An approach to the Cartan geometry
II : CR manifolds

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Introduction

One of the prominent features in the post-Oka development of the several complex variables is the extensive use of the Cauchy-Riemann partial differential equations. We also note the development of the CR geometry induced on the boundary. This geometry is introduced by E. Cartan [3] in low dimensional cases. The general case is developed by N. Tanaka [9], S.-S. Chern-J. Moser [4], S. Webster [10], and D. Burns. Jr.-S. Shnider [1]. This geometry will be the vehicle to set the Cauchy-Riemann equation geometrically.

The CR geometry is a special case of the Cartan geometry, which is regarded as a deformation of the Klein's classical geometry. Namely, for each classical geometry given as a homogenous space G/H we have the Cartan geometries modeled after G/H. For example, Riemann geometry is modeled after the euclidean geometry, which is the quotient of the group of euclidean motions by the orthogonal group. On a space X we have a Cartan geometry modeled after G/H when we have (1) a principal H-bundle E formed by frames, i.e. ways to identify up to equivalence (infinitesimally up to certain order) its neighborhood with open sets in G/H. (2) A Cartan connection on E valued in the Lie algebra of G.

CR geometry may be regarded as the case of Cartan geometry when the homogenous space is the unit ball in complex euclidean space acted by the group of holomorphic automorphisms. We constructed CR geometry in [6] from the above view point. However, we did not construct the frame bundle directly. We first construct the bundle of the frames of the first (infinitesimal) order and then we prolong it to the frame bundle. In this paper, we construct CR geometry by defining frames directly. We also write down the normal CR Cartan connections and discuss its global aspect.

§1.   The Homogenous CR manifolds

We fix a non-degenerate hermitian $n \times n$ matrix

(1) $(\underline{h}_{\alpha\bar{\beta}}), \quad \alpha, \beta = 1, ..., n.$

We consider, as our model, the CR-structure on the hypersurface $\mathcal{M}$ in $\mathbf{C}^{n+1} = \{(z^1, ..., z^n, w)\}$, given by

(2) $\Im w = \frac{1}{2}\langle z, z \rangle, \quad \langle z, z \rangle = \underline{h}_{\alpha\bar{\beta}} z^\alpha \overline{z^\beta}.$

A) We embed $\mathbf{C}^{n+1}$ in the complex projective space $\mathbf{CP}^{n+1}$ sending $(z^1, ..., z^n, w)$ to the point with the homogenous coordinate $[1, z^1, ..., z^n, w]$. The subgroup $\mathcal{G}$ of the projective group which preserves the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ acts transitively on the closure. Thus $\overline{\mathcal{M}}$ is the homogenous space on which we model our CR geometry.

B) We find that $\mathcal{G}$ decomposes to the product of the translation group and the isotropy group. Namely,

(3) $\mathcal{G} = \mathcal{L} \cdot \mathcal{H},$

(4)

$$\mathcal{L} = \left\{ l(z, x) = \begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ w & iz^* & 1 \end{pmatrix} : z = (z^1, ..., z^n)^{\mathrm{tr}}, \ w = x + \frac{i}{2}\langle z, z \rangle \right\}$$

where $(z^*)_\alpha = \underline{h}_{\alpha\bar{\beta}}\overline{z^\beta}$.

$\mathcal{H} = H/\text{center}$, where H is the group of $(n+2) \times (n+2)$ matrixes:

(5) $h = h(a, u, \beta, s) = \begin{pmatrix} a & \nu^* & b \\ 0 & u & \beta \\ 0 & 0 & 1/\bar{a} \end{pmatrix}, \quad \text{where}$

a is a non-zero complex number, u a complex $n \times n$-matrix, β is a column complex n-vector β , and s is a real number satisfying:

(6) $u^* u = I, \quad \frac{a}{\bar{a}} \det u = 1, \quad \nu = i\bar{a}u^*\beta, \quad \frac{b}{a} = s - \frac{i}{2}\langle \beta, \beta \rangle,$

$(u^*)^\alpha_\beta = \underline{h}^{\alpha\bar{\gamma}}\underline{h}_{\beta\bar{\sigma}}\overline{u^\sigma_\gamma}$, and I is the identity $n \times n$-matrix. The center is the finite group

(7) $\{h(e^{im'}, e^{im'}I, 0, 0) \ : \ m' = \frac{m}{n+2}2\pi, \quad m = 0, 1, ..., n+1\}.$

C) The Lie algebra $\mathbf{g}$ of $\mathcal{G}$ has the grading:

$$(8) \qquad \mathbf{g} = \mathbf{g}_{(-2)} + \mathbf{g}_{(-1)} + \mathbf{g}_{(0)} + \mathbf{g}_{(1)} + \mathbf{g}_{(2)}, \quad \text{where}$$

$$(9.1) \qquad \mathbf{g}_{(-2)} = \{ \{\dot{x}\}_{(-2)} = (\frac{d(l(0, s\dot{x}))}{ds})_{s=0} : \dot{x} \in \mathbf{R}\},$$

$$(9.2) \qquad \mathbf{g}_{(-1)} = \{ \{\dot{z}\}_{(-1)} = (\frac{d(l(s\dot{z}, 0))}{ds})_{s=0} : \dot{z} \in \mathbf{C}^n\},$$

$$(9.3) \qquad \mathbf{g}_{(0)} = \mathbf{R}\pi + \mathbf{R}\mu + \{\mathbf{su}(n)\}, \quad \text{where for } \dot{u} \in \mathbf{su}(n)$$

$$(9.4) \qquad \{\dot{u}\} = (\frac{dh(1, e^{s\dot{u}}, 0, 0)}{ds})_{s=0}, \quad \pi = (\frac{dh(e^s, I, 0, 0)}{ds})_{s=0},$$
$$\mu = (\frac{dh(e^{is}, e^{-\frac{2}{n}is}I, 0, 0)}{s})_{s=0},$$

$$(9.5) \qquad \mathbf{g}_{(1)} = \{ \{\dot{\beta}\}_{(1)} = (\frac{dh(1, I, s\dot{\beta}, 0)}{ds})_{s=0} : \dot{\beta} \in \mathbf{R}^m\},$$

$$(9.6) \qquad \mathbf{g}_{(2)} = \{ \{\dot{b}\}_{(2)} = (\frac{dh(1, I, 0, s\dot{b}))}{ds})_{s=0} : \dot{b} \in \mathbf{R}\},$$

$$(9.7) \qquad \dot{u} \in \mathbf{su}(n) \text{ if and only if } \underline{h}_{\sigma\bar{\gamma}}\dot{u}^{\sigma}_{\alpha} + \underline{h}_{\alpha\bar{\sigma}}\overline{\dot{u}^{\sigma}_{\gamma}} = 0.$$

$$(10) \qquad \mathbf{h} = \mathbf{g}_{(0)} + \mathbf{g}_{(1)} + \mathbf{g}_{(2)} \text{ is the Lie algebra of } H.$$

For $\dot{g} \in \mathbf{g}$ we set
$$(11)$$
$$\dot{g} = \{\dot{g}_{[-2]}\}_{(-2)} + \{\dot{g}_{[-1]}\}_{(-1)} + \dot{g}_{\pi}\pi + \dot{g}_{\mu}\mu + \{\dot{g}_{\mathbf{su}}\} + \{\dot{g}_{[1]}\}_{(1)} + \{\dot{g}_{[2]}\}_{(2)}.$$

D) In terms of the decomposition (3) the action of $g \in \mathcal{G}$ on $(z', w') \in \mathcal{M}$ is given by
$$(12)$$
$$T_{l(z,x)}(z', w') = (z' + z, w' + w + i\langle z', z\rangle), \quad \text{where } \langle z', z\rangle = \underline{h}_{\alpha\bar{\beta}}(z')^{\alpha}\overline{z^{\beta}}.$$

$$(13)$$
$$T_h(z', w') = (\frac{1}{a\lambda}(uz' + w'\beta), \frac{1}{\lambda}\frac{1}{|a|^2}w'), \quad \text{where } \lambda = 1 - i\langle uz', \beta\rangle + \frac{b}{a}w'.$$

E) The ∂_b-operators of the CR structure on $\mathcal{M}$ is generated by

(14) $$P^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - iz_*^\alpha \frac{\partial}{\partial \overline{w}}, \quad z_*^\alpha = \underline{h}_{\beta\bar\alpha} z^\beta.$$

We have

(15) $$[P^\alpha, \overline{P^\beta}] = i\underline{h}_{\beta\bar\alpha} \frac{\partial}{\partial \theta_{\mathcal{M}}}, \quad \frac{\partial}{\partial \theta_{\mathcal{M}}} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}.$$

F) The Maurer-Cartan form ω_G has the expression:

(16) $$\omega_G = Ad(h^{-1})(\{\theta_{\mathcal{M}}\}_{(-2)} + \{dz\}_{(-1)}) + \omega_H,$$

where $\omega_H = h^{-1}dh$ is the Maurer-Cartan form of H and

(17) $$\theta_{\mathcal{M}} = dx + \frac{i}{2}\langle z, dz\rangle - \frac{i}{2}\langle dz, z\rangle.$$

It then follows by calculation that using the terminology in (11)

(18) $$(\omega_G)_{[-2]} = |a|^2 \theta_{\mathcal{M}}, \quad (\omega_G)_{[-1]} = au^\star(dz - \bar{a}\beta\theta_{\mathcal{M}}).$$

Note that for matrix valued 1-forms α and β

(19) $$[\alpha, \beta] = \alpha \wedge \beta + \beta \wedge \alpha.$$

We then find that the structure equation : $d\omega_G + [\omega_G, \omega_G]/2 = 0$ is rewritten in terms of the grading (8) as

(20.1) $$d(\omega_G)_{[-2]} - i\langle(\omega_G)_{[-1]}, (\omega_G)_{[-1]}\rangle - 2(\omega_G)_\pi \wedge (\omega_G)_{[-2]} = 0.$$

(20.2) $$d(\omega_G)_{[-1]} + \{(\omega_G)_{\mathbf{su}} - ((\omega_G)_\pi + \frac{n+2}{n}i(\omega_G)_\mu)I\} \wedge (\omega_G)_{[-1]}$$
$$+ (\omega_G)_{[1]} \wedge (\omega_G)_{[-2]} = 0,$$

(20.3) $$d(\omega_G)_\pi - \Im\langle(\omega_G)_{[-1]}, (\omega_G)_{[1]}\rangle + (\omega_G)_{[2]} \wedge (\omega_G)_{[-2]} = 0,$$

(20.4) $$d(\omega_G)_\mu + \Re\langle(\omega_G)_{[-1]}, (\omega_G)_{[1]}\rangle = 0,$$

(20.5) $$d(\omega_G)_{\mathbf{su}} + (\omega_G)_{\mathbf{su}} \wedge (\omega_G)_{\mathbf{su}} + i(\omega_G)_{[1]} \wedge (\omega_G)_{[-1]}^*$$
$$- i(\omega_G)_{[-1]} \wedge (\omega_G)_{[1]}^* + \frac{2}{n}i\Re\langle(\omega_G)_{[-1]}, (\omega_G)_{[1]}\rangle = 0,$$

(20.6)
$$d(\omega_G)_{[1]} + ((\omega_G)_{\mathbf{su}} + ((\omega_G)_\pi - \frac{n+2}{n} i(\omega_G)_\mu) I) \wedge w^*_{[1]} + (\omega_G)_{[-1]} \wedge (\omega_G)_{[2]} = 0,$$

(20.7)
$$d(\omega_G)_{[2]} + i\langle (\omega_G)_{[1]}, (\omega_G)_{[1]} \rangle + 2(\omega_G)_\pi \wedge (\omega_G)_{[2]} = 0.$$

G) Note by calculation that for $g = l(z_0, w_0)h$

(21)
$$\overline{P^\alpha} T_g^\gamma(0) = \frac{1}{a} u_\alpha^\gamma, \quad \frac{\partial}{\partial \theta_{\mathcal{M}}} T_g^\alpha(0) = \frac{1}{a} \beta^\alpha,$$

$$\overline{P^\alpha} T_g^0(0) = \frac{i}{a} \underline{h}_{\gamma\bar{\sigma}} u_\alpha^\gamma \overline{z_0^\sigma}, \quad \frac{\partial}{\partial \theta_{\mathcal{M}}} T_g^0(0) = \frac{1}{|a|^2} + \frac{i}{a} \langle \beta, z_0 \rangle.$$

(22)
$$\frac{\partial}{\partial \theta_{\mathcal{M}}} \overline{P^\gamma} T_g^\alpha(0) = -\frac{b}{a} \frac{1}{a} u_\gamma^\alpha + i \underline{h}_{\sigma\bar{\nu}} u_\gamma^\sigma \frac{1}{a} \beta^\alpha \overline{\beta^\nu}.$$

H) We find by calculation that, setting

(23)
$$\mathrm{Ad}(h^{-1})(\{\dot{g}\}_{(l)}) = A(h, \dot{g}, l), \quad \text{we have}$$

$$A(h, \dot{x}, -2)_{[-2]} = |a|^2 \dot{x}, \quad A(h, \dot{x}, -2)_{[-1]} = -|a|^2 \dot{x} u^* \beta,$$
$$A(h, \dot{x}, -2)_\pi + i A(h, \dot{x}, -2)_\mu = -a\bar{b}\dot{x},$$

(24.1)
$$A(h, \dot{x}, -2)_{[\mathbf{su}]} = i|a|^2 \dot{x} (u^* \beta) \otimes (\beta^* u) + \frac{2i}{n} A(h, \dot{x}, \mu) I$$

$$A(h, \dot{x}, -2)_{[1]} = -\bar{a}b\dot{x} u^* \beta, \quad A(h, \dot{x}, -2)_{[2]} = -|b|^2 \dot{x},$$

(24.2)
$$A(h, \dot{z}, -1)_{[-2]} = 0, \quad A(h, \dot{z}, (-1))_{[-1]} = a u^* \dot{z},$$

$$A(h, \dot{z}, -1)_\pi + i A(h, \dot{z}, -1)_\mu = ia\langle \dot{z}, \beta \rangle,$$

$$A(h, \dot{z}, -1)_{[\mathbf{su}]} = -ia(u^* \dot{z}) \otimes (\beta^* u) - i\bar{a}(u^* \beta) \otimes (\dot{z}^* u) + \frac{2i}{n} A(h, \dot{z}, \mu) I,$$

$$A(h, \dot{z}, -1)_{[1]} = b u^* \dot{z} - i\bar{a}\langle \beta, \dot{z} \rangle u^* \beta, \quad A(h, \dot{z}, -1)_{[2]} = 2\Re i b\langle \dot{z}, \beta \rangle,$$

(24.3)
$$\mathrm{Ad}(h^{-1})\pi = \pi + \{u^* \beta\}_{(1)} + \{2\Re\frac{b}{a}\}_{(2)},$$

(24.4)
$$\mathrm{Ad}(h^{-1})\mu = \mu - \{\frac{n+2}{n} i u^* \beta\}_{(1)} + \{\frac{n+2}{n} \langle \beta, \beta \rangle\}_{(2)},$$

(24.5) $\text{Ad}(h^{-1})\{\sigma\} = \{u^*\sigma u\} + \{u^*\sigma\beta\}_{(1)} + \{i\langle\sigma\beta,\beta\rangle\}_{(2)}$ $(\sigma \in \mathbf{su}(n))$,

(24.6) $\text{Ad}(h^{-1})\{\gamma\}_{(1)} = \{\frac{1}{\bar{a}}u^*\gamma\}_{(1)} + \{2\Re\frac{i}{a}\langle\gamma,\beta\rangle\}_{(2)}$ $(\gamma \in \mathbf{C}^n)$,

(24.7) $\text{Ad}(h^{-1})\{s\}_{(2)} = \{\frac{s}{|a|^2}\}_{(2)}$.

§2. CR coframes of infinitesimal order 1

A) Let M be a CR manifold with non-degenerate Levi-form, given by a subbundle $T_b''M$ of ∂_b differential operators. We may identify M with a hypersurface in $\mathbf{C}^{n+1}$ passing the origin p_0 defined by an equation:

(1) $$r = 0.$$

We regard p_0 as the reference point and interested in the local aspect near p_0. Hence we may shrink M if necessary. We consider a chart $\{(z^1,...,z^n,w)\}$ of $\mathbf{C}^{n+1}$. By a holomorphic linear change of chart we may assume

(2) $$\frac{\partial r}{\partial w} - \frac{\partial r}{\partial \bar{w}} \neq 0 \text{ at } p_0, \quad \frac{\partial r}{\partial z^\alpha} = O(1).$$

We set $r_\alpha = \partial/\partial z^\alpha$, $r_{\bar\alpha} = \partial/\partial\overline{z^\alpha}$, etc. Our model is the case

(3) $$r = r_{\mathcal{M}} = \frac{1}{i}(w - \bar{w}) - \langle z,z\rangle.$$

B) The space $T_b''M$ of the $\overline{\partial}_b$ differential operators of M is generated by

(4) $$Q^\alpha = \frac{\partial}{\partial\overline{z^\alpha}} - \frac{r_{\bar\alpha}}{r_{\bar w}}\frac{\partial}{\partial\bar{w}}. \quad \text{Set}$$

(5) $$\frac{\partial}{\partial\theta_M} = \frac{2}{r_w - r_{\bar w}}\left(r_w\frac{\partial}{\partial\bar{w}} - r_{\bar w}\frac{\partial}{\partial w}\right).$$

$\partial/\partial\theta_M$ is tangential to M. $Q^\alpha, \overline{Q^\alpha}, \partial/\partial\theta_M$ form a base of the complex tangent space $\mathbf{C}TM$.

C) For a differential form λ on $\mathbf{C}^{n+1}$ we also use the same letter to denote its restriction to M. $\overline{\partial}_b$ operators and their bar generate the subbundle of complex tangent space $\mathbf{C}TM$ defined by

(6) $$\theta_M = 0, \quad \text{where } \theta_M = \frac{1}{2}\left(dw + d\bar{w} + \frac{r_\beta}{r_w}dz^\beta + \frac{r_{\bar\beta}}{r_{\bar w}}d\overline{z^\beta}\right).$$

$dz^\alpha, d\overline{z^\alpha}, \theta_M$ form a base of $\mathbf{C}T^*M$ dual to the above mentioned base of $\mathbf{C}TM$. $T''_\flat M$ is given by he equation:

$$(7) \qquad\qquad dz^\alpha = 0, \quad \theta_M = 0.$$

Since $T''M$ is closed under bracket, we see by the expression of Q^α in (2)

$$(8.1) \qquad\qquad [Q^\alpha, Q^\beta] = 0.$$

Because of the Definition of the Levi-form we may set

$$(8.2) \qquad [Q^\alpha, \overline{Q^\beta}] \equiv ic^{\alpha\bar\beta}\frac{\partial}{\partial\theta_M} \qquad (\text{mod } Q^\gamma, \overline{Q^\gamma}).$$

In view of (15) §1 and (3) we may assume that

$$(8.3) \qquad\qquad c^{\alpha\bar\beta}(p_0) = \underline{h}_{\beta\bar\alpha}.$$

Because of the above mentioned duality, when l is a function on M,

$$(9) \qquad dl = (\overline{Q^\alpha}l)dz^\alpha + (Q^\alpha l)d\overline{z^\alpha} + \frac{\partial l}{\partial\theta_M}\theta_M.$$

D) Consider a manifold N and a map $f : N \to M$. Since f is also a map into $\mathbf{C}^{n+1}$ we have in terms of the standard chart $(z^1, ..., z^n, w)$ the expression $f = (f^1, ..., f^n, f^0)$. Note that for any vector field X on N and a function l on M we have $X(l \circ f) = \langle dl, dfX \rangle \circ f$. Therefore by (9)

(10.1)

$$X(l \circ f) = (Xf^\alpha)\,(\overline{Q^\alpha}l)\circ f + (X\overline{f^\alpha})\,(Q^\alpha l)\circ f + (R_Xf)\,\frac{\partial l}{\partial\theta_M}\circ f, \quad \text{where}$$

$$(10.2) \qquad R_Xf = \frac{1}{2}(Xf^0 + X\overline{f^0} + \frac{r_\alpha}{r_w}\circ f\, Xf^\alpha + \frac{r_{\bar\alpha}}{r_{\bar w}}\circ f\, X\overline{f^\alpha}).$$

Since dfX is tangential to M,

$$(10.3) \qquad r_w\circ fXf^0 + r_{\bar w}\circ fX\overline{f^0} + r_\alpha\circ fXf^\alpha + r_{\bar\alpha}\circ fX\overline{f^\alpha} = 0.$$

Therefore we also have the expressions:

$$(10.4) \qquad \begin{aligned} R_Xf &= \frac{1}{2}(\frac{r_{\bar w} - r_w}{r_{\bar w}r_w}\circ f)\,\{(r_w\circ f)Xf^0 + (r_\alpha\circ f)Xf^\alpha\} \\ &= \frac{1}{2}(\frac{r_w - r_{\bar w}}{r_{\bar w}r_w}\circ f)\,\{(r_{\bar w}\circ f)X\overline{f^0} + (r_{\bar\alpha}\circ f)X\overline{f^\alpha}\}. \end{aligned}$$

E) Let $f : \mathcal{M} \to M$ be a map sending the origin 0 to p_0. Then by (6)

$$(11) \qquad f^*\theta_M = \frac{1}{2}(df^0 + d\overline{f^0} + \frac{r_\beta}{r_w} \circ f df^\beta + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f d\overline{f^\beta}).$$

Apply (9) to the case $N = M = \mathcal{M}$ and $l = f^0$ as well as $l = f^\beta, f^{\bar\beta}$. We then find

$$f^*\theta_M = C_f\, \theta_{\mathcal{M}} + C^0_{\alpha f}\, dz^\alpha_{\mathcal{M}} + C^0_{\bar\alpha f}\, d\overline{z^\alpha}_{\mathcal{M}}, \quad \text{where}$$

$$(12.1) \quad \begin{aligned} C_f &= \frac{1}{2}\Big(\frac{\partial f^0}{\partial\theta_{\mathcal{M}}} + \frac{\partial\overline{f^0}}{\partial\theta_{\mathcal{M}}} + \frac{r_\beta}{r_w} \circ f \frac{\partial f^\beta}{\partial\theta_{\mathcal{M}}} + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f \frac{\partial\overline{f^\beta}}{\partial\theta_{\mathcal{M}}}\Big), \\ C^0_{\alpha f} &= \frac{1}{2}\Big(\overline{P^\alpha}f^0 + \overline{P^\alpha \overline{f^0}} + \frac{r_\beta}{r_w} \circ f\, \overline{P^\alpha}f^\beta + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\, \overline{P^\alpha \overline{f^\beta}}\Big), \\ C^0_{\bar\alpha f} &= \frac{1}{2}\Big(P^\alpha f^0 + P^\alpha\overline{f^0} + \frac{r_\beta}{r_w} \circ f\, P^\alpha f^\beta + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\, P^\alpha\overline{f^\beta}\Big). \end{aligned}$$

Similarly, we find

$$f^*dz^\gamma = C^\gamma_{0f}\, \theta_{\mathcal{M}} + C^\gamma_{\alpha f}\, dz^\alpha_{\mathcal{M}} + C^\gamma_{\bar\alpha f}\, d\overline{z^\alpha}_{\mathcal{M}},$$

$$(12.2) \qquad C^\gamma_{0f} = \frac{\partial f^\gamma}{\partial\theta_{\mathcal{M}}}, \quad C^\gamma_{\alpha f} = \overline{P^\alpha}f^\gamma, \quad C^\gamma_{\bar\alpha f} = P^\alpha f^\gamma.$$

Since $r \circ f = 0$, we also have

$$(13.1) \quad r_w \circ f \frac{\partial f^0}{\partial\theta_{\mathcal{M}}} + r_{\bar w} \circ f \frac{\partial\overline{f^0}}{\partial\theta_{\mathcal{M}}} + r_\beta \circ f \frac{\partial f^\beta}{\partial\theta_{\mathcal{M}}} + r_{\bar\beta} \circ f \frac{\partial\overline{f^\beta}}{\partial\theta_{\mathcal{M}}} = 0.$$

$$(13.2) \quad r_w \circ f\, P^\alpha f^0 + r_{\bar w} \circ f\, P^\alpha\overline{f^0} + r_\beta \circ f\, P^\alpha f^\beta + r_{\bar\beta} \circ f\, P^\alpha\overline{f^\beta} = 0.$$

Set

$$(14.1) \qquad W = \frac{\partial f^0}{\partial\theta_{\mathcal{M}}} + \frac{r_\alpha}{r_w} \circ f \frac{\partial f^\alpha}{\partial\theta_{\mathcal{M}}}.$$

By the Definition of C_f in (12.1) and (13.1) we find that

$$(14.2) \qquad W + \overline{W} = 2C_f, \quad r_w \circ f\, W + r_{\bar w} \circ f\, \overline{W} = 0.$$

Hence $(r_{\bar w} - r_w) \circ f\, W = 2(r_{\bar w} \circ f)C_f$. Therefore

$$(15) \qquad C_f = \frac{r_{\bar w} - r_w}{2r_{\bar w}\, r_w} \circ f\Big(r_w \circ f \frac{\partial f^0}{\partial\theta_{\mathcal{M}}} + r_\beta \circ f \frac{\partial f^\beta}{\partial\theta_{\mathcal{M}}}\Big).$$

F) We define the CR attaching maps of M as the maps which preserve infinitesimally the defining equation (7) of our CR structure. Namely,

(16) **Definition.** $f : \mathcal{M} \to M$ is called a CR attaching map of order m when f is a diffeomorphism near 0 and

$$(16.1) \qquad C^0_{\alpha f} = O(m), \quad C^\gamma_{\bar\alpha f} = O(m), \quad C_f(0) > 0.$$

(17) **Proposition.** Let $f : \mathcal{M} \to M$ be a CR attaching map of order m. Then

$$(17.1) \qquad P^\alpha f^j = O(m) \quad for\ j = 0, 1, ..., n;\ \alpha = 1, ..., n.$$

Conversely $f : \mathcal{M} \to M$ satisfying (17.1) is a CR attaching map of order m, provided C_f given by (15) is positive at the origin. We also have

$$(17.2) \qquad r_w \circ f\, \overline{P^\alpha} f^0 + r_\beta \circ f\, \overline{P^\alpha} f^\beta = O(l).$$

Proof. Set for an arbitrary $f : \mathcal{M} \to M$

$$(18.1) \quad W^1_\alpha = P^\alpha f^0 + \frac{r_\beta}{r_w} \circ f\, P^\alpha f^\beta, \quad W^2_\alpha = P^\alpha \overline{f^0} + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\, P^\alpha \overline{f^\beta}.$$

We see by (13.2) and (12.1) that

$$(18.2) \qquad r_w \circ f\, W^1_\alpha + r_{\bar w} \circ f\, W^2_\alpha = 0, \quad W^1_\alpha + W^2_\alpha = C^0_{\bar\alpha f}.$$

In the case f is a CR attaching map of order m, we have $W^1_\alpha = O(m)$, $W^2_\alpha = O(m)$. Therefore (17.2) holds. Since $P^\alpha f^\gamma = O(m)$ by (16.1) and $W^1_\alpha = O(m)$, (17.1) also holds. The converse holds, because (17.1) implies $W^1_\alpha = O(m)$ and by the 1st formula in (18.2) we have $W^2_\alpha = O(m)$. Q.E.D.

(19) **Proposition.** For any $p \in M \subset \mathbf{C}^{n+1}$ there is an attaching map of order 3.

Proof. We may assume that p is the origin. In view of the theorem of Chern and Moser we may assume that M is given by the equation: $r = 0$, where

$$(20.1) \qquad r = \frac{1}{i}(w - \bar{w}) - \langle z, z \rangle - F(w, x^0), \quad x^0 = \frac{1}{2}(w + \bar{w})$$

where $F \equiv 0 \pmod{(z, \bar{z})^4}$. Then the map

$$(20.2) \qquad f \; : \; \mathcal{M} \ni (z, w) \to (z, w + iF(z, \frac{1}{2}(w + \bar{w})))$$

is a CR attaching map of order 3, because

$$(20.3) \qquad f^0 \equiv w \pmod{(z, \bar{z})^4}, \quad f^\alpha \equiv z^\alpha \pmod{(z, \bar{z})^4}.$$

<div align="right">Q.E.D.</div>

G) Let N be a manifold. We denote by $J_0^l(\mathcal{M}, N)$ the space of l-jets at the reference point 0 of maps of $\mathcal{M}$ into N.

(21) **Definition.** $J \in J_0^l(\mathcal{M}, M)$ *is called a CR l-jet when there is a CR attaching map f of order l representing J. Denote by $J_0^l(M)_{CR}$ the space of CR l-jets.*

Since $P^\alpha, \overline{P^\alpha}, \partial/\partial\theta_{\mathcal{M}}$ form a base of $\mathbf{C}T\mathcal{M}$, $J_0^1(\mathcal{M}, \mathbf{C}^{n+1})$ has the standard chart $(.., p^{(0)j}, ..., p_\alpha^{(1)j}, ..., p_{\bar{\alpha}}^{(1)j}, ..., p_0^{(1)j}, ...)$, where $j = 0, 1, ..., n$. Namely, for $J \in J_0^1(\mathcal{M}, \mathbf{C}^{n+1})$ represented by a map $f \; : \; \mathcal{M} \to \mathbf{C}^{n+1}$

$$(22) \qquad \begin{aligned} p^{(0)j}(J) &= f^j(0), \quad p_\alpha^{(1)j}(J) = \overline{P^\alpha} f^j(0), \\ p_{\bar{\alpha}}^{(1)j}(J) &= P^\alpha f^j(0), \quad p_0^{(1)j}(J) = \frac{\partial f^j}{\partial\theta_{\mathcal{M}}}(0). \end{aligned}$$

$J^1(\mathcal{M}, M) \subset J^1(\mathcal{M}, \mathbf{C}^{n+1})$ is the submanifold defined by

$$(23.1) \qquad p^{(0)} = (p^{(0)1}, ..., p^{(0)n}, p^{(0)0}) \in M,$$

$$(23.2) \qquad \Re(r_w(r^{(0)})p_0^{(1)0} + r_\gamma(p^{(0)})p_0^{(1)\gamma}) = 0,$$

$$(23.3) \quad r_w(p^{(0)})p_\alpha^{(1)0} + r_{\bar{w}}(p^{(0)})\overline{p_{\bar{\alpha}}^{(1)0}} + r_\gamma(p^{(0)})p_\alpha^{(1)\gamma} + r_{\bar{\gamma}}(p^{(0)})\overline{p_{\bar{\alpha}}^{(1)\gamma}} = 0.$$

Note that the map

(24)
$$J \in J_0^1(\mathcal{M}, \mathbf{C}^{n+1}) \rightarrow (p^{(0)}(J), ..., p_{\bar{\alpha}}^{(1)j}(J), ..., \Re(r_w(p^{(0)}(J))p_0^{(1)0}(J)$$
$$+ r_{\bar{\alpha}}(p^{(0)}(J))p_\alpha^{(1)0}(J)), ..., r_w(p^{(0)}(J))p_\alpha^{(1)0}(J) + r_\beta(p^{(0)}(J))p_\alpha^{(1)\beta}(J), ...)$$
$$\in M \times \mathbf{C}^{n(n+1)} \times \mathbf{R} \times \mathbf{C}^n$$

is of maximal rank. Note also that $C_{\alpha f}^0(0) = 0$ is a consequence of $p_\alpha^{(1)j} = 0$ and (23.2). In view of (17), it then follows that

(25) **Proposition.** $J_0^1(M)_{CR}$ is the subspace of $J_0^1(\mathcal{M}, M)$ defined by the equations:

(25.1)
$$p_{\bar{\alpha}}^{(1)j} = 0, \quad C^{(1)} > 0,$$

where $C^{(1)}$ is defined by

(26)
$$C^{(1)} = \frac{r_{\bar{w}} - r_w}{2r_{\bar{w}}r_w}(p^{(0)})\{r_w(p^{(0)})p_0^{(1)0} + r_\gamma(p^{(0)})p_0^{(1)\gamma}\}.$$

(27) **Proposition.** *For any* $p \in M$, *complex numbers* C_j^γ ($\gamma = 1, ..., n; j = 0, 1..., n$), *and* $C > 0$ *there is unique* $J \in J_0^1(M)_{CR}$ *such that*

(28)
$$p^{(0)}(J) = p, \quad p_j^{(1)\gamma}(J) = C_j^\gamma, \quad p_\alpha^{(1)0}(J) = -\frac{r_\gamma}{r_w}(p^{(0)})C_\alpha^\gamma,$$
$$p_0^{(1)0}(J) = \frac{2r_{\bar{w}}}{r_{\bar{w}} - r_w}(p(0)) \, C - \frac{r_\gamma}{r_w}(p(0)) \, p_0^{(1)\gamma}(J).$$

We thus have a chart $(x, ..., C_j^\gamma, ..., C)$ of $J_0^1(M)_{CR}$, called standard.

H) Because of the duality we have for an attaching map f of order 1 at $x \in M$

(29)
$$(f_* \overline{P^\alpha})_x = C_{\alpha f}^\gamma(0)(\overline{Q^\gamma})_x,$$
$$(f_* \frac{\partial}{\partial \theta}_{\mathcal{M}})_x = C_{0f}^\gamma(0)(\overline{Q^\gamma})_x + \overline{C_{0f}^\gamma}(0)(Q^\gamma)_x + C_f(0)(\frac{\partial}{\partial \theta}_M)_x.$$

We call $(f_* \overline{P^\alpha})_x, (f_* \frac{\partial}{\partial \theta}_{\mathcal{M}})_x$ the CR frame of order 1 associated to a CR 1-jet $J = j_0^1 f$. The space of CR frame of order 1 is diffeomorphic to $J_0^1(M)_{CR}$. The CR coframe $..., \omega_j^j, ...$ of order 1 associated to CR 1-jet

J at $x \in M$ is defined as the dual to a CR frame of order 1 associated to J. We then find

$$(30) \quad \omega_J^\alpha = (C^{-1})_\gamma^\alpha(J)((dz_M^\gamma)_x - \frac{C_0^\gamma(J)}{C(J)}(\theta_M)_x), \quad \omega_J^0 = \frac{1}{C(J)}(\theta_M)_x.$$

where $((C^{-1})_\gamma^\alpha(J))$ is the inverse matrix of the matrix $(C_\alpha^\gamma(J))$.

We may regard ω_J^j as a 1-form Ω^j on $J_0^1(M)_{CR}$. Hence using the standard chart

$$(31) \qquad \Omega^\alpha = (C^{-1})_\gamma^\alpha(dz_M^\gamma - \frac{C_0^\gamma}{C}\theta_M), \quad \Omega^0 = \frac{1}{C}\theta_M.$$

Remark. In the case $M = \mathcal{M}$ we see by (17)-(18) §1 that $\Omega^\alpha = (\omega_G)_{[-1]}^\alpha$, $\Omega^0 = (\omega_G)_{([-2]}$.

I) Note that the isotropy group $\mathcal{H}$ at 0 acts on $\mathcal{M}$ as a CR isomorphism group. Hence, when f is a CR attching map of oder l and $h \in H$, $f \circ T_h$ (cf. (13) §1) is a CR attaching map of order l. Therefore we have the action of h on $J_0^1(M)_{CR}$, which we denote by R_h. We then find by (21) §1 and calculation that for $J \in J_0^1(M)_{CR}$

$$(32) \quad C_\alpha^\gamma(R_h J) = C_\sigma^\gamma(J)\frac{1}{a}u_\alpha^\sigma, \quad C_0^\gamma(R_h J) = C_0^\gamma(J)\frac{1}{|a|^2} + C_\sigma^\gamma(J)\frac{1}{a}\beta^\sigma,$$

$$(33) \qquad C(R_h J) = C(J)\frac{1}{|a|^2}.$$

§3.   CR coframe of infinitesimal order 2

A) Let $f : \mathcal{M} \to M \subset \mathbf{C}^{n+1}$ be a CR attaching map of order m. Then

$$(1) \qquad f^*\theta_M = C_f\theta_\mathcal{M} + O(m). \quad \text{Hence}$$

$$(2) \quad \begin{aligned} f^*d\theta_M &= C_f\,d\theta_\mathcal{M} + dC_f \wedge \theta_\mathcal{M} + O(m-1) \\ &= iC_f <dz_\mathcal{M}, dz_\mathcal{M}> + dC_f \wedge \theta_\mathcal{M} + O(m-1). \end{aligned}$$

Since $f^*dz^\alpha = C_{\alpha f}^\gamma\, dz_\mathcal{M}^\gamma + C_{0f}^\alpha\, \theta_\mathcal{M} + O(m)$, we find that

$$(3) \qquad dz_\mathcal{M}^\gamma = C_\alpha^{\gamma f}\{f^*dz^\alpha - C_{0f}^\alpha\, \theta_\mathcal{M}\} + O(m),$$

where $(C_\gamma^{\alpha f})$ is the inverse matrix of $(C_{\gamma f}^\alpha)$. Therefore

$$
(4) \quad
\begin{aligned}
f^* d\theta_M = {} & iC_f \underline{h}_{\gamma\bar\sigma} C_\alpha^{\gamma f} \overline{C_\beta^{\sigma f}} \{ f^* dz^\alpha \wedge f^* \overline{dz^\beta} + C_{0f}^\alpha f^* \overline{dz^\beta} \\
& - \overline{C_{0f}^\beta} f^* dz^\alpha) \wedge \theta_\mathcal{M} \} + dC_f \wedge \theta_\mathcal{M} + O(m-1).
\end{aligned}
$$

For a function l on M we have by taking d of (9) §2

$$
(5) \quad \frac{\partial l}{\partial \theta_M} d\theta_M = -d(\overline{Q^\alpha} l) \wedge dz^\alpha - d(Q^\alpha l) \wedge \overline{dz^\alpha} - d\frac{\partial l}{\partial \theta_M} \wedge \theta_M.
$$

Applying (9) §2 again when l is replaced $Q^\alpha l$, etc. we find that

$$
(6) \quad
\begin{aligned}
\frac{\partial l}{\partial \theta_M} d\theta_M = {} & [Q^\beta, \overline{Q^\alpha}] l \, dz^\alpha \wedge \overline{dz^\beta} \\
& - \{ [\overline{Q^\alpha}, \frac{\partial}{\partial \theta_M}] l \, dz^\alpha + [Q^\alpha, \frac{\partial}{\partial \theta_M}] l \, \overline{dz^\alpha} \} \wedge \theta_M.
\end{aligned}
$$

Applying the above in the case $l = (w + \bar w)/2$, we find by (8.2) §2 that

$$
(7) \quad d\theta_M = ic^{\beta\bar\alpha} dz^\alpha \wedge \overline{dz^\beta} + (\overline{c^\alpha} dz^\alpha + c^\alpha \overline{dz^\alpha}) \wedge \theta_M,
$$

where

$$
(8) \quad c^{\beta\bar\alpha} = \frac{1}{2i}[Q^\beta, \overline{Q^\alpha}](w + \bar w), \quad c^\alpha = \frac{1}{2}[\frac{\partial}{\partial \theta_M}, Q^\alpha](w + \bar w).
$$

Hence

$$
(9) \quad
\begin{aligned}
f^* d\theta_M = {} & ic^{\beta\bar\alpha} \circ f \, f^* dz^\alpha \wedge f^* \overline{dz^\beta} \\
& + (\overline{c^\alpha} \circ f \, f^* dz^\alpha + c^\alpha \circ f \, f^* \overline{dz^\alpha}) \wedge f^* \theta_M.
\end{aligned}
$$

Comparing the above with (4), we find that

$$
(10) \quad c^{\beta\bar\alpha} \circ f = C_f \underline{h}_{\gamma\bar\sigma} C_\alpha^{\gamma f} \overline{C_\beta^{\sigma f}} + O(m-1),
$$

$$
(11) \quad C_f c^\alpha \circ f = ic^{\alpha\bar\beta} \circ f \, C_{0f}^\beta + \overline{C_\alpha^{\beta f}} P^\beta C_f + O(m-1).
$$

B) Denote by $J_0^2(M)$ the space of 2-jets of maps f of neighborhoods of 0 in $\mathcal{M}$ into M. When $\tilde J = j_0^2(f)$, we set

$$
(12)
$$

$$
p_{\alpha\beta}^{(2)j}(\tilde J) = \overline{P^\alpha}\, \overline{P^\beta} f^j(0), \quad p_{\bar\alpha\beta}^{(2)j}(\tilde J) = P^\alpha \overline{P^\beta} f^j(0), \quad p_{\alpha 0}^{(2)j}(\tilde J) = \overline{P^\alpha}\frac{\partial}{\partial \theta_\mathcal{M}} f^j(0),
$$

$$
p_{00}^{(2)j}(\tilde J) = \frac{\partial^2}{\partial \theta_\mathcal{M}^2} f(0), \quad C_\alpha^{(2)}(\tilde J) = P^\alpha C_f(0), \quad C_\alpha^{(2)}(\tilde J) = \overline{P^\alpha} C_f(0).
$$

Denote by $J_0^2(M)_{CR}$ the space of 2-jets of CR attaching map to M of order 2. We set

(13) $$E_1 = \rho_1^2(J_0^2(M)_{CR}) \subset J_0^1(M)_{CR}.$$

Let $(c_{\beta\bar\alpha})$ be the inverse matrix of $(c^{\beta\bar\alpha})$. We have by (10)-(11)

(14) **Proposition.** For $J = \rho_1^2(J^2) \in E_1$ with $J^2 \in J_0^2(M)_{CR}$

(15) $$\frac{1}{C(J)} p_\gamma^{(1)\alpha}(J) \underline{h}^{\gamma\bar\sigma} \overline{p_\sigma^{(1)\beta}(J)} = c_{\beta\bar\alpha}(p^{(0)}(J)),$$

(16) $$C_\beta^{(2)}(J^2) = p_\beta^{(1)\sigma}(J)\{ic^{\alpha\bar\sigma}(p^{(0)}(J))\overline{p_0^{(1)\alpha}(J)} + \overline{c^\sigma(p^{(0)}(J))}C(J)\}.$$

The action of H on $J_0^1(M)_{CR}$ (cf. (35)-(36) §2) preserves E_1. We find by (36) §2 that H acts transitively on the subspace of $J_0^1(M)_{CR}$ defined by the equation: In terms of the standard chart $(x, ..., C_j^\alpha, ..., C)$ of $J^1(M)_{CR}$

(17) $$C_\gamma^\alpha \, \underline{h}^{\gamma\bar\sigma} \overline{C_\sigma^\beta} = C \, c_{\beta\bar\alpha}(x).$$

In view of (16) we conclude that

(18) **Proposition.** E_1 is the subspace of $J_0^1(M)_{CR}$ defined by the equation (17).

C) We also find that the subgroup H_1 of H which acts as the identity transformation is given by

(19) $$a = 1, \quad u = I, \quad \beta = 0.$$

Hence H_1 is a 1 dimensional subgroup parametrized by

(20) $$s = \Re\frac{b}{a}.$$

Therefore E_1 is a principal bundle with the structure group H/H_1.

We wish to define the CR frame bundle E by the following diagram:

(21)
$$\begin{array}{ccc} J^1(M)_{CR} & \leftarrow & \tilde{J}^2 \\ \uparrow & & \downarrow \\ E_1 & \leftarrow & E \end{array}$$

where $\tilde{J}^2_{CR}$ is a suitable subspace of $J^2(M)_{CR}$. (22) §1 and (20) suggest that we use as the above downward arrow the map

$$(22) \qquad \tilde{\rho} \ : \ J^2 \to p^{(2)}_\sharp(J^2) = -\frac{1}{n}\Re(C^{-1})^\gamma_\alpha(J)p^{(2)\alpha}_{0_\gamma}(J^2).$$

D) We justify the above choice.

Since $p^{(2)}_\sharp$ may be regarded as a small deformation of $\Re(b/a)$ by (22) §1, $\tilde{\rho}$ is a projection. It remains to show that H acts on E making E a principal H-bundle. We define $\tilde{J}^2_{CR}$ as the space of 2-jets representable by a CR attaching map of order 3. We need to show that $p^{(2)}_\sharp(R_h\tilde{J})$ is a function of $p^{(2)}_\sharp(\tilde{J})$ and of h , provided $\tilde{J} \in \tilde{J}^2(M)_{CR}$.

We find by (16) §2 that for $f \ : \ \mathcal{M} \to M$

$$(23) \qquad \frac{\partial}{\partial\theta_\mathcal{M}}\overline{P^\gamma}(f^\alpha \circ T_h)(x) = \{\frac{\partial}{\partial\theta_\mathcal{M}}(\overline{P^\gamma}T^\sigma_h)(x)\}(\overline{P^\sigma}f^\alpha)(T_hx)$$
$$+ (\overline{P^\gamma}T^\sigma_h)(x)\frac{\partial}{\partial\theta_\mathcal{M}}\{(\overline{P^\sigma}f^\alpha) \circ T_h\}(x).$$

We apply (16) §2 to $\frac{\partial}{\partial\theta_\mathcal{M}}\{(\overline{P^\sigma}f^\alpha) \circ T_h\}(x)$ in the case $N = M = \mathcal{M}$ and (X,l,f) is $(\partial/\partial\theta_\mathcal{M}, \overline{P^\sigma}f^\alpha, T_h)$. We then find by (21)-(22) §2

$$(24)$$
$$p^{(2)\alpha}_{0\gamma}(R_h\tilde{J}) = \frac{1}{a}u^\sigma_\gamma\{\frac{1}{|a|^2}p^{(2)\alpha}_{0\sigma}(\tilde{J}) + \frac{1}{a}\beta^\mu p^{(2)\alpha}_{\sigma\mu}(\tilde{J}) + i\underline{h}_{\sigma\bar{\mu}}\frac{1}{\bar{a}}\overline{\beta^\mu}C^\alpha_0(J)\}$$
$$+ \{-\frac{b}{a}\frac{1}{a}u^\sigma_\gamma + i\underline{h}_{\mu\bar{\nu}}u^\mu_\gamma\frac{1}{a}\beta^\sigma\overline{\beta^\nu}\}C^\alpha_\sigma(J).$$

Therefore it is enough to show that $p^{(2)\alpha}_{\sigma\gamma}(\tilde{J})$ is a function on E_1, provided $\tilde{J}$ is represented by an attaching map of order 3.

By (10) we have for a CR attaching map f of order 3

$$(25) \qquad C_f \ c_{\phi\bar{\alpha}} \circ f = P^\nu\overline{f^\phi} \ \underline{h}^{\gamma\bar{\nu}} \ \overline{P^\gamma}f^\alpha + O(2).$$

Applying $\overline{P^\sigma}$, we find that
$$(26)$$
$$(\overline{P^\sigma}C_f)c_{\phi\bar{\alpha}} \circ f + C_f\overline{P^\sigma}(c_{\phi\bar{\alpha}} \circ f)$$
$$= (\overline{P^\sigma} \ \overline{P^\gamma}f^\alpha)\underline{h}^{\gamma\bar{\nu}}(P^\nu\overline{f^\phi}) + (\overline{P^\gamma}f^\alpha)\underline{h}^{\gamma\bar{\nu}}(\overline{P^\sigma}P^\nu\overline{f^\phi}) + O(1).$$

Hence we see by (16)
$$(27)$$
$$p^{(2)\alpha}_{\sigma\gamma}(\tilde{J})\underline{h}^{\gamma\bar{\nu}}\overline{C^\phi_\nu}(J) = C^{(2)}_\sigma(\tilde{J})c_{\phi\bar{\alpha}}(x) + C(J)C^\nu_\sigma(J)\overline{Q^\nu}c_{\phi\bar{\alpha}}(x) + iC^\alpha_\sigma(J)\overline{C^\phi_0}(J).$$

In view of (16) we now conclude that $p^{(2)\alpha}_{\gamma\sigma}$ is a function on E_1 and consequently H acts on the space E.

We write down the formula for the operation of H on $p^{(2)}_{\sharp}$. Since $C^{\alpha}_{\gamma}(R_h J) = C^{\alpha}_{\sigma}(J) u^{\sigma}_{\gamma}/a$, we find

$$(C^{-1})^{\gamma}_{\alpha}(R_h J) p^{(2)\alpha}_{0\gamma}(R_h \tilde{J}) = (C^{-1})^{\gamma}_{\alpha}(J)\{\frac{1}{|a|^2} p^{(2)\alpha}_{0\gamma}(\tilde{J})$$

$$(28) \qquad + \frac{1}{a}\beta^{\sigma} p^{(2)\alpha}_{\gamma\sigma}(\tilde{J}) + i\underline{h}_{\gamma\bar{\sigma}}\frac{1}{a}\overline{\beta^{\sigma}}C^{\alpha}_0(J)\} - n\frac{b}{a} + i < \beta, \beta > .$$

Since $C(C^{-1})^{\nu}_{\mu} = \underline{h}^{\nu\bar{\gamma}}\overline{C^{\sigma}_{\gamma}}c^{\sigma\bar{\mu}}$ we have on the other hand
(29)

$$p^{(2)\alpha}_{\gamma\sigma}(\tilde{J})(C^{-1})^{\gamma}_{\alpha}(J) = \frac{n}{C(J)}C^{(2)}_{\sigma}(\tilde{J}) + C^{\gamma}_{\sigma}(J)\{c^{\tau\bar{\alpha}}\overline{Q^{\gamma}}c_{\tau\bar{\alpha}}(x) + i\frac{\overline{C^{\alpha}_0(J)}}{C(J)}c^{\alpha\bar{\gamma}}\}.$$

We then find after some cancellation
(30)

$$p^{(2)}_{\sharp}(R_h\tilde{J}) = \frac{1}{|a|^2}p^{(2)}_{\sharp}(\tilde{J}) + \Re\frac{b}{a} - \Re\frac{1}{a}\beta^{\alpha}\{\frac{C^{(2)}_{\alpha}(\tilde{J})}{C(J)} + \frac{1}{n}C^{\gamma}_{\alpha}(J)(\overline{Q^{\gamma}}c_{\sigma\bar{\mu}}(x))c^{\sigma\bar{\mu}}(x)\}.$$

Therefore

$$(30) \quad \underline{p^{(2)}_{\sharp}} = \frac{1}{|a|^2}p^{(2)}_{\sharp} + \Re\frac{b}{a} - \Re\frac{1}{a}\beta^{\alpha}C^{\gamma}_{\alpha}\{\overline{c^{\gamma}} + ic^{\sigma\bar{\gamma}}\frac{\overline{C^{\sigma}_0}}{C} + \frac{1}{n}c^{\sigma\bar{\mu}}(\overline{Q^{\gamma}}c_{\sigma\bar{\mu}})\}.$$

§4. The normal CR Cartan Connections

Let $\omega : TE \to \mathbf{g}$ be a Cartan connection on the CR frame bundle E.

A) ω is called a CR Cartan connection (cf. (31) §2) when

$$(1) \qquad\qquad \omega^{\alpha}_{[-1]} = \Omega^{\alpha}, \quad \omega_{[-2]} = \Omega^0.$$

Let $U = \{(x)\} = \{(z, x^0)\}$ be a chart open set of M. In terms of a local trivialization $U \times H$ of E we have an expression :

$$(2) \qquad\qquad \omega = Ad(h^{-1})w + \omega_H,$$

where w is a $\mathbf{g}$-valued 1-form on U and ω_H is the Maurer-Cartan form of H regarded as a $\mathbf{h}$-valued 1-form. Its curvature form has the expression:

$$(3) \quad K = d\Omega + \frac{1}{2}[\Omega, \Omega] = Ad(h^{-1})k, \quad \text{where} \quad k = dw + \frac{1}{2}[w, w].$$

B) A local trivialization of E over U is given, using a section J : $U \to E$, by

(4) $$U \times H \ni (x, h) \to R_h J(x) \in E.$$

We find by (31) §2 that ω in (2) is a CR Cartan connection when (5)

$$w^\alpha_{[-1]}(x) = (C^{-1})^\alpha_\gamma(x)(dz^\gamma - \frac{C^\gamma_0(x)}{C(x)}\theta_M), \quad w_{[-2]} = \frac{1}{C(x)}\theta_M, \quad \text{where}$$

(6) $$J(x) = (..., C^\alpha_j(x), ..., C(x), p^{(2)}_\sharp(x))$$

is the standard chart expression of $J(x)$. We see by the above that we have to determine $w_\pi, w_\mu, w_{\mathbf{su}}, w_{[1]}, w_{[2]}$ (cf (11) §1) to determine a CR Cartan connection. We put curvature conditions so that we have CR Cartan connections unique up to isomorphism.

C) As we obtained (20) §1 we find that k in (3) has the expression:

(7.1) $$k_{[-2]} = dw_{[-2]} - i\langle w_{[-1]}, w_{[-1]} \rangle - 2w_\pi \wedge w_{[-2]}.$$

(7.2) $$k_{[-1]} = dw_{[-1]} + \{w_{\mathbf{su}} - (w_\pi + \frac{n+2}{n}iw_\mu)I\} \wedge w_{[-1]}$$
$$+ w_{[1]} \wedge w_{[-2]},$$

(7.3) $$k_\pi = dw_\pi - \Im\langle w_{[-1]}, w_{[1]} \rangle + w_{[2]} \wedge w_{[-2]},$$

(7.4) $$k_\mu = dw_\mu + \Re\langle w_{[-1]}, w_{[1]} \rangle,$$

(7.5) $$k_{\mathbf{su}} = dw_{\mathbf{su}} + w_{\mathbf{su}} \wedge w_{\mathbf{su}} + iw_{[1]} \wedge w^*_{[-1]}$$
$$- iw_{[-1]} \wedge w^*_{[1]} + \frac{2}{n}i\Re\langle w_{[-1]}, w_{[1]} \rangle,$$

(7.6) $$k_{[1]} = dw_{[1]} + i(w_{\mathbf{su}} + (w_\pi - \frac{n+2}{n}iw_\mu)I) \wedge w^*_{[1]} + w_{[-1]} \wedge w_{[2]},$$

(7.7) $$k_{[2]} = dw_{[2]} + i < w_{[1]}, w_{[1]} > + 2w_\pi \wedge w_{[2]}.$$

D) In order to carry out the program mentioned at the end of B), we set

(8) $$\underline{C} = \text{the matrix } (C^\alpha_\beta(x)), \quad \hat{C} = (..., C^\alpha_0(x), ...).$$

We also omit x in $C(x)$, etc. We see by (7) §3, (10) §3, and (5) that

$$(9) \qquad d\theta_M = iC\langle w_{[-1]}, w_{[-1]}\rangle - 2\Re(ic^{\gamma\bar{\alpha}}C_0^\alpha - Cc^\gamma)\overline{dz^\gamma} \wedge w_{[-2]}.$$

We then find that
(10)
$$dw_{[-2]} - i\langle w_{[-1]}, w_{[-1]}\rangle - q^0 \wedge w_{[-2]} = 0, \quad dw_{[-1]} + q \wedge w_{[-1]} + q_{[1]} \wedge w_{[-2]} = 0,$$

$$q^0 = \Re(-\frac{i}{C}c^{\gamma\bar{\alpha}}C_0^\alpha + c^\gamma)\overline{dz^\gamma} - \frac{1}{2}d\log C,$$

$$(11) \qquad q = \underline{C}^{-1}d\underline{C} - i\underline{C}^{-1}\hat{C} \otimes w_{[-1]}^*,$$

$$q_{[1]} = \underline{C}^{-1}\hat{C}2\Re(-\frac{i}{C}c^{\beta\bar{\alpha}}C_0^\alpha + c^\beta)\overline{dz^\beta} + Cd\frac{\underline{C}^{-1}\hat{C}}{C}.$$

(12) **Lemma.** *We can find a unique set of a complex valued 1-form b^0, an $\mathbf{su}(n)$-valued 1-form $b_{\mathbf{su}}$, a $\mathbf{C}^n$-valued 1-form $b_{[1]}$, such that*

$$(13.1) \qquad b^0, \ b_{\mathbf{su}}, \ b_{[1]} \equiv 0 \quad (\mathrm{mod} \ w_{[-1]}, \overline{w_{[-1]}}),$$

$$(13.2) \qquad \begin{aligned} &dw_{[-2]} - i\langle w_{[-1]}, w_{[-1]}\rangle - 2\Re b^0 \wedge w_{[-2]} = 0, \\ &dw_{[-1]} + (b_{\mathbf{su}} - b^0 I) \wedge w_{[-1]} + b_{[1]} \wedge w_{[-2]} = 0. \end{aligned}$$

Proof. By using the type with respect to $w_{[-1]}, \overline{w_{[-1]}}$, we check the uniqueness. To show the existence, note by (10) that $d\langle w_{[-1]}, w_{[-1]}\rangle - q^0 \wedge \langle w_{[-1]}, w_{[-1]}\rangle \equiv 0$ (mod $w_{[-2]}$). We then find

$$(14.1) \quad (dw_{[-1]}^\alpha)^{(2,0)} - \underline{h}^{\alpha\bar{\beta}}\underline{h}_{\sigma\bar{\gamma}}w_{[-1]}^\sigma \wedge \overline{q_{\beta\bar{\mu}}^\gamma}w_{[-1]}^\mu - (q^0)^{(1,0)} \wedge w_{[-1]}^\alpha = 0,$$

where $(q_\beta^\gamma)^{(0,1)} = q_{\beta\bar{\mu}}^\gamma\overline{w_{[-1]}^\mu}$. On the other hand we see by (10) that

$$(14.2) \qquad (dw_{[-1]}^\alpha)^{(1,1)} + q_{\sigma\bar{\mu}}^\alpha\overline{w_{[-1]}^\mu} \wedge w_{[-1]}^\sigma = 0.$$

Therefore we find that (13) is valid when we set

$$(b_{\mathbf{u}})_\gamma^\alpha = q_{\gamma\bar{\sigma}}^\alpha\overline{w_{[-1]}^\sigma} - \underline{h}^{\alpha\bar{\beta}}\underline{h}_{\gamma\bar{\nu}}\overline{q_{\beta\bar{\sigma}}^\nu}w_{[-1]}^\sigma,$$

$$(15) \qquad (b_{\mathbf{su}})_\beta^\alpha = (b_{\mathbf{u}})_\beta^\alpha - (b_{\mathbf{u}})_\gamma^\gamma\delta_\beta^\alpha, \quad b^0 = (q^0)^{(1,0)} + (b_{\mathbf{u}})_\gamma^\gamma,$$

$$b_{[1]} \equiv q_{[1]} - \frac{1}{C}\underline{C}^{-1}\frac{\partial C}{\partial\theta_M}w_{[-1]} \quad (\mathrm{mod} \ w_{[-2]}).$$

$$\text{Q.E.D.}$$

E) For a differential form α we set

(16) $\alpha = \alpha^+ + \alpha^{(0)} \wedge w_{[-2]}$,   where $\alpha^+, \alpha^{(0)}$ do not contain $w_{[-2]}$.

By the Lemma we find the followings:

(17) **Proposition.** $k_{[-2]} = 0$ if and only if $w_\pi^+ = \Re\, b^0$.

(18) **Proposition.** Assume that $k_{[-2]} = 0$. Then $k_{[-1]} = 0$ if and only if

$$w_{\mathbf{su}}^+ = b_{\mathbf{su}}, \quad w_\mu^+ = \frac{n}{n+2} \Im\, b^0, \quad w_{[1]}^+ = b_{[1]} + (b_{\mathbf{su}}^{(0)} - (b^0)^{(0)} I) w_{[-1]}.$$

From now on we consider only CR Cartan connections satisfying the conditions in (17) and (18). We next examine conditions $k_\pi = 0$, $k_\mu = 0$.

By taking the exterior derivative of the first equality in (13.2), we find that

(19) $(d\Re\, b^0 - \Im\, \langle w_{[-1]}, b_{[1]} \rangle) \wedge w_{[-2]} = 0$.

Therefore, we have the expression:

(20) $d\Re\, b^0 - \Im\, \langle w_{[-1]}, b_{[1]} \rangle + b_{[2]} \wedge w_{[-2]} = 0$, $b_{[2]} = b_{[2]}^+$.

Hence we find that

(21)
$$\begin{aligned} k_\pi ={}& \Im \langle w_{[-1]}, b_{[1]} - w_{[1]} \rangle + w_\pi^{(0)} i \langle w_{[-1]}, w_{[-1]} \rangle \\ & + (dw_\pi^{(0)} + w_{[2]} - b_{[2]} + 2w_\pi^{(0)} \Re b^0) \wedge w_{[-2]}. \end{aligned}$$

(22) **Proposition.** Assume that $k_{[-2]} = k_{[-1]} = 0$. Then $k_\pi = 0$ if and only if

$$w_{[2]}^+ = b_{[2]} - (dw_\pi^{(0)})^+ - 2w_\pi^{(0)} \Re b^0 - \Im \langle w_{[-1]}, w_{[1]}^{(0)} \rangle.$$

We find
(23)
$$\begin{aligned} k_\mu ={}& \frac{n}{n+2} d(\Im b^0) + \Re \langle w_{[-1]}, b_{[1]} + (w_{\mathbf{su}}^{(0)} - \frac{n+2}{n} i w_\mu^{(0)} I) w_{[-1]} \rangle \\ & + i w_\mu^{(0)} \langle w_{[-1]}, w_{[-1]} \rangle + (dw_\mu^{(0)} + \Re \langle w_{[-1]}, w_{[1]}^{(0)} \rangle + w_\mu^{(0)} 2\Re b^0) \wedge w^{[-2]}. \end{aligned}$$

By taking the exterior derivative of the 2nd formula in (13.2), we find by (20) that
(24)
$$\{(db_{\mathbf{su}} - id\Im b^0 I)^+ - \Im\langle w_{[-1]}, b_{[1]}^+\rangle I + b_{\mathbf{su}}^+ \wedge b_{\mathbf{su}}^+ + ib_{[1]}^+ \otimes w_{[-1]}^*\} \wedge w_{[-1]} = 0.$$

Then it follows that
(25)
$$\frac{n}{n+2}(d\Im b^0)^{(0,2)} - \frac{1}{2}\langle b_{[1]}^{(0,1)}, w_{[-1]}\rangle = 0.$$

Therefore we find the following: Set
(26)
$$(d(\Im b^0))^{(1,1)} = (d\Im b^0)_{\alpha\bar\beta}w_{[-1]}^\alpha \wedge \overline{w\beta_{[-1]}}, \quad \Re b^0 = (\Re b^0)_\alpha w_{[-1]}^\alpha + (\Re b^0)_{\bar\beta}\overline{w_{[-1]}^\beta},$$

$$(d\Im b^0)^{(0)} = \tilde{b}_\alpha^0 w_{[-1]}^\alpha + \tilde{b}_{\bar\alpha}^0\overline{w_{[-1]}^\alpha}, \quad b_{[1]}^\alpha = b_{[1]\gamma}^\alpha w_{[-1]}^\gamma + b_{[1]\bar\beta}^\alpha\overline{w_{[-1]}^\beta}.$$

(27) **Proposition.** *Assume that $k_{[-2]} = k_{[-1]} = k_\pi = 0$. Then $k_\mu = 0$ if and only if*

$$w_\mu^{(0)} = \frac{1}{2}\frac{n+2}{n(n+1)}(\Im b_{[1]})_\alpha^\alpha + \frac{i}{2(n+1)}\underline{h}^{\alpha\bar\beta}(d\Im b^0)_{\alpha\bar\beta},$$

$$(w_{\mathbf{su}}^{(0)})_\beta^\alpha = \frac{n}{n+2}\underline{h}^{\alpha\bar\gamma}(d\Im b^0)_{\beta\bar\gamma} - \frac{1}{n+2}\underline{h}^{\kappa\bar\gamma}(d\Im b^0)_{\kappa\bar\gamma}\delta_\beta^\alpha + \frac{1}{2}\underline{h}^{\alpha\bar\kappa}\underline{h}_{\beta\bar\gamma}\overline{b_{[1]\kappa}^\gamma}$$

$$- \frac{1}{2}b_{[1]]\beta}^\alpha + \frac{1}{n}(\Im b_{[1]})_\gamma^\gamma\delta_\beta^\alpha,$$

$$\text{with } dw_\mu^{(0)} = \tilde{w}_{\mu\alpha}w_{[-1]}^\alpha + \tilde{w}_{\mu\bar\alpha}\overline{w_{[-1]}^\alpha},$$

$$w_{[1]}^{(0)\alpha} = -2\underline{h}^{\alpha\bar\beta}\{\tilde{w}_{\mu\bar\beta} + w_\mu^{(0)}(\Re b^0)_{\bar\beta} + \frac{1}{2}\tilde{b}_{\bar\beta}^0\}.$$

Finally we put the condition:
(28)
$$\text{tr } k_{[2]} = 0,$$

where for a 2-form ϕ
(29)
$$\text{tr } \phi = \underline{h}^{\alpha\bar\beta}\phi_{\alpha\bar\beta}, \quad \phi^{(1,1)} = \phi_{\alpha\bar\beta}w_{[-1]}^\alpha \wedge \overline{w_{[-1]}^\beta}.$$

(30) **Proposition.** *tr $k_{[2]} = 0$ if and only if*

$$w_{[2]}^{(0)} = \frac{1}{n}\{itr\, dw_{[2]}^+ - tr\,\langle w_{[1]}, w_{[1]}\rangle + 2itr\,(w_\pi^+ \wedge w_{[2]}^+)\}.$$

F) Note by (23)-(24) §1 that

(31)
$$k_{[-2]} = k_{[-1]} = k_\pi = k_\mu = \operatorname{tr} k_{[2]} = 0 \text{ if and only if}$$
$$K_{[-2]} = K_{[-1]} = K_\pi = K_\mu = \operatorname{tr} K_{[2]} = 0.$$

(32) **Definition.** *A CR Cartan connection is called normal when its curvature satisfies the above conditions.*

Clearly the normality condition is a globally defined condition. We also see

(33) **Proposition.** *When we fix a chart* (z, x^0) *and a local cross-section* (4), *for arbitrary choice of* $w_\pi^{(0)}$ *there is a unique normal CR Cartan connection. The isomorphism class of the normal CR Cartan connections is unique.*

G) We next discuss the global aspect of the normal CR Cartan connetions.

Fix a chart $x = (z, x^0)$. Beside the local cross-section $J(x)$ given in (4)-(6) consider a new cross-section

(34) $\qquad \underline{J}(x) = R_{h(x)} J(x)$ for a H-valued function $h(x)$.

$\underline{J}(x)$ induces a chart $(x, \underline{h})$, which is related to the original chart (x, h) by

(35)
$$h = h(x)\underline{h}.$$

A Cartan connection (2) has the two expressions:

(36) $\qquad \omega = \operatorname{Ad}(h^{-1})w(x) + h^{-1}dh = \operatorname{Ad}(\underline{h}^{-1})\underline{w}(x) + \underline{h}^{-1}d\underline{h}.$

Therefore

(37) $\qquad \underline{w}(x) = \operatorname{Ad}(h(x)^{-1})(w(x) + h(x)^{-1}dh(x)).$

From now we omit (x) for simplicity. By the above and by (23) §1 we find that

(38.1)
$$\underline{w}_{[-2]} = |a|^2 w_{[-2]}.$$

(38.2) $\qquad \underline{w}^{\alpha}_{[-1]} = a(u^{-1})^{\alpha}_{\gamma}w^{\gamma}_{[-1]} - |a|^2(u^{-1}\beta)^{\alpha}w_{[-2]}.$

(39) $\qquad \underline{w}_{\pi} = w_{\pi} + \Re ia\langle w_{[-1]}, \beta\rangle - |a|^2 s\, w_{[-2]} + d\log|a|.$

For a 1-form ϕ set

(40) $\phi = \phi_{\alpha}w^{\alpha}_{[-1]} + \phi_{\bar{\alpha}}\overline{w^{\alpha}_{[-1]}} + \phi^{(0)}w_{[-2]} = \tilde{\phi}_{\alpha}\underline{w}^{\alpha}_{[-1]} + \tilde{w}_{\bar{\alpha}}\overline{\underline{w}^{\alpha}_{[-1]}} + \tilde{\phi}^{(0)}\underline{w}_{[-2]}.$

Then

(41) $\qquad \tilde{\phi}_{\alpha} = \phi_{\gamma}\dfrac{1}{a}u^{\gamma}_{\alpha}, \quad \tilde{\phi}^{(0)} = \dfrac{1}{|a|^2}\phi^{(0)} + \phi_{\alpha}\dfrac{1}{a}\beta^{\alpha} + \phi_{\bar{\alpha}}\dfrac{1}{\bar{a}}\overline{\beta^{\alpha}}.$

Setting $\underline{w}^{(0)}_{\pi} = \tilde{\underline{w}}^{(0)}_{\pi}$ for simplicity, we then find

(42)
$$\underline{w}^{(0)}_{\pi} = \dfrac{1}{|a|^2}w^{(0)}_{\pi} - s + 2\Re w_{\pi\alpha}\dfrac{1}{a}\beta^{\alpha} + 2\Re\dfrac{1}{a}\beta^{\alpha}(d\log|a|)_{\alpha}$$
$$+ \dfrac{1}{|a|^2}(d\log|a|)^{(0)}.$$

On the other hand we see by (30) §3, (11), and (17) that

(43) $\quad \underline{p}^{(2)}_{\sharp} = \dfrac{1}{|a|^2}p^{(2)}_{\sharp} + s - 2\Re\dfrac{1}{a}\beta^{\alpha}\{w_{\pi\alpha} + \dfrac{1}{2}(d\log C)_{\alpha} + \dfrac{1}{n}c^{\gamma\bar{\sigma}}\overline{Q^{\alpha}}c_{\gamma\bar{\sigma}}\}.$

Therefore

(44) $\qquad \underline{w}^{(0)}_{\pi} + \underline{p}^{(2)}_{\sharp} = \dfrac{1}{|a|^2}(w^{(0)}_{\pi} + p^{(2)}_{\sharp}) + R, \quad$ where

(45)
$$R = 2\Re\dfrac{1}{a}\beta^{\alpha}\{(d\log|a|)_{\alpha} - \dfrac{1}{2}(d\log C)_{\alpha} - \dfrac{1}{n}C^{\gamma}_{\alpha}c^{\nu\bar{\sigma}}\overline{Q^{\gamma}}c_{\nu\bar{\sigma}}\} + \dfrac{1}{|a|^2}(d\log|a|)^{(0)}.$$

Note that we have the standard chart $(\underline{C}, \underline{C}^{\alpha}_0, \underline{C}^{\alpha}_{\gamma})$ induced by the local cross-section $\underline{J}$. We find by (32)-(33) §2 and (30) §3 that

(46) $\qquad \underline{C} = \dfrac{1}{|a|^2}C, \quad \underline{C}^{\alpha}_0 = \dfrac{1}{|a|^2}C^{\alpha}_0 + \dfrac{1}{a}\beta^{\gamma}C^{\alpha}_{\gamma}, \quad \underline{C}^{\alpha}_{\gamma} = \dfrac{1}{a}C^{\alpha}_{\nu}u^{\nu}_{\gamma}.$

Set

(47) $\qquad\qquad U = (d\log C)^{(0)} - C^{\alpha}_0 c^{\gamma\bar{\sigma}}\overline{Q^{\alpha}}c_{\gamma\bar{\sigma}}.$

Then we see by calculation that

$$(48) \qquad\qquad \underline{U} = \frac{1}{|a|^2} U - R.$$

Therefore the condition: $w_\pi^0 + p_\sharp^{(2)} + U = 0$ is a globally defined condition. We conclude

(49) **Proposition.** *When we choose*

$$(50) \qquad\qquad w_\pi^{(0)} = -p_\sharp^{(2)} - U,$$

the normal CR Cartan connection is globally defined.

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