Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 289–299

Prolongation of holomorphic vector fields on a tube domain and its applications

Satoru Shimizu

Introduction

In general, in the study of the holomorphic equivalence problem for complex manifolds, that is to say, the problem of investigating what happens when two complex manifolds are biholomorphically equivalent, it is one of standard ways to direct our attention to biholomorphic invariant objects. As a typical and good example of such objects, we have holomorphic automorphism groups. In fact, when Poincaré showed that a ball and a polydisk in $\mathbf{C}^2$ are not biholomorphically equivalent, he looked at their holomorphic automorphism groups, and showed that the dimensions do not coincide. One of the foundations of observations like this is the pioneer result of H. Cartan that the holomorphic automorphism group of a complex bounded domain has the structure of a Lie group.

Now, when a holomorphic automorphism group has the structure of a Lie group, what advantage do we have? It seems that one advantage is that conjugacy theorems in Lie group theory can be applied. The conjugacy theorems are very powerful tools, and if they can be applied well, splendid achievements are produced. But, in order to apply the conjugacy theorems, we need to know a lot about the Lie group structure of a holomorphic automorphism group. So, since Lie algebra provides much useful information about Lie group, we are led to turning our eyes to the Lie algebra of complete holomorphic vector fields corresponding to the Lie algebra of a holomorphic automorphism group. Then, in the process of investigating such Lie algebras, we often come up against the problem of completeness, or the fundamental problem of judging whether a vector field is complete or not. In general, a judgement on the completeness of a vector field is very difficult to deal with. Actually, given a vector field, the problem of whether its integral curve is lengthened to

Received March 20, 2002

infinity or not has complicated aspects as the problem of solutions of autonomous systems in the theory of nonlinear oscillations. But, in some geometric setting, there is a nice algebraic criterion on the completeness of a vector field. In this article, we discuss such a criterion in the case of holomorphic vector fields on a tube domain. Our objects of consideration are polynomial vector fields on a tube domain T_Ω. We give a method of determining higher degree complete polynomial vector fields on T_Ω from the data on lower degree complete polynomial vector fields on T_Ω, which we call prolongation. Furthermore, we give its applications to the holomorphic equivalence problem for tube domains.

§1. Basic concepts and results on tube domains

We first recall some notation and terminology. An automorphism of a complex manifold M means a biholomorphic mapping of M onto itself. The group of all automorphisms of M is denoted by $\mathrm{Aut}(M)$. We denote by $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$ the subgroup of $\mathrm{Aut}(\mathbf{C}^n)$ consisting of all transformations of the form

$$\mathbf{C}^n \ni z \longmapsto Az + \beta \in \mathbf{C}^n,$$

where $A \in GL(n, \mathbf{R})$ and $\beta \in \mathbf{C}^n$. Two complex manifolds are said to be holomorphically equivalent if there is a biholomorphic mapping between them. For a Lie group G, we denote by G° the identity component of G and by $\mathrm{Lie}\, G$ the Lie algebra of G. If $E = \{\cdots\}$ is a subset of a vector space V over a field $\mathbf{F}$, the linear subspace of V spanned by E is denoted by $E_{\mathbf{F}} = \{\cdots\}_{\mathbf{F}}$.

We now recall basic concepts and results on tube domains. A tube domain T_Ω in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ given by $T_\Omega = \mathbf{R}^n + \sqrt{-1}\Omega$, where Ω is a domain in $\mathbf{R}^n$ and is called the base of T_Ω. Clearly, each element $\xi \in \mathbf{R}^n$ gives rise to an automorphism $\sigma_\xi \in \mathrm{Aut}(T_\Omega)$ defined by

$$\sigma_\xi(z) = z + \xi \quad \text{for } z \in T_\Omega.$$

Write $\Sigma = \mathbf{R}^n$. The additive group Σ acts as a group of automorphisms on T_Ω by

$$\xi \cdot z = \sigma_\xi(z) \quad \text{for } \xi \in \Sigma \text{ and } z \in T_\Omega.$$

The subgroup of $\mathrm{Aut}(T_\Omega)$ induced by Σ is denoted by Σ_{T_Ω}. Note that if $\varphi \in GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$, then $\varphi(T_\Omega)$ is a tube domain in $\mathbf{C}^n$, and we have $\varphi \Sigma_{T_\Omega} \varphi^{-1} = \Sigma_{T_\Xi}$, where $T_\Xi = \varphi(T_\Omega)$.

Consider a biholomorphic mapping $\varphi\colon T_{\Omega_1} \to T_{\Omega_2}$ between two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$. Then, by what we have noted above and [3, Section 1, Proposition], φ is given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$

if and only if φ is equivariant with respect to the Σ-actions. Biholomorphic mappings between tube domains equivariant with respect to the Σ-actions may be considered as natural isomorphisms in the category of tube domains. In view of this observation, we say that two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$ are affinely equivalent if there is a biholomorphic mapping between them given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$.

If the convex hull of the base Ω of a tube domain T_Ω in $\mathbf{C}^n$ contains no complete straight lines, then T_Ω is holomorphically equivalent to a bounded domain in $\mathbf{C}^n$ and, by a well-known theorem of H. Cartan, the group $\mathrm{Aut}(T_\Omega)$ of all automorphisms of T_Ω forms a Lie group with respect to the compact-open topology. The Lie algebra $\mathfrak{g}(T_\Omega)$ of the Lie group $\mathrm{Aut}(T_\Omega)$ can be identified canonically with the finite-dimensional real Lie algebra consisting of all complete holomorphic vector fields on T_Ω. Throughout this article, we are concerned with tube domains whose bases have the convex hulls containing no complete straight lines.

Let $z_1, \cdots, z_n$ be the complex coordinate functions of $\mathbf{C}^n$ and, for $j = 1, \cdots, n$, we write $\partial_j = \partial/\partial z_j$. Let D be a domain in $\mathbf{C}^n$. Then every holomorphic vector field Z on D can be written in the form

$$Z = \sum_{j=1}^{n} f_j(z)\partial_j,$$

where $f_1(z), \cdots, f_n(z)$ are holomorphic functions on D. The vector field Z is called a polynomial vector field if $f_1(z), \cdots, f_n(z)$ are polynomials in $z_1, \cdots, z_n$. The maximum value of the degrees of the polynomials $f_1(z), \cdots, f_n(z)$ is called the degree of Z. The following result is fundamental in our study.

Structure Theorem ([3, Section 2, Theorem]). *To each tube domain T_Ω in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines, there is associated a tube domain $T_{\tilde{\Omega}}$ which is affinely equivalent to T_Ω such that $\mathfrak{g}(T_{\tilde{\Omega}})$ has the direct sum decomposition*

$$\mathfrak{g}(T_{\tilde{\Omega}}) = \mathfrak{p} + \mathfrak{e}$$

for which

$$\mathfrak{p} = \{X \in \mathfrak{g}(T_{\tilde{\Omega}}) \mid X \textit{ is a polynomial vector field}\},$$

$$\mathfrak{e} = \sum_{i=1}^{r} \{E_i^+, E_i^-\}_{\mathbf{R}},$$

$$E_i^{\pm} = e^{\pm z_i}\left(\partial_i \pm \sum_{j=r+1}^{n} \sqrt{-1}a_i^j\partial_j\right), \quad i = 1, \cdots, r,$$

where r is an integer between 0 *and n and a_i^j, $i = 1, \cdots, r$, $j = r+1, \cdots, n$, are real constants.*

The integer r is called the exponential rank of the tube domain T_Ω, and is denoted by $e(T_\Omega)$. This is well-defined, because it is readily verified that if two tube domains T_{Ω_1} and T_{Ω_2} are affinely equivalent, then we have $e(T_{\Omega_1}) = e(T_{\Omega_2})$. When a tube domain T_Ω satisfies $e(T_\Omega) = 0$, we call T_Ω a tube domain with polynomial infinitesimal automorphisms.

Our main theme in this article is a study of tube domains with polynomial infinitesimal automorphisms. This is motivated by the holomorphic equivalence problem for tube domains, which we will explain below.

In terms of the notion of the affine equivalence of tube domains, the holomorphic equivalence problem for tube domains may be formulated as the problem of studying the connection between the two equivalences - the holomorphic equivalence and the affine equivalence - of tube domains. It is clear that if two tube domains in $\mathbf{C}^n$ are affinely equivalent, then they are holomorphically equivalent. What we have to ask is whether the converse assertion holds or not:

Problem. If two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$ are holomorphically equivalent, then are they affinely equivalent?

When Ω_1 and Ω_2 are convex cones in $\mathbf{R}^n$, an affirmative answer is given (see Matsushima [1]). On the other hand, when Ω_1 and Ω_2 are arbitrary domains in $\mathbf{R}^n$ whose convex hulls contain no complete straight lines, there is a simple counter example. In fact, consider the upper half plane

$$T_{(0,\infty)} = \{x + \sqrt{-1}y \in \mathbf{C} \,|\, x \in \mathbf{R},\, y > 0\}$$

and the strip

$$T_{(0,\pi)} = \{x + \sqrt{-1}y \in \mathbf{C} \,|\, x \in \mathbf{R},\, 0 < y < \pi\}$$

in the complex plane. Then the tube domains $T_{(0,\infty)}$ and $T_{(0,\pi)}$ in $\mathbf{C}$ are holomorphically equivalent, but not affinely equivalent. We can clarify what causes a phenomenon like this by making use of the Structure Theorem stated above.

Let T_{Ω_1} and T_{Ω_2} be tube domains in $\mathbf{C}^n$ whose bases Ω_1 and Ω_2 have the convex hulls containing no complete straight lines. Since the exponential rank of a tube domain is an affine invariant, it is natural to reformulate the holomorphic equivalence problem for tube domains as follows:

Problem $(*)$. If $e(T_{\Omega_1}) = e(T_{\Omega_2})$ and if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then are T_{Ω_1} and T_{Ω_2} affinely equivalent?

The counter example shown above corresponds to the case where $e(T_{\Omega_1}) \neq e(T_{\Omega_2})$, because $e(T_{(0,\infty)}) = 0$ and $e(T_{(0,\pi)}) = 1$. On the other hand, when Ω_1 and Ω_2 are bounded domains in $\mathbf{R}^n$, it is shown ([5]) that if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then we have $e(T_{\Omega_1}) = e(T_{\Omega_2})$, and T_{Ω_1} and T_{Ω_2} are affinely equivalent.

Specifying Problem $(*)$, we consider the following problem which has fundamental importance:

Problem $(**)$. If $e(T_{\Omega_1}) = e(T_{\Omega_2}) = 0$ and if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then are T_{Ω_1} and T_{Ω_2} affinely equivalent?

When Ω_1 and Ω_2 are convex cones in $\mathbf{R}^n$, we have $e(T_{\Omega_1}) = e(T_{\Omega_2}) = 0$ (see [1]), and an affirmative answer to Problem $(**)$ is given, as stated above. For an attempt to solve Problem $(**)$ in the case where T_{Ω_1} and T_{Ω_2} are arbitrary tube domains with polynomial infinitesimal automorphisms, we need a further study of the structure of $\mathfrak{g}(T_\Omega)$. The Prolongation Theorem given in the next section enables us to make a more detailed analysis of the structure of $\mathfrak{g}(T_\Omega)$ and, applying this, together with the classification result in [6] and so on, we can give an affirmative answer to Problem $(**)$ in various cases [4], [8], [9].

§2. Prolongation of complete polynomial vector fields on a tube domain and tube domains with polynomial infinitesimal automorphisms

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines. For a polynomial vector field Z on T_Ω of degree 2, we write

$$Z = \sum_{k=0}^{2} \left(X^{(k)} + \sqrt{-1} Y^{(k)} \right),$$

where $X^{(k)}, Y^{(k)}$ are polynomial vector fields whose components with respect to $\partial_1, \cdots, \partial_n$ are homogeneous polynomials in $z_1, \cdots, z_n$ with real coefficients of degree k, and set

$$\begin{aligned} Z_{[b]} &= X^{(2)} + \sqrt{-1} Y^{(1)}, \\ Z_{[a]} &= X^{(1)} + \sqrt{-1} Y^{(0)}, \\ Z_{[s]} &= X^{(0)}. \end{aligned}$$

Note that $Z = Z_{[s]} + Z_{[a]} + Z_{[b]} + \sqrt{-1}Y^{(2)}$. Our criterion on the completeness of Z is given in the following theorem.

Prolongation Theorem ([7, Section 2, Prolongation Theorem]). *Let Z be a polynomial vector field on T_Ω of degree 2. Then Z is complete on T_Ω if and only if one has $Y^{(2)} = 0$, and the vector fields $[\partial_i, Z]$, $i = 1, \cdots, n$, and $Z_{[a]}$ are all complete on T_Ω. Consequently, if Z is complete on T_Ω, then $Z_{[b]}$ is complete on T_Ω. Also, if $Z = Z_{[b]}$ and if the vector fields $[\partial_i, Z]$, $i = 1, \cdots, n$, are all complete on T_Ω, then Z is complete on T_Ω.*

The proof of this theorem is based on the fact that every infinitesimal isometry on a complete Riemannian manifold is complete. It follows from this fact that Z is complete on T_Ω if and only if the coefficient functions of Z satisfy the system of certain linear partial differential equations, and it is represented as the condition stated in the Prolongation Theorem.

Now, when we are discussing tube domains T_Ω with polynomial infinitesimal automorphisms, it is one of the key points that a polynomial gives the Taylor expansion around the origin of the function it represents. In what follows, we give some fundamental results on $\mathfrak{g}(T_\Omega)$ obtained by combining the Prolongation Theorem above with this fact.

2.1. General observations on an isotropy subalgebra of $\mathfrak{g}(T_\Omega)$

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines. We may assume without loss of generality that T_Ω contains the origin of $\mathbf{C}^n$. Every element Z of $\mathfrak{g}(T_\Omega)$ has the Taylor expansion around the origin given as

$$Z = \sum_{k=0}^{\infty} Z^{((k))},$$

where $Z^{((k))}$ is a polynomial vector field whose components with respect to $\partial_1, \cdots, \partial_n$ are homogeneous polynomials in $z_1, \cdots, z_n$ of degree k. We write

$$Z^{((1))} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_{ji}(Z) z_i \right) \partial_j,$$

where $c_{ji}(Z)$, $j, i = 1, \cdots, n$, are complex constants. Let $\mathfrak{k}$ denote the isotropy subalgebra of $\mathfrak{g}(T_\Omega)$ at the origin. Then $\mathfrak{k}$ consists of those elements Z of $\mathfrak{g}(T_\Omega)$ which satisfy $Z^{((0))} = 0$. An application of H.

Cartan's uniqueness theorem [2, Chapter 5, Proposition 1] yields the following result.

Lemma 1. *If Z is an element of $\mathfrak{k}$ and if $Z^{((1))} = 0$, then $Z = 0$.*

This result implies that the linear representation of $\mathfrak{k}$ given by

$$\mathfrak{k} \ni Z \longmapsto (c_{ji}(Z)) \in \mathfrak{gl}(n, \mathbf{C})$$

is faithful, where $\mathfrak{gl}(n, \mathbf{C})$ denotes the set of complex n by n matrices viewed as the Lie algebra of $GL(n, \mathbf{C})$. We recall here that T_Ω has the Bergman metric $ds^2_{T_\Omega}$. Using the invariance of $ds^2_{T_\Omega}$ under the action of Σ_{T_Ω}, after a suitable real linear change of coordinates we may assume that the holomorphic vector fields $\partial_1, \cdots, \partial_n$ form an orthonormal basis at the origin with respect to $ds^2_{T_\Omega}$. Then the matrix $(c_{ji}(Z))$ is a skew-Hermitian matrix for every element Z of $\mathfrak{k}$. Indeed, this follows from the fact that every automorphism of T_Ω is an isometry with respect to $ds^2_{T_\Omega}$.

2.2. Consequences of the Prolongation Theorem

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines, and suppose further that $e(T_\Omega) = 0$, or $\mathfrak{g}(T_\Omega)$ consists of all polynomial vector fields which are complete on T_Ω. Then every element Z of $\mathfrak{g}(T_\Omega)$ can be written in the form

$$(\sharp) \qquad Z = \sum_{k=0}^{\infty} Z^{(k)},$$

where $Z^{(k)}$ is a polynomial vector field whose components with respect to $\partial_1, \cdots, \partial_n$ are homogeneous polynomials in $z_1, \cdots, z_n$ of degree k. Note that, in ($\sharp$), only finitely many $Z^{(k)}$'s are not equal to zero. We may assume without loss of generality that T_Ω contains the origin, and that $\partial_1, \cdots, \partial_n$ form an orthonormal basis at the origin with respect to the Bergman metric $ds^2_{T_\Omega}$. Then ($\sharp$) gives the Taylor expansion of Z around the origin. For $k = 0, 1, 2, \cdots$, we write

$$Z^{(k)} = X^{(k)} + \sqrt{-1}Y^{(k)},$$

where $X^{(k)}, Y^{(k)}$ are polynomial vector fields whose components are homogeneous polynomials with real coefficients of degree k. We define real vector subspaces $\mathfrak{q}, \mathfrak{s}, \mathfrak{a}_*, \mathfrak{b}$ of $\mathfrak{g}(T_\Omega)$ by

$$\mathfrak{q} = \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = \sum_{k=0}^{2} Z^{(k)} = \sum_{k=0}^{2} \left(X^{(k)} + \sqrt{-1} Y^{(k)} \right) \right\},$$
$$\mathfrak{s} = \{\partial_1, \cdots, \partial_n\}_{\mathbf{R}},$$
$$\mathfrak{a}_* = \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = X^{(1)} + \sqrt{-1} Y^{(0)} \right\},$$
$$\mathfrak{b} = \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = X^{(2)} + \sqrt{-1} Y^{(1)} \right\}.$$

The Prolongation Theorem shows that $\mathfrak{q}$ has the direct sum decomposition

$$\mathfrak{q} = \mathfrak{s} + \mathfrak{a}_* + \mathfrak{b}.$$

Note that $\mathfrak{b}$ is contained in the isotropy subalgebra $\mathfrak{k}$ of $\mathfrak{g}(T_\Omega)$ at the origin. The following result on $\mathfrak{b}$ is useful for a further study of the structure of $\mathfrak{g}(T_\Omega)$.

Lemma 2 ([7, Section 4, Lemma 4.2]). *Let $Z = X^{(2)} + \sqrt{-1} Y^{(1)}$ be an element of $\mathfrak{b}$ and write*

$$Y^{(1)} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ji}(Z) z_i \right) \partial_j,$$

where $b_{ji}(Z)$, $j, i = 1, \cdots, n$, are real constants. Then the following hold.

i) $X^{(2)} = 0$ *if and only if* $Y^{(1)} = 0$.

ii) *The real n by n matrix $(b_{ji}(Z))$ is symmetric for every element Z of $\mathfrak{b}$.*

As a consequence of ii) of Lemma 2, it should be observed that, when $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{g}(T_\Omega)$, the matrices $(b_{ji}(Z))$, $Z \in \mathfrak{b}$, are simultaneously diagonalizable by a suitable orthogonal change of coordinates.

§3. An application of Lie group theory to the holomorphic equivalence problem for tube domains

The following result plays an important role in the study of the equivalence of Siegel domains.

Conjugacy Theorem (cf. Matsushima [1]). *Any two maximal triangular subalgebras of a real Lie algebra are conjugate to each other under an inner automorphism.*

As a consequence of this result, we obtain a useful observation on an application of Lie group theory to the holomorphic equivalence problem for tube domains. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two real Lie algebras. Consider a subalgebra $\mathfrak{s}_1$ of $\mathfrak{g}_1$ sucu that $ad\,X$ is nilpotent on $\mathfrak{g}_1$ for every $X \in \mathfrak{s}_1$. Then, in view of Engel's theorem, there exists a maximal triangular subalgebra $\mathfrak{t}_1$ of $\mathfrak{g}_1$ containing $\mathfrak{s}_1$. Similarly, consider a subalgebra $\mathfrak{s}_2$ of $\mathfrak{g}_2$ sucu that $ad\,X$ is nilpotent on $\mathfrak{g}_2$ for every $X \in \mathfrak{s}_2$, and let $\mathfrak{t}_2$ be a maximal triangular subalgebra of $\mathfrak{g}_2$ containing $\mathfrak{s}_2$. Note that $\mathfrak{s}_1$ is contained in the nilradical $\mathfrak{n}_1$ of $\mathfrak{t}_1$, while $\mathfrak{s}_2$ is contained in the nilradical $\mathfrak{n}_2$ of $\mathfrak{t}_2$. Suppose now that there is a Lie algebra isomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Since $\Phi(\mathfrak{t}_1)$ is a maximal triangular subalgebra of $\mathfrak{g}_2$, it follows from the Conjugacy Theorem that there exists an inner automorphism σ of $\mathfrak{g}_2$ such that $\sigma(\Phi(\mathfrak{t}_1)) = \mathfrak{t}_2$. Since $\sigma(\Phi(\mathfrak{n}_1)) = \mathfrak{n}_2$, we see that $\sigma(\Phi(\mathfrak{s}_1))$ and $\mathfrak{s}_2$ are subalgebras of $\mathfrak{n}_2$.

To apply the above observation to our study, let T_{Ω_1} and T_{Ω_2} be two tube domains in $\mathbf{C}^n$ with polynomial infinitesimal automorphisms, and set $\mathfrak{g}_1 = \mathfrak{g}(T_{\Omega_1})$ and $\mathfrak{g}_2 = \mathfrak{g}(T_{\Omega_2})$. Since $\mathfrak{g}(T_{\Omega_1})$ consists of polynomial vector fields, it follows that $ad\,X$ is nilpotent on $\mathfrak{g}(T_{\Omega_1})$ for every element X of the subalgebra $\mathrm{Lie}\,\Sigma_{T_{\Omega_1}}$ of $\mathfrak{g}(T_{\Omega_1})$ corresponding to $\Sigma_{T_{\Omega_1}}$. Therefore we can set $\mathfrak{s}_1 = \mathrm{Lie}\,\Sigma_{T_{\Omega_1}}$. Similarly, we can set $\mathfrak{s}_2 = \mathrm{Lie}\,\Sigma_{T_{\Omega_2}}$. Suppose now that T_{Ω_1} and T_{Ω_2} are holomorphically equivalent. The above observation shows that we can find a solvable subalgebra $\mathfrak{t}_1$ of $\mathfrak{g}_1$ containing $\mathfrak{s}_1$, a solvable subalgebra $\mathfrak{t}_2$ of $\mathfrak{g}_2$ containing $\mathfrak{s}_2$, and a biholomorphic mapping $\psi : T_{\Omega_1} \to T_{\Omega_2}$ between T_{Ω_1} and T_{Ω_2} such that $\Psi(\mathfrak{t}_1) = \mathfrak{t}_2$, where Ψ is a Lie algebra isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ given as the differential of the Lie group isomorphism $\mathrm{Aut}(T_{\Omega_1}) \ni g \mapsto \psi \circ g \circ \psi^{-1} \in \mathrm{Aut}(T_{\Omega_2})$. Note that both $\Psi(\mathfrak{s}_1)$ and $\mathfrak{s}_2$ are n-dimensional abelian subalgebras of the nilradical of the solvable Lie algebra $\mathfrak{t}_2$, and that if $\Psi(\mathfrak{s}_1)$ and $\mathfrak{s}_2$ are conjugate under an inner automorphism of $\mathfrak{g}_2$, then we can conclude by [3, Section 1, Proposition] that T_{Ω_1} and T_{Ω_2} are affinely equivalent. Thus the problem reduces to the investigation of certain solvable Lie algebras, and, as one direction to complete the story of our study, it seems to be important to study tube domains with solvable groups of automorphisms.

§4. A class of tube domains with solvable groups of automorphisms

Among tube domains with polynomial infinitesimal automorphisms, tube domains T_Ω whose bases Ω are convex cones are characteristic in the point that they have the property that if $\mathrm{Aut}(T_\Omega)$ is solvable, then the identity component of $\mathrm{Aut}(T_\Omega)$ necessarily consists of affine transformations. On the other hand, when Ω is an arbitrary convex domain in $\mathbf{R}^n$ containing no complete straight lines, there is a tube domain T_Ω in $\mathbf{C}^n$ such that $\mathrm{Aut}(T_\Omega)$ is solvable, but contains nonaffine automorphism, as the following theorem shows.

Theorem 1. *Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines and let $n \geq 2$. Assume that:*

i) *T_Ω is a tube domain with polynomial infinitesimal automorphisms;*
ii) *$\mathrm{Aut}(T_\Omega)$ is a solvable Lie group;*
iii) *T_Ω contains the origin of $\mathbf{C}^n$ and the orbit of $G(T_\Omega)$ through the origin has dimension $n+1$, where $G(T_\Omega) = \mathrm{Aut}(T_\Omega)^\circ$.*

Then, in the notation of Subsection 2.2, $\mathfrak{g}(T_\Omega)$ coincides with $\mathfrak{q}$. Moreover, according to the cases of a) *$\mathfrak{b} \neq \{0\}$ and* b) *$\mathfrak{b} = \{0\}$, the following hold.*

a) *One has $n \geq 3$ and, after a real linear change of coordinates in $\mathbf{C}^n$, $\mathfrak{a}_*$, $\mathfrak{b}$ and the nilradical $\mathfrak{n}$ of $\mathfrak{g}(T_\Omega)$ are given by*

$$\mathfrak{a}_* = \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}} + \mathfrak{k} \cap \mathfrak{a}_* \quad \textit{(direct sum)},$$
$$\mathfrak{b} = \{\sqrt{-1}z_1\partial_1 + z_1^2\partial_2\}_{\mathbf{R}},$$
$$\mathfrak{n} = \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}.$$

Also, any n-dimensional abelian subalgebra of $\mathfrak{n}$ is conjugate to $\mathfrak{s}$ by an inner automorphism of $\mathfrak{g}(T_\Omega)$.

b) *The nilradical $\mathfrak{n}$ of $\mathfrak{g}(T_\Omega)$ has dimension less than or equal to $n+1$. Also, any n-dimensional abelian subalgebra of $\mathfrak{n}$ coincides with $\mathfrak{s}$.*

Combining this structure theorem with the observation given in Section 3, we can give an answer to the holomorphic equivalence problem for a class of tube domains with solvable groups of automorphisms.

Theorem 2. *Let T_{Ω_1} and T_{Ω_2} be two tube domains in $\mathbf{C}^n$ whose bases Ω_1 and Ω_2 are convex domains in $\mathbf{R}^n$ containing no complete straight lines and let $n \geq 2$. Assume that:*

i) *T_{Ω_1} and T_{Ω_2} are tube domains with polynomial infinitesimal automorphisms;*
ii) *$\mathrm{Aut}(T_{\Omega_1})$ is a solvable Lie group;*
iii) *There exists a point z_0 of T_{Ω_1} such that the orbit of $G(T_{\Omega_1})$ through z_0 has dimension $n+1$.*

Under these assumptions, if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then they are affinely equivalent.

References

[1] Y. Matsushima, On tube domains, in "Symmetric Spaces", Pure and Appl. Math., Vol. 8, Dekker, New York, 1972, pp. 255–270.

[2] R. Narasimhan, "Several Complex Variables", Univ. of Chicago Press, Chicago, London, 1971.

[3] S. Shimizu, Automorphisms of tube domains, in "Geometry and Analysis on Complex Manifolds", Festschrift for Professor S. Kobayashi's 60th Birthday (T. Mabuchi, J. Noguchi, T. Ochiai, ed.), World Scientific, Singapore, New Jersey, London, Hong Kong, 1994, pp. 198–241.

[4] S. Shimizu, Holomorphic equivalence problem for two-dimensional tube domains with polynomial infinitesimal automorphisms, in "Geometric Complex Analysis", Proceedings of the third international research institute, the Mathematical Society of Japan, Hayama, 1995 (J. Noguchi, H. Fujimoto, J. Kajiwara, T. Ohsawa, ed.), World Scientific, Singapore, New Jersey, London, Hong Kong, 1996, pp. 563–568.

[5] S. Shimizu, Automorphisms and equivalence of tube domains with bounded base, Math. Ann., **315** (1999), 295–320.

[6] S. Shimizu, A classification of two-dimensional tube domains, Amer. J. Math., **122** (2000), 1289–1308.

[7] S. Shimizu, Prolongation of holomorphic vector fields on a tube domain. preprint

[8] S. Shimizu, Holomorphic equivalence problem for two-dimensional tube domains with polynomial infinitesimal automorphisms. in preparation

[9] S. Shimizu, Structure and equivalence of a class of tube domains with solvable groups of automorphisms. in preparation

Mathematical Institute
Tohoku University
Sendai, 980-8578
Japan