Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 123–126

On the non-existence of smooth Levi-flat hypersurfaces in $\mathbb{CP}_n$

Andrei Iordan

Abstract.

We prove that there exists no C^m Levi-flat real hypersurface in $\mathbb{CP}_n$ for $n \geqslant 2$ and $m \geqslant 4$. This is an improvement of the regularity in a theorem of Y.-T. Siu who proved this result for $m \geqslant 8$.

In [9] Y-T. Siu proved the following theorem:

Theorem 1 ([9]). *There exists no C^m Levi-flat real hypersurface in $\mathbb{CP}_n$ for $n \geqslant 2$ and $m \geqslant 8$.*

This theorem answers to a question raised by D. Cerveau [1]. The real analytic case of Theorem 1 was proved by A. Lins Neto [5] for $n \geqslant 3$ and by T. Ohsawa [7] for $n \geqslant 2$. The case $n \geqslant 3$ and $m \geqslant 3n/2 + 7$ of Theorem 1 was proved by Siu [8].

The proof of Theorem 1 is based on the following regularity result for the $\overline{\partial}$-operator:

Theorem 2 ([9]). *Let Ω be a domain with C^{m+1} Levi-flat boundary in $\mathbb{CP}_2$, $m \geqslant 3$. Let g be a C^{m+1} $\overline{\partial}$-closed $(0,1)$-form on Ω which is C^m up to the boundary of Ω. Then there exists u belonging to the Sobolev space $W^m(\Omega)$ such that $\overline{\partial} u = g$.*

A recent paper of G. M. Henkin and the author [4] study the regularity of the $\overline{\partial}$-operator on peudoconcave domains in $\mathbb{CP}_n$.

By using the results of [4] and Theorem 2 we prove in this note that there exists no C^m Levi-flat real hypersurface in $\mathbb{CP}_n$ for $n \geqslant 2$ and $m \geqslant 4$. The methods are the same as in [4].

Let Ω be a domain of $\mathbb{CP}_n$ and E a holomorphic hermitian vector bundle over Ω. We denote by $W^k_{(p,q)}(\Omega; E)$ the (p,q)-forms on Ω with coefficients in the Sobolev space $W^k(\Omega)$ and values in the bundle E endowed with the Sobolev norm $\|\cdot\|_k$ (or $\|\cdot\|_{k,\Omega}$), $A^\infty_{(p,q)}(\Omega; E)$ the set of $\overline{\partial}$-closed (p,q)-forms on Ω with values in E which have a C^∞ extension

Received March 30, 2002.

to $\overline{\Omega}$ and $AW^k_{(p,q)}(\Omega;E)$ the set of $\overline{\partial}$-closed (p,q)-forms contained in $W^k_{(p,q)}(\Omega;E)$.

Let $\delta(z)$ be the distance from $z \in \Omega$ to the boundary of Ω with respect to the Fubini-Study metric. A theorem of Takeuchi [10] shows that for every pseudoconvex domain Ω there exists a positive constant $\mathcal{K}_n \geqslant 1/3$ such that $i\partial\overline{\partial}(-\log\delta) \geqslant \mathcal{K}_n\omega$ where ω is the Kähler form of the Fubini-Study metric (see also [2], [6]). We denote by $L^2_{(p,q)}(\Omega;\delta^k;E)$ the set of E-valued (p,q)-forms f on Ω such that $\delta^k f$ is an L^2-form on Ω.

We say that a domain $\Omega \subsetneq \mathbb{CP}_n$ is pseudoconcave if $\mathbb{CP}_n\backslash\overline{\Omega}$ is pseudoconvex.

Let Ω_- be a pseudoconcave domain in $\mathbb{CP}_n$, k a positive integer and $f \in W^k_{(p,n-1)}(\Omega_-;\mathcal{O}(m))$ a $\overline{\partial}$-closed form. We set $\Omega_+ = \mathbb{CP}_n\backslash\overline{\Omega_-}$. We say that f verifies the moment condition of order k if there exists an extension $\widetilde{f} \in W^k_{(p,n-1)}(\mathbb{CP}_n;\mathcal{O}(m))$ of f such that $\overline{\partial}\widetilde{f} \in L^2_{(p,n)}(\Omega_+;\delta^{-k+1};\mathcal{O}(m))$ and $\int_{\Omega_+}\overline{\partial}\widetilde{f}\wedge h = 0$ for every holomorphic form $h \in L^2_{(n-p,0)}(\Omega_+;\delta^{k-1};\mathcal{O}(-m))$. Every form $f = \overline{\partial}u$ where $u \in W^{k+1}_{(p,n-2)}(\Omega_-;\mathcal{O}(m))$ verifies the moment condition of order k.

We recall here the following consequence of Theorem 7.1 and Theorem 8.7 of [4]:

Theorem 3 ([4]). *Let Ω_- be a pseudoconcave domain with Lipschitz boundary in $\mathbb{CP}_n$ and $k \geq 1$ an integer such that $2(k-1)\mathcal{K}_n - m + n + 1 > 0$. Then for every $\overline{\partial}$-closed form $f \in C^\infty_{(n,n-1)}(\overline{\Omega_-};\mathcal{O}(m))$ verifying the moment condition of order k there exists $u \in W^k_{(n,n-2)}(\Omega_-;\mathcal{O}(m)) \cap C^\infty_{(n,n-2)}(\Omega_-;\mathcal{O}(m))$ such that $\overline{\partial}u = f$ and $\|u\|_k \leq C_k \|f\|_k$, where C_k is a constant independent of f.*

We use also the following approximation lemma (Lemma 8.3 of [4]):

Lemma 1 ([4]). *Let Ω be a relatively compact domain with Lipschitz boundary in a complex manifold, E a holomorphic bundle on X. Suppose that there exists a fundamental system of neighborhoods $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of Ω with the following property: for every $\overline{\partial}$-exact form $\Phi = \overline{\partial}\psi$ with $\psi \in A^\infty_{(p,q)}(\Omega_\varepsilon;E)$, there exists $0 < \varepsilon' < \varepsilon$ and $\varphi \in W^s_{(p,q)}(\Omega_{\varepsilon'};E) \cap C^\infty_{(p,q)}(\Omega_{\varepsilon'};E)$ such that $\overline{\partial}\varphi = \Phi$ and $\|\varphi\|_{s,\Omega_{\varepsilon'}} \leq C\|\Phi\|_{s,\Omega_{\varepsilon'}}$ with C independent of Φ and ε. Then, every $f \in AW^s_{(p,q)}(\Omega;E) \cap C^\infty_{(p,q)}(\Omega;E)$ belongs to the closure of $A^\infty_{(p,q)}(\Omega;E)$ in $W^s_{(p,q)}(\Omega;E)$.*

From Theorem 3 and Lemma 1 we obtain:

Proposition 1. *Let* Ω_- *be a pseudoconcave domain with Lipschitz boundary of* $\mathbb{CP}_2$. *Then* $A^\infty(\Omega_-;\mathcal{O}(1))$ *is dense in* $AW^3(\Omega_-;\mathcal{O}(1))$.

Proof. We identify the $\mathcal{O}(1)$-valued sections of $A^\infty(\Omega_-;\mathcal{O}(1))$ with the $\mathcal{O}(4)$-valued $(2,0)$-forms of $A^\infty_{(2,0)}(\Omega_-;\mathcal{O}(4))$. Since $\mathcal{K}_2 \geqslant 1/3$, it follows that $2(k-1)\mathcal{K}_2 - m + n + 1 > 0$ for $k = 3$ and $m = 4$. Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a fundamental neighborhood system $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of $\overline{\Omega_-}$ such that Ω_ε is a pseudoconcave domain with Lipschitz boundary of $\mathbb{CP}_2$ for each $\varepsilon > 0$. Since every form $f = \overline{\partial} u$ where $u \in A^\infty_{(2,0)}(\Omega_\varepsilon;\mathcal{O}(4))$ verifies the moment condition of order 4, Proposition 1 follows from Theorem 3 and Lemma 1. Q.E.D.

Since

$$\dim A^\infty(\Omega_-;\mathcal{O}(1)) = \dim A^\infty_{(2,0)}(\Omega_-;\mathcal{O}(4)) = 3$$

(see Proposition 10.1 of [4]), from Proposition 1 we obtain:

Corollary 1. $\dim AW^3(\Omega_-;\mathcal{O}(1)) = 3$.

Theorem 4. *There exists no domain with* C^k *Levi-flat boundary in* $\mathbb{CP}_n$ *for* $n \geqslant 2$ *and* $k \geqslant 4$.

Proof. The proof is done by using Theorem 2 and an extension argument as in the proof of Proposition 4.3 of [4]. By using projections it is enough to prove the result for $n = 2$.

Let Ω be a domain with C^4 Levi-flat boundary in $\mathbb{CP}_2$, $a \in \Omega$ and $b \in \mathbb{CP}_2\backslash\overline{\Omega}$. We denote by H the complex projectiv line through a and b and we choose homogeneous coordinates $z = (z_0; z_1; z_2)$ for a point $[z] \in \mathbb{CP}_2$ such that the complex projectiv line through a and b is given by $H = \{[z]\,|\,z_0 = 0\}$. Let Ω' be an open neighborhood of $\overline{\Omega}$ which does not contain the point b and $h \in H^{0,0}(H \cap \Omega';\mathcal{O}(1))$. By [3] there exists a Stein neighborhood V of $H \cap \Omega'$ and let $\widetilde{h} \in H^{0,0}(V;\mathcal{O}(1))$ an extension of h.

Let χ be a C^∞ function on $\mathbb{CP}_2$ with support contained in V such that $\chi \equiv 1$ near $H \cap \Omega$. By identifying the sections of $\mathcal{O}(1)$ with the 1-homogeneous functions in homogeneous coordinates, $\frac{\widetilde{h}\overline{\partial}\chi}{z_0}$ defines a form $g \in C^\infty_{(0,1)}(\overline{\Omega})$. By Theorem 2 there exists $u \in W^3(\Omega)$ such that $\overline{\partial} u = g$. Then $\chi\widetilde{h} - z_0 u$ defines a section $f \in AW^3(\Omega;\mathcal{O}(1))$ such that $f = h$ on $H \cap \Omega$. This implies that $AW^3(\Omega;\mathcal{O}(1))$ is infinite dimensional and it contradicts the Corollary 1. Q.E.D.

References

[1] D. Cerveau, *Minimaux des feuilletages algébriques de P_n*, Ann. Inst. Fourier **43** (1993), 1535–1543.

[2] G. Elencwajc, *Pseudo-convexité locale dans les variétés kähleriennes*, Ann. Inst. Fourier **25** (1975), no. 2, 295–314.

[3] R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[4] G. M. Henkin and A. Iordan, *Regularity of $\overline{\partial}$ on pseudoconcave compacts and applications*, Asian J. Math. **4** (2000), no. 4, 855–884, and Erratum to : Regularity of $\overline{\partial}$ on pseudoconcave compacts and applications by G. M. Henkin and A. Iordan, Asian J. Math., 4, 855-884, 2000, to appear in Asian J. Math.

[5] A. Lins Neto, *A note on projective Levi flats and minimal sets of algebraic foliations*, Ann. Inst. Fourier **49** (1999), 1369–1385.

[6] T. Ohsawa, *Pseudoconvex domains in P^n: A question on 1-convex boundary pointsin in Analysis and Geometry in Several Complex Variables, G. Komatsu and M. Kuranishi Editors, Birkhäuser Boston*, Trends in Mathematics (1999), 239–252.

[7] ———, *Nonexistence of real analytic Levi-flat hypersurfaces in P_2*, Nagoya Math. J. **158** (2000), 95–98.

[8] Y.-T. Siu, *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension $\geqslant 3$*, Ann. of Math. **151** (2000), 1217–1243.

[9] ———, *$\overline{\partial}$-regularity for weakly pseudoconvex domains in hermitian symmetric spaces with respect to invariant metrics*, to appear in Ann. of Math.

[10] A. Takeuchi, *Domaines pseudoconvexes infinis et la métrique riemanienne dans un espace projectif*, J. Math. Soc. Japan **16** (1964), 159–181.

Institut de Mathématiques
UMMR 7586 du CNRS, case 247
Université Pierre et Marie-Curie
4 Place Jussieu
75252 Paris Cedex 05
France
E-mail address: iordan@math.jussieu.fr