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# Problems related to hyperbolicity of almost complex structures

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The contents of my talk at this conference are in two papers [4] and [5]. So the emphasis here is on what I was unable to deliver at the conference for lack of time.

## §1. Generic almost complex structures and hyperbolicity

Let $(M, J)$ be an almost complex manifold. Because of paucity of local holomorphic functions in general, there is no complex function theory on $(M, J)$. However, there is an abundant supply of holomorphic mappings from a disk of $\mathbf{C}$ into $(M, J)$ [6], and we can define the intrinsic pseudo-distance $d_M$ and hyperbolicity for an almost complex manifold $M$ exactly in the same way as in the complex manifold case.

It is obvious that if $M$ is hyperbolic, every holomorphic map $f\colon \mathbf{C} \to M$ is constant. Conversely, if $M$ is compact and if there exist no non-constant holomorphic maps from $\mathbf{C}$ into $M$, then $M$ is hyperbolic. In order to state the theorem a little more precisely, let $z$ denote the natural coordinate system in $\mathbf{C}$, and take a length function $E$ on $M$. We call a non-constant holomorphic map $f\colon \mathbf{C} \to M$ a **complex line** if

$$f^*E^2 \leq C dz d\bar{z}$$

for some constant $C$. If $f(\mathbf{C})$ is contained in a compact subset of $M$, then this condition is independent of the choice of $E$. Let $S$ be a subset (usually a domain) in $M$. We say that a complex line $f\colon \mathbf{C} \to M$ is a **limit complex line coming from** $S$ if on each disk $D_R = \{|z| < R\}$ of radius $R$ the mapping $f|_{D_R}$ is the limit of a sequence of holomorphic mappings of $D_R$ into $S$. In this case, we have $f(\mathbf{C}) \subset \bar{S}$. Trivially, every complex line in $M$ is a limit complex line coming from $M$.

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The proof for the following Brody's hyperbolicity criterion is exactly the same as in the complex case ([3; pp.100-103]).

(1.1) **Theorem.** *If a compact almost complex manifold $M$ is not hyperbolic, then there is a complex line $f\colon \mathbf{C} \to M$.*

The following almost complex version of (3.6.8) in [3; p.106] holds.

(1.2) **Theorem.** *Let $Z$ be an almost complex manifold, and $Y$ a compact almost complex submanifold of $Z$. If $Y$ is hyperbolic, there is a relatively compact neighborhood $U$ of $Y$ which is hyperbolically imbedded in $Z$.*

(1.3) **Corollary.** *Let $\pi\colon Z \to X$ be an almost complex fiber space with compact fiber. If the fiber $\pi^{-1}(p_0)$ at a point $p_0 \in X$ is hyperbolic, then in a small neighborhood of $p_0$ every fiber is hyperbolic.*

**Remark.** The infinitesimal form $F_M$ of the pseudo-distance $d_M$ can be defined as in the complex case. As we remarked in [3; p.101], for the proofs of the results above we use only the most basic properties of $F_X$ that are obvious from the definition. We need not know whether $F_X$ is upper semi-continuous and $d_M$ is the integrated form of $F_X$, although this is also an interesting question.

In view of (1.3) it seems to be reasonable to conjecture that if $(M, J_0)$ is a compact hyperbolic almost complex manifold, all nearby almost complex structures $J$ are hyperbolic. (By "nearby" we mean the first and second partial derivatives of $J$ are close to those of $J_0$). Unlike the moduli space of complex structures on a compact manifold, the set of almost complex structures (modulo diffeomorphisms) is huge and has no nice structures. So, (1.3) by itself does not prove the conjecture.

If $(M, J_0, g_0)$ is an almost Hermitian manifold with its holomorphic sectional curvature bounded by a negative constant, then for $J$ sufficiently close to $J_0$ and for the Hermitian metric $g$ defined by

$$g(u, v) = \frac{1}{2}(g_0(u, v) + g_0(Ju, Jv)),$$

the holomorphic sectional curvature remains bounded by a negative constant. On the other hand, as we have shown in [4], an almost Hermitian manifold with its holomorphic sectional curvature bounded by a negative constant is hyperbolic. So this is also another supporting evidence for the conjecture above.

A related question is hyperbolicity of a generic almost complex structure. Let $(M, J_0)$ be a compact non-hyperbolic almost complex manifold. In view of (1.1) it seems that an arbitrarily small, but suitable deformation of $J_0$ would result in a hyperbolic almost complex structure.

## §2. Automorphisms of almost complex manifolds

Generalizing the old theorem of Bochner for compact complex manifolds, Boothby, Wang and I proved in [1] that the automorphism group $\mathrm{Aut}(M, J)$ of a compact almost complex manifold $(M, J)$ is a Lie group with Lie algebra $\mathbf{aut}(M, J)$ consisting of infinitesimal automorphisms of $(M, J)$. The condition that a (real) vector field $u$ is an infinitesimal automorphism of $(M, J)$ is given by

$$(2.1) \qquad L_u(Jv) = J(L_u v) \qquad \text{for all vector fields} \quad v,$$

where $L_u$ denotes the Lie differentiation with respect to $u$. Since $L_u v = [u, v]$, the condition above may be written as

$$(2.2) \qquad [u, Jv] = J[u, v] \qquad \text{for all vector fields} \quad v.$$

In the complex case, the automorphism group is a complex Lie group. This is because if $u \in \mathbf{aut}(M, J)$, then $Ju \in \mathbf{aut}(M, J)$. However, this is not the case for almost complex manifolds.

The integrability condition for $J$ is given by vanishing of the Nijenhuis tensor $N$ defined by

$$N(u, v) = [Ju, Jv] - J[Ju, v] + J(J[u, v] - [u, Jv]).$$

So, if $u, Ju \in \mathbf{aut}(M, J)$, then $N(u, v) = 0$ for all $v$. It is now clear that we cannot expect to have, in general, a complex Lie group acting on an almost complex manifold.

Now, if $(M, J)$ is a compact hyperbolic almost complex manifold, $\mathrm{Aut}(M, J)$ is compact since it preserves the intrinsic distance $d_M$. We know that for a compact hyperbolic complex manifold $(M, J)$, the group $\mathrm{Aut}(M, J)$ is finite. The reason is that if $\dim \mathrm{Aut}(M, J) > 0$, then $\mathrm{Aut}(M, J)$ has a complex one-parameter subgroup and the action of this one-parameter subgroup gives rise to nonconstant holomorphic maps from $\mathbf{C}$ into $M$, in violation of the hyperbolicity. Clearly, this argument cannot be used in the almost complex case.

However, we can circumvent this obstacle by using a slightly modified argument. If $u \in \mathbf{aut}(M, J)$, then by (2.2) we have

$$[u, Ju] = J[u, u] = 0.$$

Hence, the one-parameter groups $e^{su}$ and $e^{tJu}$ commute. Given a point $p_0 \in M$, the map $f\colon \mathbf{C} \to M$ defined by

$$f(s + ti) = e^{su+tJu}(p_0), \qquad s + ti \in \mathbf{C}$$

is holomorphic. For a suitable choice of $p_0$ this map is nonconstant, which proves the following theorem.

(2.3) **Theorem**. *The automorphism group of a compact hyperbolic almost complex manifold is finite.*

Let $X$ and $Y$ be compact almost complex manifolds, $\mathrm{Hol}(X, Y)$ be the family of holomorphic maps from $X$ into $Y$, and $\mathrm{Sur}(X, Y)$ the family of surjective holomorphic maps from $X$ to $Y$. If $Y$ is hyperbolic, then $\mathrm{Hol}(X, Y)$ and $\mathrm{Sur}(X, Y)$ are compact. If, moreover, $X$ and $Y$ are complex manifolds, then $\mathrm{Sur}(X, Y)$ is finite. This has been proved under various additional assumptions and finally by Noguchi [7] in the most general form, see also [3; Chapter 6, §6]. The natural question is whether this holds also in the almost complex case.

At the moment, for a complete generalization there are too many obstacles. However, in some special cases it should be possible to find arguments avoiding the use of complex structures.

Consider, for example, Urata's theorem [9] which says that the family of surjective holomorphic maps *with connected fibers* from a compact complex manifold $X$ to a compact hyperbolic complex manifold $Y$ is finite. The simplified proof of this theorem by Simha [8] depends on the following two facts: (i) finiteness of $\mathrm{Aut}(Y)$ and (ii) constancy of a bounded holomorphic function on a compact complex space. The latter fact is used to show that a holomorphic map from a closed complex subspace of $X$ into a coordinate neighborhood in $Y$ is constant.

Simha's proof (which does not make us of the complex analytic structure of $\mathrm{Hol}(X, Y)$) seems to be adaptable to the almost complex case. As we have shown in (2.3) above, we have (i) in the almost complex case as well. As for (ii), from the elliptic differential equation satisfied by a holomorphic map between almost complex manifolds (see (2.2) in [1]), it is not hard to see that a holomorphic map from a compact almost complex manifold $V$ into a coordinate neighborhood in $Y$ is constant. However, we need to know this when $V$ is a fiber of a surjective holomorphic map from $X$ to $Y$, which may have singularities. In other words, we have to consider almost complex spaces (with singularities) whatever their definition may be.

If a holomorphic map $f \colon X \to Y$ from an almost complex manifold $X$ to a hyperbolic almost complex manifold $Y$ is finite-to-one, then $X$ is also hyperbolic. This is a result in metric space topology, see (1.3.14) of [3; p.13]. If we can prove something like the Stein factorization theorem for almost complex manifolds, then we would be one step closer to dropping the assumption of connected fibers from Urata's theorem.

## §3. Local hyperbolicity

One of the sufficient conditions for an almost complex manifold to be (complete) hyperbolic (in the sense that its intrinsic pseudo-distance is a (complete) distance) is that it admits a (complete) Hermitian metric with holomorphic sectional curvature bounded above by a negative constant, (see [4]).

As an application, we proved that every point of an almost complex manifold has a hyperbolic neighborhood. (In real dimension 4, the existence of a complete hyperbolic neighborhood was established by Debalme and Ivashkovich [2] by a completely different method.) In [4] I claimed that it has a *complete* hyperbolic neighborhood. However, at this conference it was pointed out by Forstnerič that the neighborhood I had constructed might not be complete. (The almost Hermitian metric constructed in [5] is a little simpler although it does not essentially differ from the one in [4].)

So the problem of constructing a *complete* hyperbolic neighborhood is still open.

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