary of the other. For a generic family of inequalities, we get a transversal superposition of these singularities when the point belongs to the boundaries of shadows but its critical inverse images in the zero level of the family are distinct. Due to the dimensions, only a superposition of two singularities of first type with transversal intersection of the boundaries is generic. In this case we obtain the normal form $v \geq |u|$ at the origin (up to a diffeomorphism).

Singularity of the type c) appears simultaneously on the boundaries of shadows of P_- and P_+. In a generic case, this is possible only when the intersection of these shadow are defined by germ of the zero level of the family of inequalities at the same point. Here the boundary of the intersection of shadows is determined by the set I, as it is easy to see. Hence, this gives normal form c) of the stationary domain (up to a diffeomorphism).

Taking into account all possible different generic position with respect to the natural fibering over the parameter of the singularities a) - c) we get exactly the list (1) of the theorem (up to parameter diffeomorphisms).

Theorem 2.1 is proved.

3.2. Singularities of maximum profit

Without loss of generality one can think that some stationary domain with typical singularities from the list (1) is fixed. A boundary point of the domain is called *singular* if at this point the singularity of the boundary is not 1) from this list. Due to Theorem 2.1 singular points of the boundary form a discrete set.

Lemma 3.1. *For a generic pair of one parameter families of inequalities and densities the derivative of the family of densities along the phase variable does not vanish at singular points of the boundary of the stationary domain.*

This lemma follows immediately from the stability of the stationary domain and Thom transversality theorem. It implies

Corollary 1. *For a generic pair of one-parametric families of inequalities and densities a singular point of the boundary of stationary domain does not provide maximum averaged profit for stationary strategies if at this point this domain has singularity* 2_-) *or* 3_-) *from the list (1).*

Corollary 2. *For a generic pair of one parameter families of inequalities and densities and a singular point* (x, p) *of the boundary of stationary domain the germ at the point* p *of the maximum averaged* *profit provided by stationary strategies which are sufficiently close to the strategy (x, p) is the germ at the origin of one of the four functions*

$$(3) \qquad 2_+)\, \sqrt{p}; \quad 3_+)\, 0,\, p \geq 0; \quad 4_\pm)\, \mp |p|; \quad 5)\, p|p|$$

up to R^+-equivalence if at this point this domain has singularity $2_+)$, $3_+)$, $4_\pm)$ *and* 5) *from the list (1), respectively.*

Thus to finish the proof of Theorem 2.2 one needs to study the singularities of the maximum averaged profit for stationary strategies which are provided either by

(a) an interior point of the stationary domain, or

(b) by a boundary point where the domain has singularity of type 1) from the list (1), or else

(c) by the competition of different stationary strategies.

Consider these three cases sequentially. For the first two cases let (x_0, p_0) be the unique stationary strategy providing the profit $A_s(p_0)$.

Case (a). For a generic family f of profit densities with one parameter at least one of the derivatives f_x, f_{xx} and f_{xxx} at any point is not zero due to Thom transversality theorem. So if the strategy (x_0, p_0) is an interior point of the stationary domain then at this point one has to have $f_x = 0$ and $f_{xx} < 0$. Due to continuity of f and closeness of S that implies that near the point p_0 that maximum is provided by the values of the family of densities on the set $f_x = 0$. Due to implicit function theorem near the point (x_0, p_0) this set is smoothly embedded curve $x = X(p)$ with some smooth map $X, X(p_0) = x_0$, due to $f_{xx}(x_0, p_0) < 0$. Hence the germ (A_s, p_0) is the germ of smooth function $f(X(p), p)$ at p_0. So it is R^+-equivalent to the germ of the zero function at the origin.

Case (b). Let the point (x_0, p_0) be the boundary point of the stationary domain with the singularity 1) from the list (1). Again due to Thom transversality theorem at least one of the derivatives f_x and f_{xx} does not vanish in a generic case.

When the derivative $f_x(x_0, p_0)$ is not zero then the germ (A_s, p_0) is the germ of the restriction of the family f to the boundary of the stationary domain near the point (x_0, p_0). Thus as above the germ (A_s, p_0) is the germ of a smooth function and it is R^+-equivalent to the germ of the zero function at the origin.

If the derivative $f_x(x_0, p_0)$ is zero then as above in the case of an interior point the derivative $f_{xx}(x_0, p_0)$ has to be negative. Due to Thom transversality theorem the differential of the restriction of the derivative f_x to the boundary do not vanish at the point (x_0, p_0) in a generic case. Consequently the profit A_s near the point p_0 is the maximum of the restrictions of the density family to the boundary of the stationary domain and to the part of the curve $f_x = 0$ of local maximums of the densities that belongs to the stationary domain.

At the point p_0 this boundary (the curve, respectively) is transversal to the natural fibering over the parameter due to the type 1) of singularity of the boundary (the inequality $f_{xx}(x_0, p_0) < 0$, respectively). Besides these boundary and curve do not tangent at this point because the differential of the restriction of the derivative f_x to the boundary do not vanish at this point. That implies that the profit A_s has singularity at the point p_0 provided by discontinuity of the second derivative, and the germ (A_s, p_0) is R^+-equivalent to the germ of function $p|p|$ at the origin.

Case (c). Due to multi jet transversality theorem in a generic case for a value p_0 of the parameter there can appear competition only of two stationary strategies. Moreover the germ of the problem at one of them s_1 has to provide the singularity 1 from Table 1. For the other strategy s_2 the value of the profit density is either equal or greater then one at the first. Otherwise there is no any competition. Consider these two subcases consequently.

In the first subcase the germ of the problem at the other strategy has to be also of type 1 from Table 1 due to multi jet transversality theorem. Moreover at the value p_0 the derivatives of best profits defined by the germs of the problem at the strategies s_1 and s_2 are different. Hence the competition gives singularity 2 from Table 1 up to R^+-equivalence.

In the second subcase the best averaged profit A_{s_2} for stationary strategies defined by the germ of the problem at the point s_2 can not provide the singularity 1 from Table 1, or the $4_\pm$) and 5) from the list (3). Otherwise there is no any competition of strategies s_1 and s_2 at the point p_0. Thus at the point p_0 the profit A_{s_2} can have up to R^+-equivalence only the singularity either 2_+) or 3_+) from the list (3). Consequently the maximum averaged profit for stationary strategies has at the point p_0 the singularities 6 and 7 from Table 1, respectively.

The stability of singularities of maximum averaged profit for stationary strategies with respect to small perturbations of generic problem follows from transversality theorems.

Finally, the stability of stationary domain up to small perturbation of generic inequality follows from the LR-stability of the map τ [11] and the stability of intersection of zero level of the inequality with the zero section of tangent bundle.

Remark 3. Besides the well-known singularities $|p|$, $\max\{0, 1 + \sqrt{p}\}$ of competition of strategies [2], [5], [8], [9], in the problem studied only one new generic singularity 7 from Table 1 appears. As we see above it is the result of the typical competition of the singularity 1 with the singularity 5 from this table.

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