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Fixed points of polynomial automorphisms of $\mathbf{C}^n$

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Abstract.

We study the fixed point indices of some polynomial automorphisms of $\mathbf{C}^n$. In particular, it is shown that, for a composition of generalized Hénon maps, the sum of the fixed point indices vanishes. A consequence is that a generic polynomial automorphism of $\mathbf{C}^2$ has a saddle fixed point.

§1. Statement of the results

A bijective map F of the space of n complex variables $\mathbf{C}^n$ onto itself defined by polynomials $f_1(x), \ldots, f_n(x)$, $x = (x_1, \ldots, x_n)$, is said to be a polynomial automorphism of $\mathbf{C}^n$. The set $\mathrm{Aut}(\mathbf{C}^n)$ of all polynomial automorphisms of $\mathbf{C}^n$ forms a group under composition. Two maps $F_1, F_2 \in \mathrm{Aut}(\mathbf{C}^n)$ are conjugate if there exists a map $G \in \mathrm{Aut}(\mathbf{C}^n)$ such that $F_2 = G^{-1} \circ F_1 \circ G$.

For a fixed point of a holomorphic map of $\mathbf{C}^n$ to itself, holomorphic Lefschetz index can be defined (see §2, also Griffiths-Harris [2]). We will study the indices for the fixed points of polynomial automorphisms, since they are important invariants under conjugation.

For the case of two variables, Friedland-Milnor [1] showed that any map in $\mathrm{Aut}(\mathbf{C}^2)$ is conjugate to either (1) an affine map, (2) an elementary map or (3) a composition $F_m \circ \cdots \circ F_1$ of generalized Hénon maps

$$F_\mu(x, y) = (y, p_\mu(y) - \delta_\mu x), \quad \mu = 1, \ldots, m,$$

where $p_\mu(y)$ are polynomials of degree ≥ 2 and $\delta_\mu \neq 0$.

We denote by H_0 the set consisting of compositions of generalized Hénon maps, and by H the set of all maps conjugate to one of the maps in H_0.

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Let Fix (F) denote the set of all fixed points of F. It was shown in [1] that, if $F \in H_0$ and $\deg F = k$, then F has k fixed points counting multiplicity. i.e., $\sum_{a \in \mathrm{Fix}(F)} \mathrm{Mult}\,(F, a) = k$.

Now we have

Theorem 1. *If $F \in H$, then we have*

$$\sum_{a \in \mathrm{Fix}(F)} \mathrm{Ind}\,(F, a) = 0.$$

We note that the formula fails in general for maps $\notin H$. A proof of this formula for a generalized Hénon map is given in [3]. A similar result for holomorphic maps on projective spaces is given in [4].

Corollary 1. *Let $F \in H$ and suppose that F has only simple fixed points a_j $(j = 1, \cdots, k)$. Let $\lambda_{j,1}, \lambda_{j,2}$ denote the eigenvalues of $F'(a_j)$. Then we have*

$$\sum_{j=1}^{k} \left(\frac{1}{1 - \lambda_{j,1}} + \frac{1}{1 - \lambda_{j,2}} \right) = k,$$

Corollary 2. *Let $F \in H$ and $\delta = \det F'$. Suppose that $|\delta| \neq 1$ or $\delta = 1$. Then (1) F has either a saddle fixed point or a multiple fixed point, and (2) F has infinitely many periodic points that are either saddle or multiple.*

The condition on δ cannot be dropped as the following example shows.

Example Let F be a Hénon map defined by

$$F(x, y) = (y, y^2 + c - \delta x).$$

Then F has at least one saddle fixed point if and only if $(\delta, c) \notin \Delta \cup \Gamma$, where $\Delta = \{(\delta + 1)^2 - 4c = 0\}$ and

$$\Gamma = \left\{ |\delta| = 1, \ \frac{c}{\delta} \text{ is real and} \sqrt{2(1 + \mathrm{Re}\,\delta)} - 1 \leq \frac{c}{\delta} < \frac{1 + \mathrm{Re}\,\delta}{2} \right\}.$$

We can generalize the index formula to maps of certain class of polynomial automorphisms of $\mathbf{C}^n$:

Theorem 2. *Let $F = F_m \circ \cdots \circ F_1$ be the composition of shift-like maps $F_\mu : \mathbf{C}^n \to \mathbf{C}^n$ $(\mu = 1, \ldots, m)$ defined by*

$$F_\mu(x_1, \ldots, x_n) = (x_2, \ldots, x_n, a_\mu x_1 + p_\mu(x_2, \ldots, x_n)),$$

where p_μ are polynomials in $n-1$ variables. Suppose that there exist ν $(2 \le \nu \le n)$ such that

$$P_\mu(x_2, \dots, x_n) = c_\mu x_\nu^{k_\mu} + \text{(lower order terms)}, \; c_\mu \neq 0.$$

Then we have $\sum_{a \in \mathrm{Fix}(F)} \mathrm{Ind}\,(F, a) = 0.$

We remark that, for general (compositions of) shift-like maps, the set $\mathrm{Fix}\,(F)$ may be non-isolated. Even if $\mathrm{Fix}\,(F)$ is isolated, the index formula does not necessarily hold.

Example Consider the map $F : \mathbf{C}^3 \to \mathbf{C}^3$ defined by

$$F(x, y, z) = (y, z, \delta x + (y - z)^2).$$

If $\delta \neq 1$, then $\mathrm{Fix}\,(F) = \{0\}$ and $\mathrm{Ind}\,(F, 0) = 1/(1-\delta)$. If $\delta = 1$, then $\mathrm{Fix}\,(F) = \{x = y = z\}$.

§2. Multiplicity and Index

Let $G : \mathbf{C}^n \to \mathbf{C}^n$ be a holomorphic map and suppose that a is an isolated zero of G. Then there exist neighborhoods U of a and V of 0 such that $G^{-1}(0) \cap U = \{a\}$ and that $G|U : U \to V$ is a branched cover. We define the zero multiplicity $\mathrm{mult}\,(G, a)$ of G at a to be the sheet number of this map $G|U$. We call that a is a simple zero of G if $\mathrm{mult}\,(G, a) = 1$, or in other words, if $\det G'(a) \neq 0$.

If a is a simple zero, we define the zero index by $\mathrm{ind}\,(G, a) = 1/\det G'(a)$. For the general case $\mathrm{ind}\,(G, a)$ is defined as follows: We set $\omega = dx_1 \wedge \cdots \wedge dx_n$ and

$$\eta = \frac{c_n}{\|x\|^{2n}} \sum_{i=1}^{n} (-1)^{i-1} \overline{x}_i d\overline{x}_1 \wedge \cdots \widehat{d\overline{x}_i} \cdots \wedge d\overline{x}_n$$

Where $c_n = \sqrt{-1}^{n^2} (n-1)!/(2\pi)^n$. We define

$$\mathrm{ind}\,(G, a) = \int_{\partial B} (G^* \eta) \wedge \omega$$

where B denotes a ball with center a of sufficiently small radius so that a is the only zero of G in B.

We will apply the following lemma in the proof of Theorem 2.

Lemma 3. *Let $G(x) = (g_1(x), \dots, g_n(x))$ be a polynomial map of $\mathbf{C}^n$ to $\mathbf{C}^n$. Suppose that g_ν is of the form*

$$g_\nu(x) = c_\nu x_{\sigma(\nu)}^{k_\nu} + \text{(lower order terms)}, \quad k_\nu \ge 2,\; c_\nu \neq 0, \quad (\nu = 1, \dots, n).$$

where σ is a permutation of $\{1, \dots, n\}$. then $\sum_{a \in G^{-1}(0)} \mathrm{ind}\,(G, a) = 0.$

To see this, we note that

$$\sum_{a \in G^{-1}(0)} \operatorname{ind}(G, a) = \int_{\partial B} (G^* \eta) \wedge \omega,$$

where B is a sufficiently large ball in $\mathbf{C}^n$. By estimating the integral, we conclude the lemma.

Now let $F : \mathbf{C}^n \to \mathbf{C}^n$ be a holomorphic map and suppose that a is an isolated fixed point of F. This is equivalent to say that a is an isolated zero of the map $Id - F$. We define the fixed point multiplicity and the fixed point index by

$$\operatorname{Mult}(F, a) = \operatorname{mult}(Id - F, a), \quad \operatorname{Ind}(F, a) = \operatorname{ind}(Id - F, a).$$

§3. Outline of the proof

3.1 To prove Theorem 2, let us first introduce the concept of vectorial shift-like map. We denote the points in $\mathbf{C}^{mn}$ as (m, n)-matrices and also as a row of column vectors: $\widehat{\xi} = (\xi_{ij}) = (\xi_1, \dots, \xi_n)$. A map $\Phi \in \operatorname{Aut}(\mathbf{C}^{mn})$ is said to be a vectorial shift-like map if it is of the form

$$\Phi(\xi_1, \dots, \xi_n) = (\xi_2, \dots, \xi_n, A\xi_1 + Q(\xi_2, \dots, \xi_n))$$

where $A \in GL(m, \mathbf{C})$ and Q is a column vector of polynomials in $m(n-1)$ variables ξ_{ij} $(1 \le i \le m; 2 \le j \le n)$.

The fixed points of Φ are of the form $\hat{b} = (b, \dots, b)$, where $b \in \mathbf{C}^m$ are the roots of the equation $A\xi + Q(\xi, \dots, \xi) = \xi$. We define a linear map $L : (\xi_1, \dots, \xi_n) \mapsto (\eta_1, \dots, \eta_n)$ by

$$\eta_\nu = \xi_\nu - \xi_{\nu+1} \ (\nu = 1, \dots, n-1) \quad \text{and} \quad \eta_n = \xi_n.$$

Then $(Id - \Phi) \circ L^{-1}$ takes the form $(\eta_1, \dots, \eta_n) \mapsto (\eta_1, \dots, \eta_{n-1}, \eta_n - A(\eta_1 + \dots + \eta_n) - Q(\eta_2 + \dots + \eta_n, \dots, \eta_n))$. The sum of the zero point indices of this map is equal to that of the map $\eta \mapsto \eta - A\eta - Q(\eta, \dots, \eta)$. If this satisfies the condition of Lemma 3, then $\sum_{\hat{b} \in \operatorname{Fix}(\Phi)} \operatorname{Ind}(\Phi, \hat{b}) = 0$.

3.2 Let $F_\mu : \mathbf{C}^n \to \mathbf{C}^n$ be holomorphic maps $(\mu = 1, \dots, m)$, and let $F = F_m \circ \dots \circ F_1$ be their composition. To study the fixed points of F, we consider the map $\hat{F} : \mathbf{C}^{mn} \to \mathbf{C}^{mn}$ defined as follows. We denote the points in $\mathbf{C}^{mn}$ by a (m, n)-matrix and also as a column of row vectors :

$\hat{x} = (x_{ij}) = {}^t(x_1, \dots, x_m)$. We define $\hat{F}$ by

$$\hat{F}(\hat{x}) = \hat{F}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} F_m(x_m) \\ F_1(x_1) \\ \vdots \\ F_{m-1}(x_{m-1}) \end{pmatrix}.$$

There is a one-to-one correspondence between the sets Fix (F) and Fix $(\hat{F})$. In fact, if a is in Fix (F), then the point $\hat{a} = {}^t(a_1, \dots, a_m)$ with $a_1 = a, a_\mu = F_{\mu-1}(a_{\mu-1})$ $(\mu = 2, \dots, m)$ is in Fix $(\hat{F})$. Conversely, if $\hat{a} = {}^t(a_1, \dots, a_m)$ is in Fix $(\hat{F})$, then a_1 is in Fix (F).

Further we can prove that, if $a \in$ Fix (F) and $\hat{a} \in$ Fix $(\hat{F})$ are corresponding fixed points, then

$$\text{Mult}\,(F, a) = \text{Mult}\,(\hat{F}, \hat{a}), \quad \text{and} \quad \text{Ind}\,(F, a) = \text{Ind}\,(\hat{F}, \hat{a}).$$

3.3 Now we apply the above obserbations to a composition $F = F_m \circ \cdots \circ F_1$ of shift-like maps F_μ. Then $\hat{F}(\hat{x})$ takes the form

$$\begin{pmatrix} x_{m2} & \cdots & x_{mn} & \delta_m x_{m1} + p_m(x_{m2}, \dots, x_{mn}) \\ x_{12} & \cdots & x_{1n} & \delta_1 x_{11} + p_1(x_{12}, \dots, x_{1n}) \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1,2} & \cdots & x_{m-1,n} & \delta_{m-1} x_{m-1,1} + p_{m-1}(x_{m-1,2}, \dots, x_{m-1,n}) \end{pmatrix}.$$

We can reduce $\hat{F}$ to a vectorial shift-like map by conjugation. To see this, consider the linear map $M : \mathbf{C}^{mn} \ni (x_{ij}) \mapsto (\xi_{ij}) \in \mathbf{C}^{mn}$ defined by $\xi_{ij} = x_{[i-j+1],j}$ where $[\ell]$ denotes the number such that $1 \leq [\ell] \leq m$ and $[\ell] \equiv \ell \mod m$. Then the conjugate $\Phi = M \circ \hat{F} \circ M^{-1}$ is a vectorial shift-like map $\Phi(\xi_1, \dots, \xi_n) = (\xi_2, \dots, \xi_n, A\xi_1 + Q(\xi_2, \dots, \xi_n))$, where

$$A\xi_1 + Q(\xi_2, \dots, \xi_n) = \begin{pmatrix} \delta_{[1-n]}\xi_{[1-n],1} + p_{[1-n]}(\xi_{[2-n],2}, \dots, \xi_{m,n}) \\ \delta_{[2-n]}\xi_{[2-n],1} + p_{[2-n]}(\xi_{[3-n],2}, \dots, \xi_{1,n}) \\ \vdots \\ \delta_{[m-n]}\xi_{[m-n],1} + p_{[m-n]}(\xi_{[1-n],2}, \dots, \xi_{m-1,n}) \end{pmatrix}.$$

The map $\eta \mapsto \eta - A\eta - Q(\eta, \dots, \eta)$ takes the form

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix} \mapsto \begin{pmatrix} \eta_1 - \delta_{[1-n]}\eta_{[1-n]} - p_{[1-n]}(\eta_{[2-n]}, \dots, \eta_m) \\ \eta_2 - \delta_{[2-n]}\eta_{[2-n]} - p_{[2-n]}(\eta_{[3-n]}, \dots, \eta_1) \\ \vdots \\ \eta_m - \delta_{[m-n]}\eta_{[m-n]} - p_{[m-n]}(\eta_{[1-n]}, \dots, \eta_{m-1}) \end{pmatrix}.$$

Under the condition of Theorem 2, this map satisfies the condition of Lemma 3. Thus Theorem 2 is proved.

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