Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 189–202

The $\bar\partial$ equation in N variables, as N varies

László Lempert

§1. Introduction

In this work we shall be concerned with solving the $\bar\partial$ equation in N dimensional balls, and the emphasis will be on understanding how the control that we have on the sup norm of the solution depends on the number of variables. The primary motivation for this line of research comes from the infinite dimensional theory of the $\bar\partial$ equation. Indeed, if it turns out that solutions of the N dimensional $\bar\partial$ equation can be estimated independently of N, one should expect that by passing to some limit a solution of the infinite dimensional $\bar\partial$ equation will be obtained as well. More on this later. However, our topic of the day is also related, perhaps only in spirit, to other areas of mathematics and beyond, where one studies systems with a large number N of degrees of freedom and investigates how properties of the system change as $N \to \infty$. One example would be statistical physics, another algorithmic complexity.

In the next section of the present work we first review the relevant estimates for the $\bar\partial$ equation available in the literature. None of them is known to be optimal; on the other hand they all involve N exponentially. In fact, exponential dependence on the dimension seems to be the rule in analysis and geometry, even beyond the theory of the $\bar\partial$ equation. This will be discussed at some length in section 2. Nevertheless we shall find one instance (Theorems 2.1 and 2.2) when the exponentially diverging estimates can be converted into dimension free estimates. As a consequence we obtain that on the level of $(0,1)$ forms the equation $\bar\partial u = f$ is solvable in pseudoconvex open subsets of the Banach space l^1 of summable sequences. This was already proved in [L1,2] for local resp. global solvability. Our treatment here does overlap with that of [L1], but is simpler. In addition, it gives a stronger result: in Theorem 4.2 the regularity assumption on f is weaker than Hölder continuity, while [L1] dealt with Lipschitz continuous f. This stronger result is sharp in that

Received February 16, 2002
Research partially supported by an NSF grant.

in l^1 mere continuity of f is not sufficient for the solvability of $\bar\partial u = f$, see [L1, Theorem 9.1].

Acknowledgement. I am grateful to the referee for his careful reading of this paper and for helpful suggestions.

§2. The estimates

2.1. Rather than studying the $\bar\partial$ equation just in Euclidean balls, we fix $p \in [1,\infty)$ and consider

$$B_{N,p}(R) = B_N(R) = \{z \in \mathbb{C}^N : \|z\|_p < R\}, \text{ where}$$

$$\|z\|_p = \|z\| = \left(\sum_{\nu=1}^{N} |z_\nu|^p\right)^{1/p}, \qquad z = (z_\nu).$$

Given a $k \in [0,\infty)$, $r \in (0,1]$ and a closed form $f \in C^k_{0,1}(B_N(1))$, we want to solve the equation

$$\bar\partial u = f|B_N(r) \tag{2.1}$$

with estimate

$$|u|_{C^0(B_N(r))} \le c_N |f|_{C^k(B_N(1))}, \tag{2.2}$$

where c_N is independent of f, but may depend on p, k, r—that we think of as fixed—, and of course on N. The norm on the left hand side of (2.2) is $\sup_{B_N(r)} |u|$. The more general C^k norms on the right must be defined with a little care, since various seemingly natural choices behave somewhat differently as $N \to \infty$. The correct definition is gotten by using the Banach space structure of $(\mathbb{C}^N, \|\ \ \|_p)$ only, ignoring coordinates. Thus, when $(X, \|\ \ \|)$ is any Banach space and $\Omega \subset X$ is open, for $0 < k < 1$ and $u : \Omega \to \mathbb{C}$ one writes

$$|u|_{C^k(\Omega)} = \sup_\Omega |u| + \sup_{z \ne \zeta \in \Omega} \frac{|u(z) - u(\zeta)|}{\|z - \zeta\|^k}.$$

For $k \ge 1$, $|u|_{C^k(\Omega)}$ is defined inductively: one thinks of du as a function on $\Omega \times B$, $B \subset X$ the unit ball, and sets $|u|_{C^k(\Omega)} = \sup_\Omega |u| + |du|_{C^{k-1}(\Omega \times B)}$. Similarly, a 1-form f on Ω is a function on $\Omega \times B$, and the $C^k(\Omega \times B)$ norm of this function is what is meant by $|f|_{C^k(\Omega)}$.

Back to (2.1), (2.2), the question is how c_N depends on N—the hope being that it does not. There are various ways to solve (2.1) with estimates: the Hilbert space methods of Hörmander or, in case of smooth boundary, of Kohn; and integral formulas. Integral formulas of Grauert-Lieb, Henkin, Øvrelid, and others directly estimate $|u|_{C^0(B_N(1))}$, especially in the strongly pseudoconvex case $p = 2$, while Hörmander and Kohn only estimate the $L^2(B_N(1))$ norm of a solution, which then has to be converted into sup norm on smaller balls $B_N(r)$, $r < 1$. When one works one's way through the constants that occur, all the above methods give $c_N \approx \gamma^N$ with $\gamma = \gamma(p,k,r) > 1$ for $r < 1$. (For infinite dimensional applications it suffices to consider arbitrarily small but fixed $r > 0$. However, it is of some interest to see what happens to $\gamma(p,k,r)$ as $r \to 1$. The Hilbert space methods yield $\gamma(p,k,r)$ that blows up as $(1-r)^{-1}$, while integral formulas, at least some of the time, yield $\gamma(p,k,r)$ that is uniformly bounded. For example one can take $\gamma(p,k,r) = 2$ when $p = 1$ or 2.)

2.2. Now an exponentially diverging c_N is not what we were after, but it is noteworthy that three different methods and their variants all produce such constants in (2.2). In fact, looking even beyond the theory of the $\bar\partial$ equation it seems that the natural place for the number of variables is in the exponent. A host of examples suggests the following general if vague principle: *In geometrical and analytical results the number of dimensions appears in the exponent, as c^N (or not at all, if $c = 1$).*

Here are some instances of this principle.

1° Scaling of volume in N dimensions, probably the source of all other examples: if $D \subset \mathbb{R}^N$ and $\lambda > 0$ then $\mathrm{Vol}(\lambda D) = \lambda^N \mathrm{Vol}\, D$.

2° The singularity of the harmonic Green function in N dimensions

$$G(x,y) \sim \mathrm{const}|x-y|^{2-N}, \qquad x \to y.$$

3° Weyl's law for the number $s(x)$ of eigenvalues $< x$ of the Laplacian on a compact N-dimensional Riemannian manifold: $s(x) \sim \mathrm{const}\; x^{N/2}$, $x \to \infty$.

4° With $L \to X$ a holomorphic line bundle over a compact base, the Euler characteristic $\chi(L^{\otimes m})$ is a polynomial in m of degree $\le N = \dim X$.

5° Sobolev's embedding theorem $W^{m,p}(\mathbb{R}^N) \subset C(\mathbb{R}^N)$, provided $m > N/p$. Here it takes a little arguing to get N in the exponent. For instance, when $p = 2$, the Sobolev space $W^{m,p}$ for the critical value $m = N/2$ consists of those $f \in L^2(\mathbb{R}^N)$ whose Fourier transform $\hat f$ satisfies

$$\int |\hat f(\xi)|^2 (1+|\xi|^2)^{N/2} d\xi < \infty :$$

N indeed appears exponentially.

There are many more examples, but counterexamples as well. One counterexample we have just glossed over occurs in 2° above. Indeed, the constant there also depends on N: its expression contains $\Gamma(N/2-1)$, in addition to N in an exponent. This in itself is nothing to seriously worry about, though. The occurence of $\Gamma(N/2-1)$ has to do with the particular normalization of the translation invariant measure one uses in $\mathbb{R}^N$, so that a different normalization would lead to const $\equiv 1$. This little manipulation, however, exposes the fact that the ratio of the volumes of the unit ball and the unit cube in $\mathbb{R}^N$ also contains N inside the Γ function, an exception to the principle formulated above that should be taken more seriously.

To sum up: even if the dimension does not always appear in the exponent, it seems to do so extensively. This phenomenon definitely deserves some explanation. It indicates that dimensional dependence is subject to generals laws that should be uncovered and analyzed. The analysis in the present paper is of this kind, in the context of the $\bar\partial$ equation. We shall show that in one instance it is possible to start with exponentially diverging c_N in (2.1), (2.2), and convert this into a dimension independent estimate by means of some rather soft analysis.

2.3. The main result is

Theorem 2.1. *Let $p = 1$. Given $k > 0$ there is a number a such that for any N and any closed $f \in C^k_{0,1}(B_N(1))$ equation (2.1) has a solution u satisfying*

$$|u|_{C^0(B_N(r))} \leq a|f|_{C^k(B_N(1))}, \tag{2.3}$$

provided $r = 10^{-3}$.

Once (2.3) is known, it is routine to improve it to a similar estimate of $|u|_{C^k(B_N(r))}$, or even $|u|_{C^{k+1}(B_N(r))}$ when $k \notin \mathbb{N}$, at the price of scaling a and r by a dimension independent factor. In some ways Theorem 2.1 is sharp. It would not hold when $k = 0$, nor would it hold for all $k > 0$ if $p > 1$ (the proof of [L1, Theorem 9.1] shows both). On the other hand, it might very well be true for arbitrary p and $k + 1 > \lceil p \rceil$ (= the least integer $\geq p$).

However, there is a norm better suited to the problem than Hölder norms C^k, which we now proceed to define. Let $D \subset \mathbb{C}^N$ be a bounded domain, with $(x, y) \in \mathbb{C}^N \times \mathbb{C}^N$ associate the map

$$\varphi_{xy} : \overline{B_1(1)} \ni s \mapsto x + sy \in \mathbb{C}^N, \tag{2.4}$$

and let $\Omega = \{(x,y) : \varphi_{xy}(\overline{B_1(1)}) \subset D\}$. Given $f \in C^0_{0,1}(D)$, for each $(x,y) \in \Omega$ try to solve the equation $\bar\partial v_{xy} = \varphi^*_{xy} f$. If this can be done with $v_{xy} \in C^1(B_1(1))$ depending continuously on x, y, put

$$[f]_D = |f|_{C^0(D)} + \inf_{\{v_{xy}\}} \sup\{\|y\|^{-1}|v_{xy}|_{C^1(B_1(1))} : (x,y) \in \Omega,\ y \neq 0\},$$

the inf taken over all families $\{v_{xy}\}$ as above. Otherwise define $[f]_D = \infty$. This norm transforms simply under affine maps $\alpha : \mathbb{C}^n \to \mathbb{C}^N$ of form $\alpha(x) = Ax + b$, A linear and injective:

$$[\alpha^* f]_{\alpha^{-1}D} \leq \|A\| [f]_D. \tag{2.5}$$

Here $\|A\|$ is the operator norm of A induced by the l^1 norms on $\mathbb{C}^n$, $\mathbb{C}^N$. To verify (2.5) note that $\alpha \circ \varphi_{xy} = \varphi_{\alpha(x),Ay}$. Hence with the family v_{xy} in the definition of $[f]_D$, $w_{xy} = v_{\alpha(x),Ay}$ will be a corresponding family for $\alpha^* f$. Since

$$\|y\|^{-1}|w_{xy}|_{C^1(B_1(1))} \leq \|A\|\ \|Ay\|^{-1}|v_{\alpha(x),Ay}|_{C^1(B_1(1))},$$

and moreover

$$|\alpha^* f|_{C^0(\alpha^{-1}D)} \leq \|A\|\ |f|_{C^0(D)},$$

(2.5) follows. In particular, (2.5) applied with homotheties α shows that $[f]$ is homogeneous of order 1, i.e. (diam $D)[f]_D$ is scale invariant. The significance of this norm is that $[\bar\partial u]_D < \infty$ implies u is (locally) C^1, as one easily shows using one variable Cauchy representations for the holomorphic function $\varphi^*_{xy} u - v_{xy}$.

If f is a Hölder continuous form then

$$[f]_{B_N(R)} \leq \text{const}|f|_{C^k(B_N(R))}, \qquad k > 0,$$

with dimension independent constant, since an admissible v_{xy} can be gotten by taking the Cauchy transform of (a C^k extension of) $\varphi^*_{xy} f$. Therefore Theorem 2.1 follows from

Theorem 2.2. *Let $p = 1$. There is a constant a such that for all closed $f \in C^0_{0,1}(B_N(1))$ (2.1) has a solution u with*

$$|u|_{C^0(B_N(r))} \leq a[f]_{B_N(1)}, \quad r = 10^{-3}. \tag{2.6}$$

Moreover, u can be chosen to depend linearly on f.

As explained above, u will be C^1 when the right hand side of (2.6) is finite. Conversely, if $\bar\partial u = f$ has a solution $u \in C^1(B_N(1))$ then $[f]_{B_N(1)} < \infty$: indeed, one can take $v_{xy} = \varphi^*_{xy} u - u(x)$.

Since from this point on only C^0 norms will matter, we shall abbreviate $|\ |_{C^0(D)} = |\ |_D$. We shall also drop the superscript from $C^0(D)$, $C^0_{0,1}(D)$. Finally, we shall write $B_N = B_N(1)$.

§3. Proofs

3.1. To prove Theorem 2.2 we shall start with the exponentially diverging estimate (2.2), where $c_N = \gamma^N$, and, as promised, we shall convert it into a dimension independent estimate. While standard by now, the proof of (2.2) is not easy: whether derived by Hilbert space techniques or by Cauchy–Fantappiè formulas, it requires serious analysis. In comparison, conversion to a dimension independent estimate will be smooth sailing, involving some combinatorics and some routine analysis of the one dimensional $\bar\partial$ operator. The only nonstandard analytical component concerns a certain property of holomorphic functions in $B_N(R)$, to which we now turn.

For the rest of the paper, $p = 1$. Let $\#z$ denote the number of nonzero coordinates of $z \in \mathbb{C}^N$.

Theorem 3.1. *Suppose* $h \in \mathcal{O}(B_N(R))$ *satisfies* $|h(z)| \le q^{\#z}$ *with some* $q > 1$. *Then*

$$|h(z)| \le \frac{R}{R - eq\|z\|}, \qquad \text{if} \quad eq\|z\| < R. \tag{3.1}$$

It is here that an estimate, exponential in dimension, is turned into a dimension independent one. Indeed, the assumption means that $\sup_P |h| \le q^{\dim P}$ for each coordinate plane P; and one concludes that near 0 $h(z)$ can be bounded irrespective of the dimension of the coordinate plane in which z sits.

Proof. We shall assume $R = 1$; the general case will then follow by a substitution $z = Rz'$. Expand h in a homogeneous series $\sum h_m$, where

$$h_m(z) = \int_0^1 h(e^{2\pi i t} z) e^{-2\pi i m t}\, dt, \qquad m = 0, 1, \ldots.$$

Clearly $|h_m(z)| \le q^{\#z}$ if $z \in B_N$. With each h_m associate the symmetric m-linear form

$$P_m(z^1, \ldots, z^m) = \frac{1}{2^m m!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \ldots \epsilon_m h_m\Big(\sum_{j=1}^m \epsilon_j z^j\Big), \tag{3.2}$$

then $h_m(z) = P_m(z, \dots, z)$. If each z^j is a possibly rotated basis vector of form $(0, \dots, e^{i\theta}, \dots, 0)$ and $z = \sum \epsilon_j z^j$, then $\|z\|$, $\#z \le m$. Hence $|h_m(z)| \le q^m m^m$ and (3.2) implies

$$|P_m(z^1, \dots, z^m)| \le q^m m^m / m! \le e^m q^m.$$

The same must hold if each z^j is in the convex hull of rotated basis vectors, i.e. whenever $\|z^j\| \le 1$. (It is here that $p = 1$ is essential.) This in turn implies $|h_m(z)| \le e^m q^m \|z\|^m$, and (3.1) follows.

The theorem would be outright false if $p > 1$, as $h(z) = \sum z_\nu$ shows.

3.2. The point of departure in the proof of Theorem 2.2 is the estimate from [L1, Corollary 3.2], a simple consequence of Hörmander's L^2 estimate [Ho, Theorem 4.4.2].

Proposition 3.2. *If $f \in C_{0,1}(B_n(R))$ is closed, $\bar\partial u = f$ has a solution $u \in C(B_n(R))$ that satisfies*

$$\begin{aligned}|u(z)| &\le 2(1 + 2\sqrt{n}) R \left(\frac{R}{R - \|z\|}\right)^n |f|_{B_n(R)} \\ &\le 3R \left(\frac{2R}{R - \|z\|}\right)^n |f|_{B_n(R)}.\end{aligned}$$

In particular u can be chosen to be the solution with minimal $L^2(B_n(R))$ norm, in which case it will depend linearly on f.

First we shall improve this to an estimate that is still exponential but in $\#z$ rather than in n:

Proposition 3.3. *If $f \in C_{0,1}(B_N(R))$ is closed, the equation $\bar\partial u = f$ has a solution $u \in C(B_N(R))$ that satisfies*

$$|u(z)| \le 3R \left(\frac{5R}{R - \|z\|}\right)^{\#z} |f|_{B_N(R)}. \tag{3.3}$$

Again, u can be chosen to depend linearly on f.

Proof. We shall take $R = 1$. For a subset $\mathcal{P} \subset \{1, \dots, N\}$ let $P = \{z \in \mathbb{C}^N : z_\nu = 0 \text{ if } \nu \notin \mathcal{P}\}$ denote the corresponding coordinate plane and $B_\mathcal{P} = B_N \cap P$; and similarly with $\mathcal{Q}$, Q. Let π_ν denote the projection of $\mathbb{C}^N$ on the ν'th coordinate hyperplane, so that $\prod_{\nu \notin \mathcal{P}} \pi_\nu$ is projection on P.

By Proposition 3.2 for each $\mathcal{P}$ there is a $u_{\mathcal{P}} \in C(B_{\mathcal{P}})$ solving $\bar{\partial} u_{\mathcal{P}} = f|B_{\mathcal{P}}$ such that

$$|u_{\mathcal{P}}(z)| \leq 3\left(\frac{2}{1-\|z\|}\right)^{|\mathcal{P}|} |f|_{B_N}. \tag{3.4}$$

If there were a $u \in C(B_N)$ with $u|B_{\mathcal{P}} = u_{\mathcal{P}}$ for all $\mathcal{P}$, this u would satisfy (3.3). While there is no reason for such a u to exist, there is a simple way to produce u for which $u|B_{\mathcal{P}} \approx u_{\mathcal{P}}$.

Quite generally, suppose we are given a system of l-forms $u_{\mathcal{P}} \in C_l(B_{\mathcal{P}})$, $\mathcal{P} \subset \{1, \ldots, N\}$. Define

$$u = \sum_{\mathcal{P}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*)\Big(\prod_{\nu \notin \mathcal{P}} \pi_\nu^*\Big) u_{\mathcal{P}} \in C_l(B_N). \tag{3.5}$$

We shall need the following properties of this operation.

1° If $u_{\mathcal{P}} = v|B_{\mathcal{P}}$ with some $v \in C_l(B_N)$ then $u = v$.

2° The operation (3.5) commutes with $\bar{\partial}$.

3° If $\bar{\partial} u_{\mathcal{P}} = f|B_{\mathcal{P}}$ with some $f \in C_{l+1}(B_N)$ then $\bar{\partial} u = f$.

4° If $\mathcal{Q} \subset \{1, \ldots, N\}$ then

$$u|B_{\mathcal{Q}} = \sum_{\mathcal{P} \subset \mathcal{Q}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*)\Big(\prod_{\nu \notin \mathcal{P}} \pi_\nu^*\Big) u_{\mathcal{P}}|B_{\mathcal{Q}}.$$

To verify 1° replace $u_{\mathcal{P}}$ by v in (3.5) and note that on $C_l(B_N)$

$$\sum_{\mathcal{P}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*) \prod_{\nu \notin \mathcal{P}} \pi_\nu^* = \prod_{\nu=1}^{N} (1 - \pi_\nu^* + \pi_\nu^*) = 1.$$

2° is obvious and 3° follows from 1° and 2°. Finally, observe that $\pi_\mu^* \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*) = 0$ when $\mu \in \mathcal{P}$ so that

$$\Big(\prod_{\mu \notin \mathcal{Q}} \pi_\mu^*\Big) u = \sum_{\mathcal{P} \subset \mathcal{Q}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*)\Big(\prod_{\nu \notin \mathcal{P}} \pi_\nu^*\Big) u_{\mathcal{P}},$$

which is equivalent to 4°.

Now apply (3.5) with our $u_{\mathcal{P}}$ initially constructed. By 3° $\bar{\partial} u = f$. Also, if $z \in B_N$ and $\mathcal{Q} = \{\nu : z_\nu \neq 0\}$ then one can estimate $u(z)$ using 4°, collecting the contributions of $\mathcal{P}$ of fixed cardinality i, and applying

(3.4):

$$\begin{aligned}
|u(z)| &\le \left(\sum_{\mathcal{P}\subset\mathcal{Q}}\prod_{\nu\in\mathcal{P}}(1-\pi_\nu^*)(\prod_{\nu\notin\mathcal{P}}\pi_\nu^*)|u_{\mathcal{P}}|\right)(z)\\
&\le \sum_{i=0}^{|\mathcal{Q}|}\binom{|\mathcal{Q}|}{i}2^i\cdot 3\left(\frac{2}{1-\|z\|}\right)^i|f|_{B_N} = 3\left(1+\frac{4}{1-\|z\|}\right)^{|\mathcal{Q}|}|f|_{B_N}\\
&\le 3\left(\frac{5}{1-\|z\|}\right)^{\#z}|f|_{B_N},
\end{aligned}$$

as claimed.

If f of Proposition 3.3 vanishes on a hyperplane, one can choose u that also vanishes there:

Proposition 3.4. *Let $0\le\rho<R$ and suppose a closed $g\in C_{0,1}(B_N(R))$ vanishes when restricted to the hyperplane $z_N=\rho$. Then the equation $\bar\partial w=g$ has a solution $w\in C(B_N(R))$ that vanishes on the hyperplane and satisfies*

$$|w(z)|\le 4R\left(\frac{5R^2}{(R-\rho)(R-\|z\|)}\right)^{\#z+1}|g|_{B_N(R)}. \tag{3.6}$$

Proof. Again we take $R=1$. Define

$$\pi(z)=\frac{z'}{1-z_N}\in\mathbb{C}^{N-1},\qquad z=(z',z_N)\in B_N,$$

and check that $\|\pi(z)\|\le\|z\|$. If $\epsilon:B_{N-1}\to B_N$ denotes the embedding $\epsilon(z')=((1-\rho)z',\rho)$ then $\pi\circ\epsilon=\mathrm{id}$. By Proposition 3.3 there is a $v\in C(B_N)$ that satisfies $\bar\partial v=g$ and

$$|v(z)|\le 3\left(\frac{5}{1-\|z|}\right)^{\#z}|g|_{B_N}.$$

Now $\bar\partial\epsilon^*v=\epsilon^*g=0$ so that $w=v-\pi^*\epsilon^*v$ also solves $\bar\partial w=g$. In addition, w vanishes on the hyperplane $z_N=\rho$. Since

$$\|\epsilon\pi(z)\|=(1-\rho)\|\pi(z)\|+\rho\le(1-\rho)\|z\|+\rho$$

and $\#\epsilon\pi(z) \le \#z + 1$, one can estimate $w(z) = v(z) - v(\epsilon\pi(z))$:

$$|w(z)| \le 3\left(\left(\frac{5}{1-\|z\|}\right)^{\#z} + \left(\frac{5}{(1-\rho)(1-\|z\|)}\right)^{\#z+1}\right)|g|_{B_N}$$
$$\le 4\left(\frac{5}{(1-\rho)(1-\|z\|)}\right)^{\#z+1}|g|_{B_N}.$$

3.3. Propositions 3.2, 3.3, and 3.4 would hold for all $p \ge 1$, with modified constants. For the proof of the next, key proposition $p = 1$ is essential.

Proposition 3.5. *If $f \in C_{0,1}(B_N)$ is closed and $Z \in B_N(1/6)$, the equation $\bar\partial U = f|B_N(1/6)$ has a solution $U \in C(B_N(1/6))$ that satisfies*

$$|U(z)| \le c\|z - Z\| q^{\#z}[f]_{B_N}, \qquad \|z\| < 1/6. \tag{3.7}$$

One can take $q = 16$, $c = 10^5$.

Proof. The claim is true when $N = 0$; we shall prove it for general N by induction. Assume it true with N replaced by $N - 1$, and also assume without loss of generality that $Z_N = \rho \ge 0$. We shall borrow π, ϵ from the previous proof.

The inductive hypothesis applied with $f' = \epsilon^* f$ gives a $U' \in B_{N-1}(1/6)$ that satisfies $\bar\partial U' = f'|B_{N-1}(1/6)$ and

$$|U'(z')| \le c\|z' - \pi(Z)\| q^{\#z'}[f']_{B_{N-1}}, \qquad \|z'\| < 1/6. \tag{3.8}$$

Set $g = f - \pi^* f'$, and apply Proposition 3.4 with $R = 5/6$, to obtain a solution of $\bar\partial w = g|B_N(5/6)$ that satisfies $w(\cdot, Z_N) = 0$ and

$$|w(z)| \le 65 \cdot q^{\#z}|g|_{B_N(5/6)}, \qquad \|z\| < 1/2, \tag{3.9}$$

with $q = 16$. If

$$U = \pi^* U' + w \tag{3.10}$$

then $\bar\partial U = \pi^* f' + f - \pi^* f' = f$. It remains to estimate U in terms of $[f]_{B_N}$.

By (2.5) $[f']_{B_{N-1}} = [\epsilon^* f]_{B_{N-1}} \le (1 - Z_N)[f]_{B_N}$. Also

$$\begin{aligned}\|\pi(z) - \pi(\zeta)\| &= \left\|\frac{z' - \zeta'}{1 - \zeta_N} + \frac{(z_N - \zeta_N)z'}{(1-\zeta_N)(1-z_N)}\right\| \\ &\le \frac{\|z' - \zeta'\| + \|z\|\,|z_N - \zeta_N|}{|1 - \zeta_N|}.\end{aligned} \tag{3.11}$$

Hence (3.8) implies

$$|U'(\pi(z))| \le c(\|z' - Z'\| + |z_N - Z_N|/6)q^{\#z}[f]_{B_N}, \tag{3.12}$$

for $\|z\| < 1/6$. Next $|d\pi|_{B_N(5/6)} \le 6$ by (3.11), whence $|\pi^* f'|_{B_N(5/6)} \le 6|f'|_{B_{N-1}} \le 6[f]_{B_N}$ and $|g|_{B_N(5/6)} \le 7[f]_{B_N}$. Thus by (3.9)

$$|w(z)| \le 460 q^{\#z}[f]_{B_N}, \qquad \|z\| < 1/2. \tag{3.13}$$

This can be refined as follows. If $\|z\| < 1/6$, consider the map

$$\varphi = \varphi_{xy} : \overline{B}_1 \ni s \mapsto ((1 - s/4)\pi(z), s/4) \in B_N(1/2).$$

Then $\bar\partial\varphi^* w = \varphi^* g = \varphi^* f$, since φ maps into a fiber of π. By the definition of $[f]_{B_N}$, there is a $v = v_{xy}$ such that $\bar\partial v = \varphi^* f$ and $|v|_{C^1(B_1)} \le [f]_{B_N}$. Thus $h = \varphi^* w - v$ is holomorphic. Since the hyperbolic distance between $4z_N, 4Z_N \in B_1(2/3) \subset B_1$ is $\le 8|z_N - Z_N|$, Schwarz's lemma implies

$$\begin{aligned} |h(4z_N) - h(4Z_N)| &\le 8|z_N - Z_N|\ |h|_{B_1} \\ &\le 8|z_N - Z_N|(|\varphi^* w|_{B_1} + |v|_{B_1}). \end{aligned}$$

Now $v(4z_N) - v(4Z_N)$ can also be estimated in terms of $z_N - Z_N$, therefore

$$\begin{aligned} w(z) &= w(\varphi(4z_N)) - w(\varphi(4Z_N)) \\ &= h(4z_N) - h(4Z_N) + v(4z_N) - v(4Z_N) \end{aligned}$$

too. All added up one obtains for $\|z\| < 1/6$

$$\begin{aligned} |w(z)| &\le |z_N - Z_N|(8|\varphi^* w|_{B_1} + 12|v|_{C^1(B_1)}) \\ &\le 7 \cdot 10^4 |z_N - Z_N| q^{\#z}[f]_{B_N} \end{aligned}$$

by (3.13), taking into account that $\#\varphi(s) \le \#z + 1$. Thus by (3.10), (3.12)

$$|U(Z)| \le c\left\{\|z' - Z'\| + \left(\frac{1}{6} + \frac{7}{c}10^4\right)|z_N - Z_N|\right\} q^{\#z}[f]_{B_N},$$

and (3.7) follows, provided $c \ge 10^5$.

3.4. Theorem 2.2 is now easily proved.

We shall verify that u given in Proposition 3.3, with $R = 1$, satisfies (2.6). Take an arbitrary $Z \in B_N(r)$ and construct U as in Proposition 3.5. Then with $h = u - U \in \mathcal{O}(B_N(1/6))$ and $z \in B_N(1/6)$ we have

$$|h(z)| \leq (3 \cdot 6^{\#z} + c \cdot 16^{\#z})[f]_{B_N} \leq 2c \cdot 16^{\#z}[f]_{B_N}$$

by (3.3), (3.7). Hence from Theorem 3.1, applied with $R = 1/6$

$$|u(Z)| = |h(Z)| \leq \frac{2c}{1 - 96e\|Z\|}[f]_{B_N} \leq 4c[f]_{B_N}, \qquad \|Z\| < r,$$

and (2.6) holds with $a = 4 \cdot 10^5$.

3.5. Above we have not insisted on sharp constants, and indeed it is possible to obtain somewhat stronger results. First off, if integral formulas are used rather than L^2 estimates, it is possible to show that in Proposition 3.2 the base of the exponential can be taken to be 2. With a little more care in subsequent estimates in Proposition 3.5 one could replace $1/6$ by an arbitrary $\rho < 1$ and q by an arbitrary number > 5. As a consequence, r of Theorem 2.2 can be anything $< 1/(5e)$. It would be of interest to know whether one can take r arbitrarily close to 1, or perhaps even equal to 1. I don't believe this is possible, even if $[f]_{B_N}$ is replaced by $|f|_{C^k(B_N)}$, as long as k is fixed. If I am right, phase transition would occur in the Cauchy–Riemann equations: there would be a critical radius $r_0 \in (0,1)$ such that for closed $f \in C^k_{0,1}(B_N)$ the equation $\bar{\partial}u = f|B_N(r)$ can be solved with dimension independent bounds on u if $r < r_0$, but not if $r > r_0$. In the latter regime $|u|_{B_N(r)}$ would diverge exponentially.

§4. Infinite dimensions

Now we shall see what Theorem 2.2 implies about the $\bar{\partial}$ equation in infinite dimensions. Let Γ be an arbitrary set and

$$l^1(\Gamma) = \{z : \Gamma \to \mathbb{C} \mid \sum_{\gamma \in \Gamma} |z(\gamma)| = \|z\| < \infty\}.$$

Given an open $D \subset l^1(\Gamma)$ and $f \in C_{0,1}(D)$ closed we ask if there is a $u \in C^1(D)$ that solves $\bar{\partial}u = f$. (For basics of $\bar{\partial}$ in Banach spaces see [L1,2].) In [L1] we showed how to pass from finite dimensional estimates for $\bar{\partial}$ to infinite dimensional results. This can be done with the improved estimates of Theorem 2.2, and we obtain the following result. If $x, y \in l^1(\Gamma)$, $s \in \bar{B}_1$, define $\varphi_{xy}(s) = x + sy$ as in (2.4), and $\Omega = \{(x,y) : \varphi_{xy}(\bar{B}_1) \subset D\}$.

Theorem 4.1. *Suppose each $(\xi, \eta) \in \Omega$ has a neighborhood Ω_0 such that if $(x, y) \in \Omega_0$, the equation $\bar\partial v_{xy} = \varphi^*_{xy} f$ can be solved with $v_{xy} \in C^1(B_1)$ depending continuously on x,y. Then in a neighborhood of an arbitrary $z \in D$ the equation $\bar\partial u = f$ is solvable with u a C^1 function.*

Global solvability can also be obtained:

Theorem 4.2. *If Γ is countable and D is pseudoconvex then $\bar\partial u = f$ has a solution $u \in C^1_{\text{loc}}(D)$ if and only if the hypothesis of Theorem 4.1 is satisfied.*

Thus solvability or nonsolvability of $\bar\partial u = f$ depends only on solvability on one dimensional slices.

Theorem 4.2 follows from Theorem 4.1 and the main result of [L2]. Indeed, if f satisfies the hypothesis then D can be covered by open sets V so that some $u_V \in C^1(V)$ solves $\bar\partial u_V = f|V$. By [L2, Theorem 0.1] the holomorphic cocycle $(u_V - u_W)$ is exact, hence of form $(h_V - h_W)$ with $h_V \in \mathcal{O}(V)$. It follows that $u(z) = u_V(z) - h_V(z)$ if $z \in V$ defines the required solution $u \in C^1_{\text{loc}}(D)$.

Very little is known about solving the $\bar\partial$ equation in Banach spaces other than l^1, or for forms of higher degree. Patyi in [P] gives an example of a Banach space in which $\bar\partial u = f$ is not solvable for some closed C^∞ form f. It would be of great interest to explore the solvability of the $\bar\partial$ equation in classical Banach spaces such as l^p, $L^p[0, 1]$, $C[0, 1]$.

References

[FK] G. B. Folland, J. J. Kohn, "The Neumann problem for the Cauchy-Riemann complex", Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1972.

[GL] H. Grauert and I. Lieb, Das Ramirezsche Integral und die Lösung der Gleichung $\overline{\partial} f = \alpha$ im Bereich der beschränkten Formen, Proc. Conf. Complex Analysis, Rice University, 1969, Rice University Studies, **56** (1970), 29–50.

[H] G. M. Henkin, Integral representation of functions holomorphic in strictly pseudoconvex domains and some applications, Mat. Sb., **82** (1970), 300–308; Engl. translation: Math. USSR Sb., **11** (1970), 273–281.

[Hö] L. Hörmander, "An Introduction to Complex Analysis in Several Variables", 3rd edition, North Holland, Amsterdam, 1990.

[L1] L. Lempert, The Dolbeault complex in infinite dimensions II, J. Amer. Math. Soc., **12** (1999), 775–793.

[L2] ———, The Dolbeault complex in infinite dimensions III, Sheaf cohomology in Banach spaces, Invent. Math., **142** (2000), 579–603.

[O] N. Øvrelid, Integral representation formulas and L^p estimates for the $\overline{\partial}$-equation, Math. Scand., **29** (1971), 137–160.

[P] I. Patyi, On the $\overline{\partial}$-equation in a Banach space, Bull. Soc. Math. France, **128** (2000), 391–406.

Department of Mathematics
Purdue University
West Lafayette, IN 47907
USA