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The monodromy covering of the versal deformation of cyclic quotient surface singularities

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Abstract.

We give a short survey on some new (and old) results on deformations of cyclic quotient surface singularities which are mainly contained in the doctoral thesis of STEPHAN BROHME.

§1. Introduction

By studying the special case of cyclic quotient surface singularities several general aspects of deformation theory of complex–analytic singularities have been detected, e. g. the existence of many components of the base space of the versal deformation (which we also call the *versal base space* for short) and their monodromy coverings and the existence of embedded components. Even more: the (necessarily) smooth reduced components, the deformations thereon including the discriminant and the adjacencies and the monodromy coverings can be explicitly described and are very well understood (c. f. [4], [5], [6], [7], [8], [11], [16]; see also [2]).

The versal base space itself has - in the first interesting case of embedding dimension $e = 5$ - quite simple equations ([13], [14]). Later, ARNDT [1] calculated those equations for embedding dimension 6 and gave a "quasi–algorithmic" structure theorem for the general case (see also [9] and [10] for another approach for the much wider classes of *rational surface singularities with reduced fundamental cycle* and *sandwiched singularities*). In his dissertation, BROHME [6] proposes an explicit algorithm to produce equations in the cyclic case which are closer related to the continued fractions than those given in [9] and proves that his algorithm really leads to correct equations up to embedding dimension 8. It should also be mentioned that MIYAJIMA [12] has done some calculations on the versal deformation space by means of the deformation theory of CR–structures.

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However, all these sets of equations are extremely complicated (therefore, they are not reproduced here due to lack of space). In particular, it is almost impossible to draw any geometric conclusions from them. Despite the beautiful "picture method" of DE JONG and VAN STRATEN, there was in my opinion a "satisfactory" construction of the versal deformation - in the case of cyclic quotients - in terms of combinatorics, i. e. in terms of the *continued fraction* associated to such a singularity still missing. In order to remedy this unpleasant situation, I sketched in August 1996 an explicit construction of (a finite covering of) the *reduced* versal deformation space (the main idea is already contained in [15]). In the following I shall state the result after some preparatory notions and remarks; a proof is contained in [6]. Due to explicit computer algebra calculations via *Singular* in small embedding dimensions with the help of Brohme's equations, I am convinced that also the embedded components can successfully be "attached" to this construction.

This work would not have been possible without the pioneering work of JAN CHRISTOPHERSEN, JAN STEVENS and KURT BEHNKE on the component structure of the deformation space of the cyclic quotients.

§2. Some notions

Recall that a quotient surface singularity is given by natural numbers n, q with $1 \leq q < n$ and $\gcd(n, q) = 1$ which determine the singularity $X_{n,q}$ as the quotient of $\mathbb{C}^2$ by the linear action of the group $C_{n,q} \subset \mathrm{GL}(2, \mathbb{C})$ generated by the diagonal matrix $\mathrm{diag}(\zeta_n, \zeta_n^q)$ where ζ_n denotes a primitive n–th root of unity. It is well–known that all quotients of $\mathbb{C}^2$ by a finite cyclic group are of this form (up to analytic isomorphism), and $X_{n,q} \cong X_{n,q'}$ if and only if $q = q'$ or $qq' \equiv 1$. Moreover, the embedding dimension $e = e_{n,q} = \mathrm{emb}\, X_{n,q}$ is equal to

$$e_{n,q} = 3 + \sum_{k=1}^{\ell} (b_k - 2)$$

with the coefficients b_k of the Hirzebruch–Jung continued fraction expansion

$$\frac{n}{q} = b_1 - \underline{1 \rfloor b_2} - \cdots - \underline{1 \rfloor b_\ell}\,, \quad b_k \geq 2$$

or, resp., $e_{n,q} = r + 2$, $r = r_{n,q}$ the codimension of $X_{n,q}$, where

$$\frac{n}{n-q} = a_1 - \underline{1 \rfloor a_2} - \cdots - \underline{1 \rfloor a_r}\,, \quad a_j \geq 2\,.$$

(Note that we changed our notations of [14] in accordance with the work of JAN CHRISTOPHERSEN [7] and JAN STEVENS [16]). In other words, the system $(a_1, \ldots , a_r)$ of exponents (as well as the system of selfintersection numbers $b_1, \ldots , b_\ell$) is an analytic invariant of the singularity up to reversal of the order.

The $r\,(r+1)/\,2$ equations for the singularity $X_{n,q}$ can be written down with these exponents in *quasideterminantal* form (see e. g. [15]). For our construction, the r *leading* equations

$$x_0\, x_2 - x_1^{a_1} = 0\,, \quad x_1\, x_3 - x_2^{a_2} = 0, \ldots , x_{r-1}\, x_{r+1} - x_r^{a_r} = 0$$

are of special importance as well as the *last* one:

$$x_0\, x_{r+1} - x_1^{a_1-1}\, x_2^{a_2-2} \cdot \ldots \cdot x_{r-1}^{a_{r-1}-2}\, x_r^{a_r-1} = 0\,.$$

It follows from the work of Christophersen and Stevens that for fixed $r \geq 2$ there exist only finitely many so-called *r–chains* (representing zero) $(k_1, \ldots , k_r) \in \mathbb{N}_+^r$ such that the reduced components of the versal deformation space of a cyclic quotient surface singularity $X_{n,q}$ of codimension r are in $1:1$ correspondence to those r–chains $\underline{k} = (k_1, \ldots , k_r)$ satisfying $\underline{k} \leq \underline{a} := (a_1, \ldots , a_r)$, i. e.

$$k_j \leq a_j\,, \quad j = 1, \ldots , r\,.$$

Before we proceed further we recall the definition of r–chains $\underline{k}$ by Christophersen. Define $\alpha_0 = 0\,,\ \alpha_1 = 1$ and inductively $\alpha_{j+1} = k_j\, \alpha_j - \alpha_{j-1}\,,\ j = 1, \ldots , r$. Then $\underline{k}$ is an r–chain if $\alpha_j \geq 1\,,\ j = 1, \ldots , r$, and $\alpha_{r+1} = 0$. This is equivalent to saying that the continued fraction

$$k_1 - \underline{\ 1\ }\!\rfloor\overline{\,k_2\,} - \cdots - \underline{\ 1\ }\!\rfloor\overline{\,k_r\,}$$

is well defined and has the value 0. Let us list here all these chains for the cases $r = 2,\ 3,\ 4$ together with their corresponding α–series which are also necessary for understanding the construction.

r	$\underline{k} = (k_1, \ldots , k_r)$	$\underline{\alpha} = (\alpha_1, \ldots , \alpha_r)$
2	(1, 1)	(1, 1)
3	(1, 2, 1)	(1, 1, 1)
	(2, 1, 2)	(1, 2, 1)
4	(1, 2, 2, 1)	(1, 1, 1, 1)
	(1, 3, 1, 2)	(1, 1, 2, 1)
	(2, 1, 3, 1)	(1, 2, 1, 1)
	(2, 2, 1, 3)	(1, 2, 3, 1)
	(3, 1, 2, 2)	(1, 3, 2, 1)

We will write K_r for the set of r–chains; its cardinality is the famous CATALAN number
$$\frac{1}{r}\binom{2(r-1)}{r-1}.$$

To each $\underline{k} \in K_r$ we can associate a certain "cross and circle" diagram $\nabla_{\underline{k}}$ which can be used as a *format* for a cyclic quotient surface singularity $X_{n,q}$ of codimension r and exponents $\underline{a} = (a_1, \dots, a_r)$: if $\underline{k} \leq \underline{a}$ holds, then $\nabla_{\underline{k}}$ determines in a completely algorithmic manner a system of equations $P_{ij}^{\nabla_{\underline{k}}}$, $0 \leq i, j \leq r+1$, $i+1 < j$ with $P_{i-1,i+1}^{\nabla_{\underline{k}}}$, $i = 1, \dots, r$, always being the leading equations as above (for more details, see [5] and [15]). The *last* equation is of the form $P_{0,r+2}^{\nabla_{\underline{k}}} = x_0\, x_r - x_1^{\alpha_1 (a_1 - k_1)} \cdot \ldots \cdot x_r^{\alpha_r (a_r - k_r)}$. Moreover, for different $\underline{k}$, the last equations are also different. The quasideterminantal format belongs to the r–chain $(1, 2, 2, \dots, 2, 1)$.

§3. The construction

We first describe the main features of our construction in the special case of cyclic *double points* and then in general. The situation for the A_{n-1}–singularity is extremely simple. It can be described by the linear action on $\mathbb{C}^2$ of the subgroup $C_{n,n-1} \subset \mathrm{SL}(2, \mathbb{C})$ which is generated by the diagonal matrix $\mathrm{diag}(\zeta_n, \zeta_n^{-1})$. Since u^n, uv, v^n are generating polynomials of the invariant ring $\mathbb{C}[u, v]^{C_{n,n-1}}$, the singularity is given by the equation
$$x_0\, x_2 = x_1^n \quad \text{in} \quad \mathbb{C}^3 .$$

To find a nice family $\mathcal{Y} \to T$ we replace the polynomial on the righthand side with a *generic* product of linear factors:

$$(*) \qquad x_0\, x_2 = (x_1 + t_1) \cdot \ldots \cdot (x_1 + t_n) .$$

Interpreting this equation as giving a hypersurface $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^n$, the projection to the second factor $T = \mathbb{C}^n$ yields an n–parameter deformation $\mathcal{Y} \to T$ of $X_{n,n-1}$ on which the symmetric group $\mathfrak{S}_n$ on n symbols acts. Dividing out the action of $\mathfrak{S}_n$, we get the deformation

$$(**) \quad x_0\, x_2 = x_1^n + s_1\, x_1^{n-1} + \cdots + s_n , \quad (s_1, \dots, s_n) \in S = \mathbb{C}^n ,$$

where $s_j = s_j(t_1, \dots, t_n)$ denotes the j^{th} elementary symmetric function in the elements $t_1, \dots, t_n$, e. g. $s_1 = t_1 + \cdots + t_n, \dots, s_n = t_1 \cdot \ldots \cdot t_n$. It is well–known that restriction of $(*)$ to the hyperplane $H = \{t_1 + \cdots + t_n = 0\}$ gives the (minimal) versal deformation $x_0\, x_2 = x_1^n + s_2\, x_1^{n-2} + \cdots + s_n$, and it is easily checked that the stabilizer subgroup of $\mathfrak{S}_n$ on H is isomorphic to $\mathfrak{S}_{n-1}$, the Weyl group of A_{n-1}–type playing here the role of the monodromy group.

The lesson to be learned by this example is not to try to construct a *minimal* family from the beginning. In fact, our base space $T_{n,q}$ for general cyclic quotients $X_{n,q}$ will be too large; but it is canonically a product of vector spaces, and minimizing the family means just to restrict to hyperplanes as above in some or all of these vector spaces.

We now explain our Ansatz. We make the leading equations completely generic in a fully symmetric way by taking the risk to not getting the minimal family (former attempts sacrificed the symmetry because of minimality and got lost in a not manageable mess of unnecessary conditions). To be more precise, we start with equations of type

$$\begin{aligned} x_0\,(x_2+t_2^{(r)}) &= (x_1+t_1^{(1)})\cdot(x_1+t_1^{(2)})\cdot\ldots\cdot(x_1+t_1^{(a_1)}) =: X_1^{(a_1)},\\ (x_1+t_1^{(\ell)})\,(x_3+t_3^{(r)}) &= (x_2+t_2^{(1)})\cdot\ldots\cdot(x_2+t_2^{(a_2)}) = X_2^{(a_2)},\\ &\vdots\\ (x_{r-2}+t_{r-2}^{(\ell)})\,(x_r+t_r^{(r)}) &= X_{r-1}^{(a_{r-1})},\\ (x_{r-1}+t_{r-1}^{(\ell)})\,x_{r+1} &= X_r^{(a_r)}\,. \end{aligned}$$

Here, of course, the upper indices (r) and (ℓ) are standing for "right" and "left" (not to be confused with the numbers r and ℓ). For inductive reasons one even should x_0 and x_{r+1} replace by $x_0+t_0^{(\ell)}$ and $x_{r+1}+t_{r+1}^{(r)}$, resp. In order to minimalize we have later to put again $t_0^{(\ell)}=t_{r+1}^{(r)}=0$ and $\sum_{k=1}^{a_j}t_j^{(k)}=0,\ j=1,\ldots,r$. Concerning the Weyl group or monodromy group, we introduce $W:=W_1\times\cdots\times W_r$, where $W_j\cong\mathfrak{S}_{a_j}$ denotes the symmetric group on a_j elements acting on the variables $t_j^{(1)},\ldots,t_j^{(a_j)}$ by permutation (and on the others including $t_j^{(r)}$, $t_j^{(\ell)}$, if existing, trivially).

Our goal is to construct a W-invariant deformation of $X_{n,q}$ over a subspace of the vector space of all t-parameters. In order to do so, we follow formally for all r-chains $\underline{k}\le\underline{a}$ the "pattern" of the format $\nabla_{\underline{k}}$. This leads to *meromorphic* equations. More precisely, it will turn out that to each $\underline{k}$ there correspond further r-tuples $\underline{\lambda}=(\lambda_1,\ldots,\lambda_r)$, $\underline{\rho}=(\rho_1,\ldots,\rho_r)$ with $\lambda_r=0$, $\rho_1=0$ independently of $\underline{a}$ such that the last equation becomes

$$(x_0+t_0^{(\ell)})\,(x_{r+1}+t_{r+1}^{(r)}) = \frac{X_1^{(a_1)\alpha_1}}{(x_1+t_1^{(\ell)})^{\lambda_1}}\cdot\frac{X_2^{(a_2)\alpha_2}}{(x_2+t_2^{(\ell)})^{\lambda_2}\,(x_2+t_2^{(r)})^{\rho_2}}\cdot$$
$$\cdots\frac{X_{r-1}^{(a_{r-1})\alpha_{r-1}}}{(x_{r-1}+t_{r-1}^{(\ell)})^{\lambda_{r-1}}\,(x_{r-1}+t_{r-1}^{(r)})^{\rho_{r-1}}}\cdot\frac{X_r^{(a_r)\alpha_j}}{(x_r+t_r^{(r)})^{\rho_r}}\,.$$

We now put

$$t_1^{(\ell)} = t_1^{(1)} = \cdots = t_1^{(\lambda_1)} \quad \text{and} \quad t_r^{(r)} = t_r^{(1)} = \cdots = t_r^{(\rho_r)}$$

or correspondingly with all other combinations of equations we get by the action of $W_1 \times \cdots \times W_r$ on the righthand side. In the middle terms $2 \leq j \leq r-1$, we set $t_j^{(r)} = t_j^{(\ell)} = t_j^{(1)} = \cdots = t_j^{(\lambda_j+\rho_j)}$ or etc. for $\alpha_j = 1$; for $\alpha_j > 1$, we can choose $t_j^{(r)}$ and $t_j^{(\ell)}$ independently as before.

It is easily seen that all equations of type $\nabla_{\underline{k}}$, not only the last one, are then in fact *holomorphic* on the corresponding linear subspaces since the exponents λ_j and ρ_j satisfy sufficiently good properties.

§4. The main result

By the construction of the preceding section, we can attach to any cyclic quotient surface singularities $X = X_{n,q}$ a (reduced) subspace

$$T = T_{n,q} \subset \mathbb{C}^N, \quad N = N_{n,q},$$

consisting of a huge bunch of linear subspaces on which a subgroup $W = W_{n,q}$ of the symmetric group $\mathfrak{S}_N$ acts in a canonical way such that the following is satisfied (for details, see [6]).

i) On each component T' of T there lives a canonical deformation $\mathcal{Y}'$ of X;
ii) for two such components T', T'' these deformations $\mathcal{Y}'$, $\mathcal{Y}''$ coincide on the intersection $T' \cap T''$ thus defining a deformation

$$\mathcal{Y} = \mathcal{Y}_{n,q} = \bigcup \mathcal{Y}' \longrightarrow T,$$

iii) W acts equivariantly in a canonical way on $\mathcal{Y} \to T$;
iv) if $W' = W'_{n,q}$ denotes the stabilizer subgroup of W on a component T' of T, then W' acts as a reflection group such that

$$S' = T'/W'$$

is a smooth component of $S := T/W$, and W' acts also on $\mathcal{Y}' = \mathcal{Y}|T' \to T'$ equivariantly, inducing a deformation $\mathcal{X}' = \mathcal{Y}'/W' \to T'/W' = S'$;
v) each component S' is a component of the (reduced) base space of $X_{n,q}$, and all of these appear precisely once such that $\mathcal{X} \to S := T/W$ is the (reduced) versal deformation of $X_{n,q}$.

Remarks. 1. If the exponents a_j are big enough, the base space of the versal deformation of $X_{n,q}$ is *stable*, i. e. a product of a fixed space, depending only on the embedding dimension, with a smooth factor, as is well–known by the work of THEO DE JONG and DUCO VAN STRATEN [10] (the conditions $a_j \geq r-1$, $j = 1, \ldots, r$, should suffice). Hence, in these cases we have the maximal number of irreducible components.
2. On each component $\mathcal{Y}'$, the quotient mapping $\mathcal{Y}' \to \mathcal{Y}'/W' = \mathcal{X}'$ is the monodromy covering of $\mathcal{X}'$ in the sense of BEHNKE and CHRISTOPHERSEN [4]. Hence one may call the family $\mathcal{Y} \to T$ the *monodromy covering* of the versal deformation $\mathcal{X} \to S$ with *monodromy group* W. It is quite unclear to which extend the existence of such a family is a special feature of the cyclic quotient singularities only.
3. The highly symmetric "Ansatz" which is leading to our family is also interesting and promising with respect to other aspects of (cyclic) quotient surface singularities. It should, e. g. help to put the *toric structures* on the components together in an intelligent manner.

§5. Embedded components

With his equations, Brohme was able to carry out some calculations with *Singular*; e. g. for $e = 7$ and the (generic) exponents $(4, 4, 4, 4, 4)$, there are 11 extra *embedded* components in addition to the 14 reduced ones, 8 of them "supported" on the Artin component, 3 on other components of highest dimension. For smaller exponents there are in general fewer embedded components. It turns out that the result has a combinatorial description, too. One has to regard the following 5–chains:

$$\begin{gathered}(2, 2, 2, 2, 2),\\ (1, 3, 2, 2, 2), (3, 1, 3, 2, 2), (2, 3, 1, 3, 2), (2, 2, 3, 1, 3), (2, 2, 2, 3, 1),\\ (3, 2, 2, 2, 2), (2, 3, 2, 2, 2), (2, 2, 3, 2, 2), (2, 2, 2, 3, 2), (2, 2, 2, 2, 3).\end{gathered}$$

Then embedded components correspond to chains which are smaller than the sequence of the a_j and are supported (on the monodromy covering) on easily describable *linear* subspaces of nonembedded components.

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