Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 147–157

Ideals of multipliers

Joseph J. Kohn

Ideals of multipliers were introduced in [8] to find conditions on domains in complex manifolds under which subellipticity of the $\bar{\partial}$-Neumann problem holds. Similar ideals were used to study subellipticity on of $\Box_b$ on CR manifolds (see [9]). In [10] such ideals are used to study the situation when subellipticity breaks down but regularity still holds. Ideals of holomorphic multipliers in a somewhat different context have been used by Nadel (see [15]) and by Siu (see [16]) to prove global theorems in algebraic geometry. Here we will be concerned with the ideals that arise in the study of local regularity. We will briefly explain the use of subelliptic estimates then we define local and microlocal multipliers and show how to use them to derive subelliptic estimates. We also discuss the use of subelliptic multipliers when subellipticity fails. Finally we show how subelliptic multipliers give rise to invariants of complex analytic varieties.

§1. Definitions

A **CR manifold** is a compact C^∞ manifold M of dimension $2n+1$ endowed with an **integrable CR structure** which consists of a subbundle $T^{1,0}(M)$ of the complexified tangent bundle $\mathbb{C}T(M)$ satisfying the following. The complex fiber dimension of $T^{1,0}(M)$ is n,

$$T^{1,0}(M) \cap \overline{T^{1,0}(M)} = \{0\},$$

and if L and L' are local sections of $T^{1,0}(M)$ then $[L, L'] = LL' - L'L$ is also a local section of $T^{1,0}(M)$.
Let $\mathcal{A}_b^{p,q}$ denote the (p,q)-forms on M, let

$$\bar{\partial}_b : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q+1}$$

denote the corresponding exterior derivative, and let $\bar{\partial}_b^* : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q-1}$ denote the L_2 adjoint of $\bar{\partial}_b$.

Received March 16, 2002.
Research was partially supported by NSF Grant DMS#9801626.

We define the **complex energy form** Q_b on $\mathcal{A}_b^{p,q}$ by

$$Q_b(\varphi, \psi) = (\bar{\partial}_b\varphi, \bar{\partial}_b\psi) + (\bar{\partial}_b^*\varphi, \bar{\partial}_b^*\psi),$$

where (,) denotes the L_2 inner product on forms. We define the **complex laplacian**

$\Box_b : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q}$ by $\Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b$. Let

$$\mathcal{H}_b^{p,q} = \{\varphi \in L_2^{p,q} \mid \Box_b\varphi = 0\}$$

Note that if $\alpha \perp \mathcal{H}_b^{p,q}$ and then $\Box_b\varphi = \alpha$ if and only if $Q_b(\varphi, \psi) = (\alpha, \psi)$, for all ψ.

If $u \in C_0^\infty(\mathbb{R}^m)$ and if $s \in \mathbb{R}$ we define $\|u\|_s$ the **Sobolev** s**-norm** of u by

$$\|u\|_s^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 dV$$

If $u \in C^\infty(M)$ we define $\|u\|_s$ by choosing a partition of unity $\{\zeta_\nu\}$ which is subordinate to a covering by coordinate charts and set $\|u\|_s^2 = \sum \|\zeta_\nu u\|^2$.

§2. Subelliptic estimates

If $P \in M$ we say that a **subelliptic estimate** for (p, q)-forms holds at P if there exists a neighborhood U of P and constants C and ε such that

$$(\bullet_q) \qquad \|\varphi\|_\varepsilon^2 \leq C(Q_b(\varphi, \varphi) + \|\varphi\|^2),$$

for all $\varphi \in \mathcal{A}_b^{p,q}$ with support in U.

The above estimate has the following consequences (see [12]).

1. If $\Box_b\varphi = \alpha$ and if α is C^∞ on U then φ is C^∞ on U.

2. $\mathcal{H}_b^{p,q} \subset C^\infty(M)$.

3. If α is a (p, q)-form which is C^∞ on U and if ψ is a $(p, q-1)$-form orthogonal to $\mathcal{H}_b^{p,q-1}$ such that $\bar{\partial}_b\psi = \alpha$ then ψ is C^∞ on U.

4. Let $S_b : L_2^{p,q-1} \to \mathcal{N}^{p,q-1}(\bar{\partial}_b)$, where $\mathcal{N}^{p,q-1}(\bar{\partial}_b)$ denotes the null space of $\bar{\partial}_b$ and S_b the orthogonal projection. If $\theta \in L_2^{p,q-1}$ with θ in C^∞ on U then $S_b\theta$ is C^∞ on U.

Duality

Let $\{L_1, \dots, L_n\}$ be an orthonormal basis for $(1,0)$ vector fields on a neighborhood $U \subset M$ of P and let $\{L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n, T\}$ be a basis for the complex vector fields on U with $\bar{T} = -T$. Let $\{\omega_1, \dots, \omega_n\}$ be the dual basis of $(1,0)$ forms. Then if $\varphi \in \mathcal{A}_b^{0,q}$ with support in U we have $\varphi = \sum \varphi_I \bar{\omega}^I$ where I runs through the strictly in creasing q-tuples of integers between 1 and n and where $\bar{\omega}^I = \bar{\omega}_{i_1} \wedge \cdots \wedge \bar{\omega}_{i_q}$. We define $F^q\varphi \in \mathcal{A}^{0,n-q}$ by

$$F^q\varphi = \sum \epsilon^I_{I'} \bar{\varphi}_I \bar{\omega}^{I'},$$

where I' denotes the increasing $(n-q)$-tuple consisting of integers between 1 and n which are not in I, and $\epsilon^I_{I'}$ is defined by

$$\epsilon^I_{I'} \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_n = \bar{\omega}^I \wedge \bar{\omega}^{I'}.$$

Then we have:

$$\bar{\partial}_b F^q \varphi = F^{q-1} \bar{\partial}_b^* \varphi + \sum a_{IJ} \bar{\varphi}_I \bar{\omega}^J$$

and

$$\bar{\partial}_b^* F^q \varphi = F^{q+1} \bar{\partial}_b \varphi + \sum b_{IJ} \bar{\varphi}_I \bar{\omega}^J.$$

Hence,

$$Q_b(\varphi, \varphi) = Q_b(F^q\varphi, F^q\varphi) + 0(\|\varphi\|^2).$$

Therefore, since $\|\varphi\|_\varepsilon = \|F^q\varphi\|_\varepsilon$, we conclude that $(\bullet_q)$ holds if and only if $(\bullet_{n-q})$ holds.

Microlocalization

Let $\{x_1, \dots, x_{2n}, t\}$ be real coordinates on U with origin at P such that

$$\frac{\partial}{\partial x_j} = \Re(L_j|_P), \ \frac{\partial}{\partial x_{j+n}} = \Im(L_j|_P),$$

and $\frac{\partial}{\partial t} = \sqrt{-1}T$. Let $\{\xi_1, \dots, \xi_{2n+1}\}$ denote the dual coordinates. If $u \in C_0^\infty(U)$ we have the microlocal decomposition $u = u^+ + u^- + u^0$, where $\mathcal{F}u^+$ and $\mathcal{F}u^-$ have supports in conical neighborhoods of $(0, \dots, 0, 1)$ and $(0, \dots, 0, -1)$, respectively and $\mathcal{F}u^0$ is supported in the union of the unit ball and the complement of the above conical neighborhoods (here $\mathcal{F}$ denotes the Fourier transform).

Let $U' \supset \bar{U}$ be a small neighborhood and let $\zeta \in C_0^\infty(U')$ with $\zeta = 1$ on U. Then we have

$$(\bullet_q^0) \qquad \|\zeta\varphi^0\|_1^2 \leq C(Q_b(\zeta\varphi^0, \zeta\varphi^0) + \|\varphi\|^2),$$

for all $\varphi \in C_0^\infty(U)$. Thus to prove $\bullet_q$ it suffices to establish the corresponding estimates $(\bullet_q^+)$ and $(\bullet_q^-)$ for $\|\zeta\varphi^+\|_\varepsilon^2$ and for $\|\zeta\varphi^-\|_\varepsilon^2$, respectively.

Let $\Omega \subset X$ be a domain in a complex manifold X which has a smooth boundary M and such that $\bar{\Omega}$ is compact. We then say that the $\bar{\partial}$-Neumann problem for (p,q)-forms at $P \in M$ is subelliptic if there exists a neighborhood U of P and constants ε and C such that

$$(\bullet\bullet_q) \qquad |||\varphi|||_\varepsilon^2 \leq C(Q(\varphi,\varphi) + |||\varphi|||^2),$$

for all $\varphi \in Dom(\bar{\partial}^*) \cap \mathcal{A}^{p,q}$ with support in $U \cap \bar{\Omega}$. Here $\mathcal{A}^{p,q}$ denotes the space of (p,q)-forms in $C^\infty(\bar{\Omega})$,

$$Q(\varphi,\varphi) = ((\bar{\partial}\varphi, \bar{\partial}\varphi)) + ((\bar{\partial}^*\varphi, \bar{\partial}^*\varphi)),$$

and $|||\ |||$, $((\ ))$, $|||\ |||_\varepsilon$ denote the L_2 norm, the L_2 inner product, and the Sobolev ε-norm on $\bar{\Omega}$, respectively. The estimate $(\bullet\bullet_q)$ has the following consequences (see [12]).

1. If $\Box\varphi = \alpha$ and if α is C^∞ on $U \cap \bar{\Omega}$ then φ is C^∞ on $U \cap \bar{\Omega}$. Here $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ with domain consisting of $\{\varphi \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*) \mid \bar{\partial}\varphi \in Dom(\bar{\partial}^*)$ and $\bar{\partial}^*\varphi \in Dom(\bar{\partial})\}$.

2. $\mathcal{H}^{p,q} \subset C^\infty(\bar{\Omega})$, where $\mathcal{H}^{p,q} = \{\varphi \mid \Box\varphi = 0\}$, is finite dimensional.

3. If α is a (p,q)-form which is C^∞ on $U \cap \bar{\Omega}$ and if ψ is a $(p,q-1)$-form orthogonal to $\mathcal{N}^{p,q-1}(\bar{\partial})$, where $\mathcal{N}^{p,q-1}(\bar{\partial})$ denotes the null space of $\bar{\partial}$, such that $\bar{\partial}_b\psi = \alpha$ then ψ is C^∞ on $U \cap \bar{\Omega}$.

4. If $B : L_2^{p,q-1}(\Omega) \to \mathcal{N}^{p,q-1}(\bar{\partial})$ is the orthogonal projection and if $\theta \in L_2^{p,q-1}(\Omega)$ with θ in C^∞ on U then $B\theta$ is C^∞ on $U \cap \bar{\Omega}$.

Denote by M the boundary of Ω and suppose that in a neighborhood of M there exists a real valued function r such that $r < 0$ in Ω which on M satisfies $r = 0$ and $dr \neq 0$. Let $\{z_1, \ldots, z_{n+1}\}$ be local holomorphic coordinates with origin at $P \in M$ such that $r_{z_i}(P) = 0$ for $i = 1, \ldots, n$ and $r_{z_{n+1}}(P) = 1$. Let

$$L_i = \frac{\partial}{\partial z_i} - r_{z_i}\frac{\partial}{\partial z_{n+1}}$$

and

$$T = r_{\bar{z}_{n+1}}\frac{\partial}{\partial z_{n+1}} - r_{z_{n+1}}\frac{\partial}{\partial \bar{z}_{n+1}}.$$

Theorem 2.1. *The $\bar{\partial}$-Neumann problem on Ω is subelliptic for (p,q)-forms at P, that is $(\bullet\bullet_q)$ holds, if and only if $(\bullet_q^+)$ holds on M.*

§3. Local and microlocal multipliers

Definition 3.1. *If $P \in M$ a* **subelliptic multiplier** *for (p,q)-forms at P is a germ of a C^∞ function f such that there exists a neighborhood U of P and positive constants ε and C so that*

$$(*_q) \qquad \|f\varphi\|_\varepsilon^2 \leq C(Q_b(\varphi,\varphi) + \|\varphi\|^2),$$

for all $\varphi \in \mathcal{A}_b^{p,q}$ with support in U. Note that this estimate is independent of p.

Let $\mathcal{I}_q$ denote the set of subelliptic multipliers. $\mathcal{I}_q$ satisfies the following.

1. $\mathcal{I}_q$ is an ideal.

2. ${}^{\mathbb{R}}\sqrt{\mathcal{I}_q} \subset \mathcal{I}_q$, here ${}^{\mathbb{R}}\sqrt{\mathcal{I}_q}$ denotes the real radical of $\mathcal{I}_q$ consisting of all germs g such that there exist $m \in \mathbb{Z}^+$ and $f \in \mathcal{I}_q$ with $|g|^m \leq |f|$.

Analogously we define $\mathcal{I}_q^+$ and $\mathcal{I}_q^-$ by the estimates

$$(*_q^+) \qquad \|f\zeta\varphi^+\|_\varepsilon^2 \leq C(Q_b(\zeta\varphi^+,\zeta\varphi^+) + \|\zeta'\varphi^+\|^2)$$

and

$$(*_q^-) \qquad \|f\zeta\varphi^-\|_\varepsilon^2 \leq C(Q_b(\zeta\varphi^-,\zeta\varphi^-) + \|\zeta'\varphi^-\|^2).$$

Then $\mathcal{I}_q = \mathcal{I}_q^+ \cap \mathcal{I}_q^-$ and we have that: $(\bullet_q^+)$ holds if and only if $1 \in \mathcal{I}_q^+$, $(\bullet_q^-)$ holds if and only if $1 \in \mathcal{I}_q^-$, $(\bullet_q)$ holds if and only if $1 \in \mathcal{I}_q$, and $(\bullet\bullet_q)$ holds if and only if $1 \in \mathcal{I}_q^+$.

These ideals satisfy the following duality property

$$\mathcal{I}_q^+ = \mathcal{I}_{n-q}^-.$$

This follows since $\|\varphi^+\| = \|(\bar{\varphi})^-\|$ and

$$\|f\zeta\varphi^+\|_\varepsilon = \|f\zeta(F_q\varphi)^-\|_\varepsilon + O(\|\varphi\|)$$

and

$$Q_b(\zeta\varphi^+,\zeta\varphi^+) = Q_b(\zeta(F_q\varphi)^-,\zeta(F_q\varphi)^-) + O(\|\varphi\|^2).$$

Pseudoconvexity

We define the **Levi form** in an open set $U \subset M$ to be the hermitian form $\mathcal{L}_P$ on $T_P^{1,0}$, for each $P \in U$ defined as follows. Let γ be a real one form in U such that $\gamma \neq 0$ and $\gamma(L) = 0$ for all $L \in T^{1,0}$. Then we set $\mathcal{L}(L,L') = \sqrt{-1} < d\gamma, L \wedge \bar{L}' >$. Then M is **pseudoconvex** if it can be covered by open sets on which $\mathcal{L}$ is positive semi-definite. In terms of the above basis we have $\mathcal{L}(L_i, L_j) = c_{ij}$ and

$$[L_i, \bar{L}_j] = c_{ij}T \quad \text{mod } (L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n).$$

If M is pseudoconvex in a neighborhood of P we will construct a sequence of ideals

$$\mathcal{I}^+_{q,k} \subset \mathcal{I}^+_{q,k+1} \subset \mathcal{I}^+_q .$$

We define the quadratic form c_{IJ}, with q-tuples I and J, by

$$c_{IJ} = \sum_{i,j,K} \epsilon^{iK}_I \epsilon^{jK}_J c_{ij},$$

where K runs over all ordered $(q-1)$-tuples. Each of the coefficients ϵ^{iK}_I is either $0, 1$, or -1 defined as follows. First, if $i \notin K$ we denote by $\langle iK \rangle$ the ordered q-tuple containing i and the elements of K. Then we define

$$\epsilon^{iK}_I = \begin{cases} 0 & \text{if } i \in K \\ 0 & \text{if } \langle iK \rangle \neq I \\ \operatorname{sgn} \left\langle {iK \atop I} \right\rangle & \text{if } \langle iK \rangle = I, \end{cases}$$

where $\operatorname{sgn} \left\langle {iK \atop I} \right\rangle$ denotes the sign of the permutation $\{i, K\} \to I$. We observe the following.

A. If $(c_{ij}) \geq 0$ then $(c_{IJ}) \geq 0$.

B. If $(c_{ij}) \geq 0$ then $\sqrt[\mathbb{R}]{(\det c_{IJ})}$ equals the real radical of the ideal generated by the $(n-q+1) \times (n-q+1)$ subdeterminants of (c_{ij}).

Integration by parts gives.

$$\sum (c_{IJ} T\varphi_I, \varphi_J) + \sum \|\bar{L}_i \varphi_I\|^2 = Q_b(\varphi, \varphi) + \textit{ error}.$$

Substituting $F^q\varphi$ for φ and conjugating we get

$$-\sum (c_{I'J'} T\varphi_I, \varphi_J) + \sum \|L_i \varphi_I\|^2 = Q_b(\varphi, \varphi) + \textit{ error}.$$

Substituting $\zeta\varphi^+$ and $\zeta\varphi^-$ for φ in the first and second equation, respectively; we obtain

$$\|(\det c_{IJ})\zeta\varphi^+\|^2_{\frac{1}{2}} \leq C(Q_b(\zeta\varphi^+, \zeta\varphi^+) + \|\varphi\|^2)$$

and

$$\|(\det c_{I'J'})\zeta\varphi^-\|^2_{\frac{1}{2}} \leq C(Q_b(\zeta\varphi^-, \zeta\varphi^-) + \|\varphi\|^2).$$

Hence the $(n-q+1)\times(n-q+1)$ subdeterminants of (c_{ij}) are in $\mathcal{I}_q^+$ and the $(q+1)\times(q+1)$ subdeterminants of (c_{ij}) are in $\mathcal{I}_q^-$.

Given germs of C^∞ functions $f_1,\dots,f_n$ we define $n\times 2n$ matrix $M(f_1,\dots,f_n)$ by

$$M(f_1,\dots,f_n)=\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \\ L_1 f_1 & L_2 f_1 & \dots & L_n f_1 \\ L_1 f_2 & L_2 f_2 & \dots & L_n f_2 \\ \vdots & \vdots & \ddots & \vdots \\ L_1 f_n & L_2 f_n & \dots & L_n f_n \end{pmatrix}.$$

Denote by $(Det^j M(f_1,\dots,f_n))$ the ideal generated by the $j\times j$ subdeterminants of $M(f_1,\dots,f_n)$.

Theorem 3.2. *If the $f_1,\dots,f_n$ are in $\mathcal{I}_q^+$ then $Det^{n-q+1}M(f_1,\dots,f_n) \subset \mathcal{I}_q^+$ and if the $f_1,\dots,f_n$ are in $\mathcal{I}_q^-$ then $Det^{q+1}M(f_1,\dots,f_n)\subset\mathcal{I}_q^-$.*

We define $\mathcal{I}_{q,k}^+$ by induction on k:

$$\mathcal{I}_{q,1}^+ = \sqrt[\mathbb{R}]{(Det^{n-q+1}M(0))}$$

and

$$\mathcal{I}_{q,k+1}^+ = \sqrt[\mathbb{R}]{(\mathcal{I}_{q,k}^+,\ \mathcal{D}^{n-q+1}(\mathcal{I}_{q,k}^+))},$$

where $\mathcal{D}^{n-q+1}(\mathcal{I}_{q,k}^+)$ is the set of all $(n-q+1)\times(n-q+1)$ subdeterminants of $M(f_1,\dots,f_n)$ for all n-tuples $(f_1,\dots,f_n)$ in $\mathcal{I}_{q,k}^+$. Similarly we define $\mathcal{I}_{q,k}^-$ by:

$$\mathcal{I}_{q,1}^- = \sqrt[\mathbb{R}]{(Det^{q+1}M(0))}$$

and

$$\mathcal{I}_{q,k+1}^- = \sqrt[\mathbb{R}]{(\mathcal{I}_{q,k}^-,\ \mathcal{D}^{q+1}(\mathcal{I}_{q,k}^-))}.$$

We then have:

$\mathcal{I}_{q,k}^+ \subset \mathcal{I}_{q,k+1}^+ \subset \mathcal{I}_q^+$,

$\mathcal{I}_{q,k}^+ \subset \mathcal{I}_{q+1,k}^+$, and

$\mathcal{I}_{q,k}^+ = \mathcal{I}_{n-q,k}^-$.

Hence if we set

$$\mathcal{I}_{q,k} = \mathcal{I}^+_{q,k} \cap \mathcal{I}^-_{q,k} = \mathcal{I}^+_{\min\{q,n-q\},k} \subset \mathcal{I}_q.$$

we conclude that if for some k

$$(**_q) \qquad 1 \in \mathcal{I}_{q,k}$$

then the subelliptic estimate $(\bullet_q)$ holds. The condition $(**_q)$ is called **finite ideal q-type**.

The conjecture is that $(**_q)$ is a necessary condition for the subelliptic estimate $(\bullet_q)$. Generalizing the work of Greiner (see [7]) this can be established when (c_{IJ}) and $(c_{I'J'})$ are diagonalizable on U. This diagonalizability condition implies that it is not necessary to use radicals in deriving $1 \in \mathcal{I}^+_{q,k}$. More generally Catlin (see [1]) has shown that subellipticity for the $\bar{\partial}$-Neumann problem is equivalent to the condition of **D'Angelo finite q-type**. The passage from the $\bar{\partial}$-Neumann problem to CR manifolds is routine. Thus the problem is to prove that finite ideal q-type is equivalent to finite D'Angelo type (see [4]). It is easy to prove that finite D'Angelo q-type implies finite ideal q-type, so the problem is to prove the converse. In case the CR manifold is real analytic this follows by use of methods developed by Diederich and Fornaess (see [5]).

§4. When subellipticity fails

The Fedii example in $\mathbb{R}^2$ is

$$Eu = -\frac{\partial^2 u}{\partial x^2} - a(x)\frac{\partial^2 u}{\partial t^2} = f,$$

where $a(x) > 0$ when $x \neq 0$ (see [F]). This equation is always hypoelliptic, it is elliptic if and only if $a(0) > 0$ and it is subelliptic if and only if $a(x) > c|x|^m$. The best way to see this is to note that a is a subelliptic multiplier in the sense that:

$$\|au\|_1^2 \leq C((\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + (a\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t})) = C(Eu, u).$$

In the Kusuoka and Stroock example (see [13]) in $\mathbb{R}^3$

$$E = -\frac{\partial^2}{\partial x^2} - a(x)\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}$$

where $a(x) > 0$ when $x \neq 0$, E is hypoelliptic if and only if

$$\lim_{x \to 0} x \log a(x) = 0.$$

Generalization of the Fedii example on $\mathbb{R}_x^n \times \mathbb{R}_t^m$, $E = E_1 + c(x,t)E_2$, where

$$E_1 = -\sum a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$E_2 = -\sum b_{ij}(x,t)\frac{\partial^2}{\partial t_i \partial t_j},$$

$(a_{ij}) \geq 0$, $(b_{ij}) \geq 0$, and the E_1 and E_2 are uniformly subelliptic on $\mathbb{R}_x^n$ and on $\mathbb{R}_t^m$, respectively. Then E is hypoelliptic whenever there exists a manifold $S \subset \mathbb{R}^n \times \mathbb{R}^m$ which is transversal to $\mathbb{R}_x^n$ and $c(x,t) > 0$ whenever $(x,t) \notin S$.

The analogous statement for $\Box_b$ for (p,q)-forms would be that $\Box_b$ is hypoelliptic if there exists $f \in \mathcal{I}_q$ and a manifold $S \subset M$ of holomorphic dimension $\leq \min\{n-q-1,\ q-1\}$ such that $f \neq 0$ outside of S.

Christ (see [3]) has shown this does not hold in general but it does hold in case $M \subset \mathbb{C}^{n+1}$ given by a defining function r with special symmetries (see [K4]), such as: $r = \Re(z_{n+1}) - F(\sum |z_i|^2)$.

To find estimates for the $\bar{\partial}$-Neumann problem for pseudoconvex domains in $\mathbb{C}^2$, Christ has used the method of superlogarithmic estimates (see [2]), developed by Morimoto (see [14]). Christ's results can easily be generalized to the study of $\Box_b$ on (p,q)-forms on pseudo convex CR manifolds when the quadratic forms c_{IJ} and $c_{I'J'}$ are diagonalizable. More generally the result (prOven in [10]) is:

Theorem 4.1. *$\Box_b$ is hypoelliptic if there exists $f \in \mathcal{I}_q$ and a manifold $S \subset M$ of holomorphic dimension $\leq \min\{n-q-1,\ q-1\}$ such that $f \neq 0$ outside of S and*

$$\lim_{x \to S} d(x,S)\log|f(x)| = 0,$$

where $d(x,S)$ denotes the distance from x to S.

To prove this theorem in general we need the following localization lemma.

Lemma 4.2. *If M is pseudoconvex if $P \in S \subset M$ with S a submanifold of holomorphic dimension $\leq \min\{n-q-1,\ q-1\}$, Then there exists a neighborhood U of P such that if*
$S_a = \{Q \in U \mid dist(Q,S) \leq a$ then there exists $C > 0$ such that

$$\|\varphi\|_{S_a}^2 \leq C(a^2 Q_b(\varphi,\varphi) + \|\varphi\|_{M-S_a}^2 + \|\varphi\|_{-1}^2),$$

for all $\varphi \in \mathcal{A}_b^{p,q}$ with support in U. Here $\|..\|_X$ denotes the L_2-norm over X.

§5. Multipliers associated with singularities

Let $\{h_1, \ldots, h_m\}$ be holomorphic functions defined in a neighborhood of the origin in $\mathbb{C}^n$, with $h_j(0) = 0$. Let $M \subset \mathbb{C}^{n+1}$ be a pseudoconvex CR manifold which near the origin is defined by

$$\Re(z_{n+1}) = \sum |h_j(z_1, \ldots, z_n)|^2.$$

If $\mathcal{G}$ is an ideal of germs of holomorphic functions in $\mathbb{C}^n$ at the origin we define $\mathbf{B}(\mathcal{G})$ to be the set of all $n \times p$ matrices with $p \geq n$

$$B(g_1, \ldots, g_p) = \begin{pmatrix} g_{1z_1} & \cdots & g_{1z_n} \\ \vdots & \ddots & \vdots \\ g_{pz_1} & \cdots & g_{pz_n} \end{pmatrix},$$

for all n-tuples in $\mathcal{G}$. Let $\mathcal{D}^j(\mathbf{B}(\mathcal{G}))$ denote the ideal generated by the set of all $j \times j$ subdeterminants of $B(g_1, \ldots, g_p)$ for all $B(g_1, \ldots, g_p) \in \mathbf{B}(\mathcal{G})$.

Set

$$J_1^q(\mathcal{G}) = \sqrt{\mathcal{D}^{n-q}(\mathbf{B}(\mathcal{G}))}\ .$$

Inductively we define

$$J_{k+1}^q(\mathcal{G}) = J_1^q(\mathcal{G}, J_1^q(\mathcal{G}), \ldots, J_k^q(\mathcal{G})).$$

Let $\mathcal{H} = (h_1, \ldots, h_m)$, the ideal generated by $h_1, \ldots, h_m$. The following result shows how the ideals $J_k^q(\mathcal{H})$ determine subellipticity on M.

Proposition 5.1. *$1 \in J_k^q(\mathcal{I}_{q,k}^+)$ if and only if $1 \in \mathcal{I}_{q,k}^+$.*

Denoting by $V(\mathcal{H})$ the variety of $\mathcal{H}$, we have

$$\dim V(\mathcal{H}) = q \iff \begin{cases} 1 \in J_k^{q+1}(\mathcal{H}) & \text{for some } k \\ 1 \notin J_k^q(\mathcal{H}) & \text{for all } k. \end{cases}$$

Suppose $\dim V(\mathcal{H}) = q$ let k_0 be the least k for which $1 \in J_k^{q+1}(\mathcal{H})$. Note that $k_0 = 1$ if and only if 0 is not a singular point and that $V(\mathcal{H}, J_1^{q+1}(\mathcal{H}))$ is the singular variety of $V(\mathcal{H})$. If $q_1 = \dim V(\mathcal{H}, J_1^{q+1}(\mathcal{H}))$ we let k_1 be the least k for which $1 \in J_k^{q_1+1}(\mathcal{H}, J_1^{q+1}(\mathcal{H}))$. We continue defining $k_2, k_3, \ldots$ and these numbers are invariants of the singularity.

References

[1] D. Catlin, Subelliptic estimates for the $\overline{\partial}-$Neumann problem on pseudoconvex domains, *Ann. of Math.* 126 (1987), 131–191; Necessary conditions for the subellipticity of the $\overline{\partial}-$Neumann problem, *Ann. of Math.* 117 (1983), 147–171.

[2] M. Christ, Hypoellipticity in the infinitely degenerate regime, *Complex Analysis and Geometry*, Proc. Conf. Ohio State Univ., Walter de Gruyer, New York 2001, 59–84.

[3] M. Christ, Spiraling and nonhypoellipticity, *Complex Analysis and Geometry*, Proc. Conf. Ohio State Univ., Walter de Gruyer, New York 2001, 85–101.

[4] J. P. D'Angelo, Real hypersurfaces, orders of contact and applications, *Ann. of Math.* 115 (1982), 615–637.

[5] K. Diederich and J. E. Fornaess, Pseudoconvex domains with real-analytic boundary, *Ann. of Math.*, 107(1978), 371–384.

[6] V. S. Fedii, On a criterion for hypoellipticity, *Mat. Sb.* 14 (1971), 15–45.

[7] P. C. Greiner, On subelliptic estimates of the $\bar{\partial}$-Neumann problem in $\mathbb{C}^2$, *J. Diff. Geom.*, 9(1974), 239-250.

[8] J. J. Kohn, Subellipticity of the $\overline{\partial}-$Neumann problem on pseudoconvex domains: sufficient conditions, *Acta Math.* 142 (1979), 79–122.

[9] J. J. Kohn, Microlocalization of CR structures, *Proceedings Several Complex Variables, Hangzhou Conference 1981*, Birkhauser, Boston 1984, 29–36, .

[10] J. J. Kohn, Superlogarithmic estimates on pseudoconvex domains and CR manifolds, *Ann. of Math.* to appear (2002).

[11] J. J. Kohn, Hypoellipticity at non-subelliptic points, *Complex Analysis and Geometry*, Proc. Conf. Ohio State Univ., Walter de Gruyer, New York 2001, 161–170.

[12] J. J. Kohn and L. Nirenberg, Non–coercive boundary value problems, *Comm. Pure Appl. Math.*, 18 (1965), 459–510.

[13] S. Kusuoka and D. Stroock, Applications of the Mallavain Calculus II, *J. Fac. Sci. Univ. Tokyo*, 1–76.

[14] Y. Morimoto, A criterion for hypoellipticity of second order differential operators, *Osaka J. Math.*, vol. 24 (1987), 651–675.

[15] A. Nadel, Multiplier ideal sheaves and the existence of Kähler-Einstein metrics of positive curvature, *Ann. of Math.* 132(1989), 549–596.

[16] Yum-Tong Siu, Very ampleness part of Fujita's conjecture and multiplier ideal sheaves of Kohn and Nadel, *Complex Analysis and Geometry*, Proc. Conf. Ohio State Univ., Walter de Gruyer, New York 2001, 171–191.

Mathematics Department
Princeton University
Princeton, NJ 08544-1000
U.S.A.