Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 339–345

Nikulin's $K3$ surfaces, adiabatic limit of equivariant analytic torsion, and the Borcherds Φ-function

Ken-Ichi Yoshikawa

Abstract.

In this note, we prove that the "adiabatic limit" of the equivariant analytic torsion of a Nikulin's $K3$ surface converges to the value of norm of the Borcherds Φ-function at its period point after a certain renormalization.

§0. Introduction

Let $\pi\colon M \to B$ be a submersion of compact Riemannian manifolds. Let g_M and g_B be Riemannian metrics on M and B, respectively. For $0 < \epsilon < \infty$, set $g_{M,\epsilon} := g_M + \epsilon^{-1}\pi^* g_B$. Let $T(g_M)$ be a geometric object depending on the metric g_M. The limit of $T(g_{M,\epsilon})$ as $\epsilon \to 0$ is called the adiabatic limit of T. The adiabatic limits of various geometric objects have been studied by many authors. In this note, we study a variant of this problem. (Although we will not discuss here, the work of Berthomieu-Bismut ([B-B]) seems to be very related to our subject.)

Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic $K3$ surface. Let $\iota\colon X \to X$ be a holomorphic involution acting non-trivially on canonical forms on X. Let κ_X and $\kappa_{\mathbb{P}^1}$ be Kähler classes on X and $\mathbb{P}^1$, respectively. By Yau ([Ya]), the Kähler class $\kappa_{X,\epsilon} := \kappa_X + \epsilon^{-1}\pi^*\kappa_{\mathbb{P}^1}$ carries uniquely a Ricci-flat Kähler form ω_ϵ. We study the equivariant analytic torsion ([Bi]) of $(X, \iota, \omega_\epsilon)$ as $\epsilon \to 0$ in the case where (X, ι) is a class of $K3$ surfaces studied by Nikulin ([N]). As a result, we recover the Borcherds Φ-function of dimension 26 restricted to a certain locus of dimension 10.

Although we talked a little about the adiabatic limit of the invariant introduced in [Yo] at the conference, we will focus on that subject in this short note.

Received March 27, 2002
We would like to thank Professor Jean-Michel Bismut for very stimulating discussion concerning the subject of this note during our stay at Université Paris-Sud in September, 2001.

§1. Nikulin's $K3$ surfaces

Let X be a $K3$ surface with canonical bundle K_X. Let $\eta_X \in H^0(X, K_X)$ be a nowhere vanishing holomorphic 2-form on X. Then $H^2(X, \mathbb{Z})$ equipped with the intersection pairing is isometric to the $K3$-lattice

$$\mathbb{L}_{K3} := U \oplus U \oplus U \oplus E_8 \oplus E_8, \tag{1.1}$$

where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 is the negative definite lattice associated with the Cartin matrix of type E_8. An isometry $\phi\colon H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ is called a marking of X, and the pair (X, ϕ) is called a marked $K3$ surface.

Set

$$\Omega := \{[x] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C});\ \langle x, x\rangle = 0,\ \langle x, \bar{x}\rangle > 0\}. \tag{1.2}$$

For a marked $K3$ surface (X, ϕ), the point $[\phi(\eta_X)] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C})$ is called the period of (X, ϕ). Then one can verify that $[\phi(\eta_X)] \in \Omega$.

Definition 1.1. Let $\iota\colon X \to X$ be a holomorphic involution acting non-trivially on $H^0(X, K_X)$, i.e., $\iota^*\eta_X = -\eta_X$. The pair (X, ι) is called a *Nikulin's $K3$ surface* if the ι^*-invariant part of $H^2(X, \mathbb{Z})$ is isometric to the lattice $\Lambda := U \oplus E_8(2)$. Here $E_8(2)$ denotes the lattice of rank 8 whose intersection form is twice of that on E_8.

Nikulin's $K3$ surfaces are constructed as follows:

Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth cubic curves in general position. Then C_1 meets C_2 transversally at 9 points; $C_1 \cap C_2 = \{p_1, p_2, \cdots, p_9\}$. Let $\mathbb{P}^2[9] \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at these 9 points. Then $\mathbb{P}^2[9]$ is the blowing-up of the base points of the pencil spanned by C_1, C_2.

Fix homogeneous polynomials $f_1(z)$, $f_2(z)$ defining C_1, C_2, respectively. Then $\mathbb{P}^2[9]$ admits the elliptic fibration $\pi\colon \mathbb{P}^2[9] \to \mathbb{P}^1$ with fiber $\pi^{-1}(s:t) = \{[z] \in \mathbb{P}^2;\ sf_1(z) + tf_2(z) = 0\}$. Hence, $\mathbb{P}^2[9]$ is a rational elliptic surface.

Let $\widetilde{C}_1, \widetilde{C}_2 \subset \mathbb{P}^2[9]$ be the proper transform of C_1, C_2, respectively. Then the divisor $\widetilde{C}_1 + \widetilde{C}_2$ is the member of the double anti-canonical system $|-2K_{\mathbb{P}^2[9]}|$. Let $X_{C_1+C_2}$ be the double covering of $\mathbb{P}^2[9]$ with branch divisor $\widetilde{C}_1 + \widetilde{C}_2$. Let $\iota_{C_1+C_2}\colon X_{C_1+C_2} \to X_{C_1+C_2}$ be the non-trivial covering transformation. By the canonical bundle formula, $X_{C_1+C_2}$ is a $K3$ surface. By the rationality of $\mathbb{P}^2[9]$, $\iota_{C_1+C_2}$ acts non-trivially on $H^0(X_{C_1+C_2}, K_{X_{C_1+C_2}})$. Since the fixed point set of $\iota_{C_1+C_2}$ is identified with $C_1 + C_2$, it follows from Nikulin's classification of the fixed point set ([N, Th. 4.2.2]) that $(X_{C_1+C_2}, \iota_{C_1+C_2})$ is a Nikulin's $K3$ surface.

Let $\pi_{C_1+C_2}\colon X_{C_1+C_2} \to \mathbb{P}^1$ be the elliptic fibration associated to the linear system $|\widetilde{C}_1|$. Since the image of every member of $|\widetilde{C}_1|$ by $\iota_{C_1+C_2}$ is again a member of $|\widetilde{C}_1|$, there exists an involution $i_{\mathbb{P}^1}$ on $\mathbb{P}^1$ such that

$$
\begin{array}{ccc}
X_{C_1+C_2} & \xrightarrow{\;p\;} & \mathbb{P}^2[9] \\
{\scriptstyle \pi_{C_1+C_2}}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathbb{P}^1 & \xrightarrow[q]{} & \mathbb{P}^1
\end{array}
\tag{1.3}
$$

is a commutative diagram, where $p\colon X_{C_1+C_2} \to \mathbb{P}^2[9] = X_{C_1+C_2}/\iota_{C_1+C_2}$ and $q\colon \mathbb{P}^1 \to \mathbb{P}^1 = \mathbb{P}^1/i_{\mathbb{P}^1}$ are the natural projections.

§2. The moduli space of Nikulin's $K3$ surfaces

Define an involution I_Λ on $\mathbb{L}_{K3}$ by

$$I_\Lambda(a,b,c,x,y) = (a,-b,-c,y,x) \qquad (a,b,c \in U,\ x,y \in E_8). \tag{2.1}$$

Then Λ is the invariant part of I_Λ. Let L be the anti-invariant part of I_Λ. Then L is the orthogonal complement of Λ in $\mathbb{L}_{K3}$, and

$$L = U \oplus U \oplus E_8(2). \tag{2.2}$$

Let (X,ι) be a Nikulin's $K3$ surface. Since the embedding $\Lambda \hookrightarrow \mathbb{L}_{K3}$ is unique up to an automorphism of $\mathbb{L}_{K3}$, there exists a marking ϕ of X such that $\phi \circ \iota^* \circ \phi^{-1} = I_\Lambda$. A marking with this property is called a marking of a Nikulin's $K3$ surface. By Definition 1.1, the period of a marked Nikulin's $K3$ surface lies in the following subset of Ω:

$$\Omega_\Lambda := \{[x] \in \mathbb{P}(L \otimes \mathbb{C});\ \langle x,x\rangle = 0,\ \langle x,\bar{x}\rangle > 0\}. \tag{2.3}$$

Then Ω_Λ consists of two connected components $\Omega_\Lambda^{\pm}$, each of which is isomorphic to a symmetric bounded domain of type IV of dimension 10. However, the period mapping omits the divisor $\mathcal{D}_\Lambda$ of Ω_Λ described as follows: For $l \in L$ with $l^2 := \langle l,l\rangle < 0$, set $H_l := \{[x] \in \Omega_\Lambda;\ \langle x,l\rangle = 0\}$. Let $\mathcal{D}_\Lambda$ be the discriminant locus of Ω_Λ:

$$\mathcal{D}_\Lambda := \bigcup_{d \in L,\, d^2=-2} H_d. \tag{2.4}$$

Let $O(L)$ be the isometry group of the lattice L. Then $O(L)$ acts naturally on Ω_Λ and preserves $\mathcal{D}_\Lambda$. In [Yo, Th. 1.8], we proved:

Theorem 2.1. *The coarse moduli space of Nikulin's $K3$ surfaces is isomorphic to the analytic space $\mathcal{M}_\Lambda^0 := (\Omega_\Lambda \setminus \mathcal{D}_\Lambda)/O(L)$ via the period mapping.*

§3. The restriction of the Borcherds Φ-function to Ω_Λ

In [Bo], Borcherds introduced a remarkable automorphic form on the 26-dimensional symmetric bounded domain of type IV associated with the even unimodular lattice $II_{2,26} := U \oplus U \oplus E_8 \oplus E_8 \oplus E_8$. His automorphic form is called the *Borcherds Φ-function* and is denoted by Φ. We refer to [Bo, Th. 10.1 and §10 Example 2] for more details about the Borcherds Φ-function.

Since $L \subset II_{2,26}$, one can restrict the Borcherds Φ-function to Ω_Λ. This automorphic form on Ω_Λ is denoted by Φ_Λ:

$$\Phi_\Lambda := \Phi|_{\Omega_\Lambda}. \tag{3.1}$$

Then we proved in [Yo, Lemma 8.5] that Φ_Λ is an automorphic form on Ω_Λ of weight 12 with zero divisor $\mathcal{D}_\Lambda$.

Fix a vector $\ell \in L \otimes \mathbb{R}$ such that $\ell^2 \geq 0$. The pointwise length of Φ_Λ is defined by

$$\|\Phi_\Lambda\|^2([z]) := \left(\frac{\langle z, \bar{z}\rangle_L}{|\langle z, \ell\rangle_L|^2}\right)^{12} |\Phi_\Lambda([z])|^2 \qquad ([z] \in \Omega_\Lambda). \tag{3.2}$$

Then $\|\Phi_\Lambda\|^2$ is an $O(L)$-invariant C^∞-function on Ω_Λ and is regarded as a function on $\mathcal{M}^0_\Lambda$.

§4. Equivariant analytic torsion of Nikulin's $K3$ surfaces

In [Bi], Bismut established the foundations of the theory of equivariant analytic torsion and equivariant Quillen metrics. Here, we recall his construction in the simplest case. We refer to [Bi] for more details about equivariant analytic torsion and equivariant Quillen metrics.

Let Y be a compact Kähler manifold. Let $\theta: Y \to Y$ be a holomorphic involution. Let $\mathbb{Z}_2 \subset \mathrm{Aut}(Y)$ be the subgroup generated by θ. Let γ_Y be a $\mathbb{Z}_2$-invariant Kähler metric on Y. Let $\square_q$ be the $\bar{\partial}$-Laplacian acting on $(0,q)$-forms on Y with respect to γ_Y. Let $\sigma(\square_q)$ be the spectrum of $\square_q$. For $\lambda \in \sigma(\square_q)$, let $E_q(\lambda)$ be the vector space of eigenforms of $\square_q$ with eigenvalue λ. Then $\mathbb{Z}_2$ preserves $E_q(\lambda)$.

For $g \in \mathbb{Z}_2$ and $s \in \mathbb{C}$, set $\zeta_q(g)(s) := \sum_{\lambda \in \sigma(\square_q)\setminus\{0\}} \mathrm{Tr}(g|_{E_q(\lambda)})\lambda^{-s}$. Classically, $\zeta_q(g)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim Y$, admits a meromorphic continuation to $\mathbb{C}$, and is holomorphic at $s = 0$.

Definition 4.1. For $g \in \mathbb{Z}_2$, the *equivariant analytic torsion* of (Y, γ_Y) is defined by

$$\log \tau_{\mathbb{Z}_2}(Y, \gamma_Y)(g) := \sum_{q \geq 0} (-1)^{q+1} \zeta_q'(g)(0). \tag{4.1}$$

When $g = 1$, $\tau_{\mathbb{Z}_2}(Y, \gamma_Y)(1)$ coincides with the Ray-Singer analytic torsion of (Y, γ_Y) and is denoted by $\tau(Y, \gamma_Y)$.

§5. The adiabatic limit of $\tau_{\mathbb{Z}_2}$ for Nikulin's $K3$ surfaces

Let (X, ι) be a Nikulin's $K3$ surface. Let $C_1 + C_2$ be the set of fixed points of ι. Then C_1 and C_2 are mutually disjoint elliptic curves. Let $[(X, \iota)] \in \mathcal{M}^0_\Lambda$ be the $O(L)$-orbit of the period of (X, ι). By the $O(L)$-invariance of $\|\Phi_\Lambda\|$, the value $\|\Phi_\Lambda([X, \iota])\|$ makes sense.

Let $\pi\colon X \to \mathbb{P}^1$ be the elliptic fibration associated with the free linear system $|C_1|$. Then the image of an arbitrary fiber of π by ι is again a fiber of π, and ι induces an involution $i_{\mathbb{P}^1}$ on $\mathbb{P}^1$ verifying (1.3).

Let κ_X be an ι-invariant Kähler class on X. Let $\kappa_{\mathbb{P}^1}$ be a Kähler class on $\mathbb{P}^1$. For $0 < \epsilon < +\infty$, set

$$\kappa_\epsilon := \kappa_X + \epsilon^{-1}\pi^*\kappa_{\mathbb{P}^1}. \tag{5.1}$$

Then $\{\kappa_\epsilon\}_{0<\epsilon<+\infty}$ is a family of ι-invariant Kähler classes on X. Notice that the Kähler class on the fiber induced from κ_ϵ is independent of ϵ. By Calabi-Yau ([Ya]), there exists uniquely an ι-invariant Ricci-flat Kähler form ω_ϵ in κ_ϵ:

$$\mathrm{Ric}(\omega_\epsilon) \equiv 0, \quad \iota^*\omega_\epsilon = \omega_\epsilon, \quad [\omega_\epsilon] = \kappa_\epsilon \qquad (0 < \epsilon < +\infty). \tag{5.2}$$

Let $\mathrm{Vol}(X, \omega_\epsilon) := \int_X \omega_\epsilon^2/2!$ be the volume of (X, ω_ϵ). Let $F \in H_2(X, \mathbb{Z})$ be the class of fibers of $\pi\colon X \to \mathbb{P}^1$. Set $\mathrm{Vol}(F, \kappa|_F) := \int_F \kappa|_F$ and $\mathrm{Vol}(\mathbb{P}^1, \kappa_{\mathbb{P}^1}) := \int_{\mathbb{P}^1} \kappa_{\mathbb{P}^1}$. By (5.1) and the projection formula, we get

$$\mathrm{Vol}(X, \omega_\epsilon) = \mathrm{Vol}(X, \kappa) + \epsilon^{-1}\mathrm{Vol}(F, \kappa|_F)\mathrm{Vol}(\mathbb{P}^1, \kappa_{\mathbb{P}^1}). \tag{5.3}$$

The following is the main result of this note:

Theorem 5.1. *There exists a constant $C \neq 0$ depending only on the lattice Λ such that*

$$\lim_{\epsilon \to 0} \tau_{\mathbb{Z}_2}(X, \omega_\epsilon)(\iota) \cdot \mathrm{Vol}(X, \omega_\epsilon) = C\, \|\Phi_\Lambda([(X, \iota)])\|^{-\frac{1}{6}}. \tag{5.4}$$

Proof. For $\tau \in \mathbb{H}$, let $\Delta(\tau) = e^{2\pi i\tau}\prod_{n>0}(1 - e^{2\pi i n\tau})^{24}$ be the Jacobi-Δ function. Set $\|\Delta(\tau)\|^2 := (\mathrm{Im}\,\tau)^{12}|\Delta(\tau)|^2$, which is a $SL_2(\mathbb{Z})$-invariant function on $\mathbb{H}$. Let $[C_i] \in \mathbb{H}/SL_2(\mathbb{Z})$ be the period of the elliptic curve C_i. By the $SL_2(\mathbb{Z})$-invariance of $\|\Delta(\tau)\|$, the value $\|\Delta([C_i])\|$ is independent of the choice of a representative of $[C_i]$ in $\mathbb{H}$.

By [Yo, Th. 5.2 and Th. 8.7], there exists a constant $C_\Lambda \neq 0$ depending only on the lattice Λ such that

$$\tau_{\mathbb{Z}_2}(X,\omega_\epsilon)(\iota)\cdot\mathrm{Vol}(X,\omega_\epsilon)\prod_{i=1}^{2}\tau(C_i,\omega_\epsilon|_{C_i})\cdot\mathrm{Vol}(F,\kappa_\epsilon|_F) = C_\Lambda\,\|\Phi_\Lambda([(X,\iota)])\|^{-\frac{1}{6}}\cdot\prod_{i=1}^{2}\|\Delta([C_i])\|^{-\frac{1}{6}}. \tag{5.5}$$

By [G-W, Th. 5.6], the family of Kähler forms $\{\omega_\epsilon|_{C_i}\}_{0<\epsilon<1}$ converges in arbitrary C^k-topology to the *flat* Kähler form ω_{C_i} on C_i with Kähler class $\kappa|_{C_i}$. Hence, we deduce from the anomaly formula for Quillen metrics that

$$\lim_{\epsilon\to 0}\tau(C_i,\omega_\epsilon|_{C_i}) = \tau(C_i,\omega_{C_i}),\qquad \mathrm{Vol}(F,\kappa_\epsilon|_F) = \mathrm{Vol}(C_i,\omega_{C_i}). \tag{5.6}$$

Since ω_{C_i} is flat, Kronecker's limit formula yields that

$$\tau(C_i,\omega_{C_i})\cdot\mathrm{Vol}(C_i,\omega_{C_i}) = \|2^{12}\Delta([C_i])\|^{-\frac{1}{6}}. \tag{5.7}$$

The result follows from (5.5), (5.6), (5.7). Q.E.D.

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Graduate School of Mathematical Sciences
University of Tokyo
Tokyo 153-8914
JAPAN
yosikawa@ms.u-tokyo.ac.jp