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Short $\mathbb{C}^k$

John Erik Fornæss

§1. Introduction

One of Oka's main contributions was to solve the Levi problem.

There are various ways to generalize the Levi Problem. The Union problem is one: Let $\Omega_0 \subset \Omega_1 \subset \cdots \subset \cup\Omega_n = \Omega$. Suppose that each Ω_j is Stein. Is Ω Stein? To approach the Union Problem, one can try at first to understand the simplest cases of Ω.

Example 1.1. *Long* $\mathbb{C}^2$. *Suppose that each* Ω_j *is biholomorphic to* $\mathbb{C}^2$. *Then we call* Ω *a long* $\mathbb{C}^2$. *It is an open question whether all long* $\mathbb{C}^2$ *are actually biholomorphic to* $\mathbb{C}^2$.

Example 1.2. (*Fornæss*, ([*F*, 1976])) *In dimension* 3 *and higher it can happen that* Ω *is not Stein and that each* Ω_n *is biholomorphic to a ball.*

This left open the question in dimension 2.

Theorem 1.3. (*Fornæss-Sibony*, ([*FS*, 1981])) *Suppose that each* Ω_j *is biholomorphic to the unit ball in* $\mathbb{C}^2$. *If the (infinitesimal) Kobayashi metric of* Ω *is not identically zero, then* Ω *is biholomorphic to the ball or to* $\Delta \times \mathbb{C}$, *where* Δ *is the unit disc.*

Recall that the (infinitesimal) Kobayashi metric of Ω vanishes identically if and only if for all $p \in \Omega$ and any tangent vector ξ to Ω at p and for any $R > 0$, there exists a holomorphic map $f : \Delta = \{z \in \mathbb{C}; |z| < 1\} \to \Omega$ so that $f(0) = p$ and $f'(0) = R\xi$.

This theorem left still open the case when the Kobayashi metric vanishes identically. The most obvious example of such a case is when $\Omega = \mathbb{C}^2$. However, the question remaining was whether there was any

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other possibility (Diederich-Sibony ([DS,1979]). In this paper we show that indeed there are other such Ω. In fact, such Ωs occur quite naturally in dynamics. In random iteration, basins of attraction can be such domains. Under iteration of fixed maps, they occur as sublevel sets of Green functions.

Fix an integer $d \geq 2$. For any $\eta > 0$, let $\mathrm{Aut}_{d,\eta}$ denote the set of polynomial automorphisms F of $\mathbb{C}^k, k \geq 2$, of the form $F(z_1, \dots, z_k) = (z_1^d + P_1(z_1, \dots, z_k), P_2(z_1, \dots, z_k), \dots, P_k(z_1, \dots, z_k))$ where each P_j is a polynomial of degree at most $d-1$ and where each coefficient is at most η in modulus.

An example is $F(z) = (z_1^d + \eta z_k, \eta z_1, \dots, \eta z_{k-1})$.

Suppose that $F_n \in \mathrm{Aut}_{d,\eta_n}, \eta_n = a_n^{d^n}, n = 1, 2, \dots, 1 > a_1 \geq a_2 \geq \cdots \lim_{n\to\infty} a_n = a_\infty \geq 0$, and set $F(n) = F_n \circ \cdots \circ F_1$. Let Ω denote the set of points $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ such that $F(n)(z) \to 0$ as $n \to \infty$.

Theorem 1.4. *The set Ω has the following properties:*
(i) Ω is a nonempty, open, connected set in $\mathbb{C}^k$,
(ii) $\Omega = \cup_{j=1}^{\infty} \Omega_j \supset \cdots \supset \Omega_\ell \supset \cdots \supset \Omega_1$. Each Ω_j is biholomorphic to the unit ball $B^k(0,1)$.
(iii) The infinitesimal Kobayashi metric of Ω vanishes identically.
(iv) There is a plurisubharmonic function $\psi : \mathbb{C}^k \to [\log a_\infty, \infty)$ such that $\Omega = \{\psi < 0\}$ and ψ is nonconstant on Ω.

The reason that Ω fails to be biholomorphic to $\mathbb{C}^k$ is that there is a nonconstant bounded plurisubharmonic function on Ω. In some sense this means that Ω is "too small" to be all of $\mathbb{C}^k$. So we might call such an Ω a short $\mathbb{C}^k$.

Next, we mention some more details about the function ψ.

Theorem 1.5 (=Theorem 3.4). *The set $U = \{z \in \mathbb{C}^k; \psi(z) > \log a_\infty\}$ is open and ψ is pluriharmonic on U.*

Theorem 1.6. *The function ψ has no critical points on $\{\psi > \log a_\infty\}$.*

This shows that the level sets of ψ are foliated by complex hypersurfaces Σ.

Theorem 1.7. *Any leaf $\Sigma(z^0)$ of $\{\psi = c > \log a_\infty\}$ can be exhausted by relatively open sets U biholomorphic to $B^{k-1}(0,1)$. Each such U is Runge in $\mathbb{C}^k$. Moreover the intrinsic infinitesimal Kobayashi metric of $\Sigma(z^0)$ vanishes identically.*

Theorem 1.8. *Each leaf* $\Sigma(z^0)$ *of* $\{\psi = c > \log a_\infty\}$ *is dense in* $\{\psi = c\}$.

It is a little harder to get good control on the set $\{\psi = \log a_\infty\}$. We investigate here only a special case with very rapidly decreasing coefficients where one can get pluripolar sets with only one singular point.

Theorem 1.9 (=Theorem 3.10). *Let* $F_n(z,w) = (z^2 + a_n w, a_n z)$. *Suppose that* $|a_n| \searrow 0$ *sufficiently rapidly. Then* $\{\psi = -\infty\} =: P$ *has the following shape:* $P \setminus (0)$ *is closed in* $\mathbb{C}^2 \setminus (0)$ *and is laminated by Riemann surfaces.*

On the contrary, when one lets the coefficients decrease at a slightly slower pace than in Theorem 1.4, Ω is biholomorphic to $\mathbb{C}^2$.

Theorem 1.10 (=Theorem 3.11). *Let* $F_n = (z^2 + a_n w, a_n z)$. *Suppose that* $0 < |a_n| < c < 1$ *and* $|a_{n+1}| \geq |a_n|^t$ *for some* $1 < t < 2$. *Then the basin of attraction of* 0 *is biholomorphic to* $\mathbb{C}^2$.

Theorem 1.11 (=Theorem 3.7). *For every* $c > \log a_\infty$, *the sublevel sets* $\{\psi < c\}$ *is a short* $\mathbb{C}^k$.

The same result is valid for other maps, such as for example iterations of any given fixed Hénon map. The next result shows also that "short" $\mathbb{C}^k$, Ω, might contain subsets biholomorphic to $\mathbb{C}^k$.

Theorem 1.12 (=Theorem 3.8). *Let* H *be a Hénon map, and let* G^+ *be the pluricomplex Green function,* $G^+(z) = \lim_{n\to\infty} \frac{\log^+ \|H^n(z)\|}{d^n}$, $d =$ *degree* H. *Then for every* $c > 0$, $\{G^+ < c\}$ *is a "short"* $\mathbb{C}^2$.

The plan of the paper is to first prove Theorem 1.4 in Section 2. Then in Section 3 we prove some of the other results above. Due to lack of space the remaining theorems and also other results in this direction will be published elsewhere.

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§2. Proof of Theorem 1.4.

Proof of Theorem 1.4.
(i): Let τ denote the maximum possible number of terms of a polynomial of degree $d-1$ in k variables. Let $\Delta^k(0,c)$ denote the polydisc in $\mathbb{C}^k$ with center at the origin and polyradius $(c,c,\dots,c), c > 0$. Suppose that $z \in \Delta^k(0,c), 0 < c < 1$, and assume that $F \in \mathrm{Aut}_{d,\eta}, F = (z_1^d +$

$P_1, \dots, P_k)$. Then $|P_i(z)| \leq \tau\eta, i = 1, \dots k$ and $|z_1^d| \leq c^d$. It follows that $F(\Delta^k(0,c)) \subset \Delta^k(0, c^d + \tau\eta)$. Pick $c, c', 0 < c < c' < 1$ and set $c_\ell = c(c')^\ell$. If $\ell \geq 0$, we have that $d^\ell \geq \ell + 1$. We show that if $n \geq n_0$, n_0 large enough and $\ell \geq 0$, then $F_{n+\ell}(\Delta^k(0, c_\ell)) \subset \Delta^k(0, c_{\ell+1})$:

$$\begin{aligned}
\log(\tau\eta_{n+\ell}) &= \log\tau + d^{n+\ell}\log a_{n+\ell} \\
&\leq \log\tau + (\ell+1)d^n \log a_1 \\
&= [\log\tau + (\ell+1)\frac{d^n}{2}\log a_1] + (\ell+1)\frac{d^n}{2}\log a_1. \\
&\quad \text{If } n \geq n_0 : \\
\log(\tau\eta_{n+\ell}) &< \log c(1-c) + (\ell+1)\log c'. \\
\tau\eta_{n+\ell} &< c(1-c)(c')^{\ell+1}. \\
\tau\eta_{n+\ell} &< c(c')^{\ell+1} - (c(c')^\ell)^d. \\
c_\ell^d + \tau\eta_{n+\ell} &< c_{\ell+1}.
\end{aligned}$$

It follows that if $n \geq n_0$ and $\ell \geq 0$, then $F_{n+\ell}(\Delta^k(0, c_\ell)) \subset \Delta^k(0, c_{\ell+1})$.

Set $\Omega_n := \{z \in \mathbb{C}^k; F(n)(z) \in \Delta^k(0,c)\}$. It follows that if $n \geq n_0$, $\Omega_n \subset \Omega_{n+1}$ and that $F(n+\ell)(z) \to 0$ uniformly on Ω_n when $\ell \to \infty$. Hence we have that $\Omega \supset \cup_{n \geq n_0}\Omega_n$ and the union is increasing. Suppose next that $z \in \Omega$. Then $F(n)(z) \to 0$ and hence $F(n)(z) \in \Delta^k(0,c)$ for some $n \geq n_0$. Hence $z \in \cup_{n \geq n_0}\Omega_n$. This proves (i).

(ii): We set $U_n = \{z \in \mathbb{C}^k; \|F(n)(z)\| < c\}$. Then $U_n \subset \Omega_n$. If $z \in \Omega_n, n \geq n_0$, then $F(n+\ell)(z) \in \Delta^k(0, c(c')^\ell) \subset B(0,c)$ for a fixed $\ell \geq 1, \forall n \geq n_0$. Hence $\Omega_n \subset U_{n+\ell}$. Therefore $\Omega = \cup_{m \geq 0} U_{n_0+m\ell} \supset \cdots \supset U_{n_0}$, writing Ω as an increasing union of balls. This proves (ii).

(iii): Fix (p,ξ), $p \in \Omega$ and ξ a tangent vector to Ω at p. Pick $R > 0$. Then $p_n = F(n)(p) \to 0$. Set $F'(n)(\xi) = \xi_n$. Then $\xi_n \to 0$ also. Define $\zeta_n : \Delta = \{\tau \in \mathbb{C}; |\tau| < 1\} \to \mathbb{C}^2$, $\zeta_n(\tau) = p_n + \tau R\xi_n$. If n is large enough, $\zeta_n(\Delta) \subset \Delta^k(0,c)$. This implies that $\zeta = F(n)^{-1} \circ \zeta_n : \Delta \to \Omega_n \subset \Omega$. Moreover $\zeta(0) = p, \zeta'(0) = F^{-1}(n)'(R\xi_n) = R\xi$. Hence (iii) is proved.

(iv): We use a modification of the Green function construction introduced by Hubbard ([H]). Write $F(n) = (f_1^n, \dots, f_k^n)$.

We define $\phi_n : \mathbb{C}^k \to \mathbb{R}$ by $\phi_n(z) = \max\{|f_1^n|, \ldots, |f_k^n|, \eta_n\}$. Each ϕ_n is a continuous function on $\mathbb{C}^k$.

Lemma 2.1. $\psi_n := \frac{\log \phi_n}{d^n} \to \psi$, *$\psi$ plurisubharmonic on* $\mathbb{C}^k$.

Proof: We show first that $\phi_{n+1} \leq (\tau + 1)\phi_n^d$.

(a): $\phi_n(z) \leq 1$:

$$\begin{aligned}
\phi_{n+1}(z) &= \max\{|(f_1^n)^d + P_1(f_1^n, \ldots, f_k^n)|, |P_2|, \ldots, |P_k|, \eta_{n+1}\} \\
&\leq \max\{\phi_n^d + \tau\eta_{n+1}, \tau\eta_{n+1}, \eta_{n+1}\} \\
&\leq \max\{\phi_n^d + \tau\eta_n^d, \eta_n^d\} \\
&\leq (\tau + 1)\phi_n^d.
\end{aligned}$$

(b): $\phi_n(z) > 1$:

$$\begin{aligned}
\phi_{n+1}(z) &= \max\{|(f_1^n)^d + P_1(f_1^n, \ldots, f_k^n)|, |P_2|, \ldots, |P_k|, \eta_{n+1}\} \\
&\leq \max\{\phi_n^d + \tau\eta_{n+1}\phi_n^{d-1}, \tau\eta_{n+1}\phi_n^{d-1}, \eta_{n+1}\} \\
&\leq \max\{\phi_n^d + \tau\phi_n^{d-1}, 1\} \\
&\leq (\tau + 1)\phi_n^d.
\end{aligned}$$

Hence $\frac{\log \phi_{n+1}}{d^{n+1}} \leq \frac{\log(\tau+1)}{d^{n+1}} + \frac{\log \phi_n}{d^n}$ which implies that the sequence

$$\{\frac{\log \phi_n}{d^n} + \sum_{j>n} \frac{\log(\tau + 1)}{d^{j+1}}\}$$

is monotonically decreasing and the limit is a plurisubharmonic function $\psi \geq \log a_\infty$, (a priori it is possible that $\psi \equiv -\infty$ but we will show that this cannot happen). For simplicity we say that $\{\frac{\log \phi_n}{d^n}\}$ is <u>almost</u> monotonically decreasing to the limit ψ.

Lemma 2.2. $\Omega = \{\psi < 0\}$.

Proof of the Lemma: Assume that $\psi(z) < 0$. Then for all large n and some constant $s < 0$,

$$\frac{\log \phi_n(z)}{d^n} < s < 0.$$

Hence $\phi_n(z) < e^{d^n s}$ which implies that $|f_j^n(z)| < e^{d^n s}, j = 1, \dots, k$ and hence $F(n)(z) \to 0$, so $z \in \Omega$.

Next assume that $z \in \Omega$. Then $F(n)(z) \in \Delta(0, c)$ for all large n. This implies that $\psi_n(z) < 0$ for all large n and hence that $\psi(z) \leq 0$.

Next, let $z^n = F^{-1}(n)(0)$. So for $n \geq n_0, z^n \in \Omega$. Then $\phi_n(z^n) = \eta_n$. Therefore

$$\begin{aligned}
\psi(z^n) &\leq \frac{\log \phi_n(z^n)}{d^n} + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&= \frac{\log \eta_n}{d^n} + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&= \log a_n + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&\leq \log a_1 + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&\leq \log a_1 + \frac{\log(\tau+1)}{d^n} \\
&< 0
\end{aligned}$$

for all large enough n. Since $\psi \leq 0$ on Ω and $\psi(z) < 0$ at some point in Ω, it follows from the subaveraging principle that $\psi < 0$ everywhere on Ω.

It remains only to show the ψ is not constant on Ω. Suppose that $\psi_{|\Omega} \equiv \alpha < 0$. First note that Ω is not all of $\mathbb{C}^2$. For example, it is easy to estimate that $F(n)(z)$ goes to infinity for any $z = (x, 0, \dots, 0), x >> 1$ since the z_1 coordinate of the iterates grows much faster than any of the other coordinates. Pick a point $z^0 \in \Omega$. Then there exists a number $R > 0$ so that the ball $B(z^0, R) \subset \Omega$ while there is a point $p \in \partial B(z^0, R) \cap \partial\Omega$. By the above lemma we know that $\psi(p) \geq 0$. By the subaveraging property of plurisubharmonic functions, $\psi(p)$ is bounded above by the average on any small ball $B(p, \epsilon)$. Since $\psi = \alpha < 0$ on almost half the ball and since ψ is upper semicontinuous, this leads to a contradiction when ϵ is small enough. This contradiction shows that ψ is nonconstant on Ω. This proves (iv). (We have also ruled out here that $\psi \equiv -\infty$, as promised.)

§3. Proofs of further results.

In this section we study in more detail the properties of Ω and its defining function ψ as given in Theorem 1.4. Hubbard ([H]) introduced a filtration of $\mathbb{C}^2$ which has proved very useful in the investigation of Hénon maps. We use the natural generalization of this filtration to $\mathbb{C}^k$.

Definition 3.1 (Filtration). *Set $R := 2\tau + 2$.*

$$\begin{aligned} V &:= \Delta(0,R) \\ V^+ &:= \{z \in \mathbb{C}^k; |z_1| \geq R, \max\{|z_2|,\ldots,|z_k|\} \leq |z_1|\} \\ V^- &:= \{z \in \mathbb{C}^k; \max\{|z_2|,\ldots,|z_k|\} \geq R, |z_1| \leq \max\{|z_2|,\ldots,|z_k|\}\}. \end{aligned}$$

The basic properties of this filtration is given in the following Lemma. Fix any integer $n > n_0$, where n_0 is large enough. (More precisely, we will need $3\tau\eta_{n_0-1}^{d-1} \leq 1, \tau R^{d-2}\eta_{n_0} \leq 1, (*)$.) For $z = (z_1,\ldots,z_k)$, set $z' = (z_1',\ldots,z_k') = F_n(z)$. Set $m = \max\{|z_2|,\ldots,|z_k|\}, m' = \max\{|z_2'|,\ldots,|z_k'|\}$.

Lemma 3.2. *Assume $n > n_0$.*
(i) Suppose $z \in V^+$. Then $z' = F_n(z) \in int(V^+)$ and $|z_1'| > 2|z_1|$.
(ii) If $z \in V$, then $z' \in V \cup V^+$.
(iii) If $z \in V^-$, then $m' \leq \tau m^{d-1}\eta_n$.
(iv) Suppose that $|z_1| \geq 2\tau\sigma \max\{m, \eta_{n-1}\}$ for some $\sigma \geq 1$. Then $|z_1'| \geq 3\tau\sigma \max\{m', \eta_n\}$ and $||z_1'| - |z_1|^d| \leq \tau\eta_n \max\{1, |z_1|^{d-1}\}$.

The proof is straightforward and will be omitted.
We show next that no orbit can stay in V^- forever.

Lemma 3.3. *Suppose that $z \in \mathbb{C}^k$. Then there exists an integer $n = n(z)$ so that $F(n)(z) \in V \cup V^+$ for all $n \geq n(z)$.*

Proof: By Lemma 3.2, (i) and (ii), it suffices to show that for some $n > n_0$, $F(n)(z) \in V \cup V^+$. Suppose to the contrary that $F(n_0 + 1)(z) =: z^1, \ldots, F(n_0 + \ell)(z) =: z^\ell, \cdots \in V^-$ for all $\ell \geq 1$. Let $m_\ell := \max\{|z_2^\ell|,\ldots,|z_k^\ell|\}$. Then applying Lemma 3.2, (iii), we obtain

$$\begin{aligned} m_{\ell+1} &\leq \tau m_\ell^{d-1}\eta_{n_0+\ell} \\ &\leq \tau m_\ell^{d-1} a_1^{d^\ell} \\ \frac{\log m_{\ell+1}}{d^{\ell+1}} &\leq \frac{\log\tau}{d^{\ell+1}} + \frac{\log m_\ell}{d^\ell} + \log a_1 \end{aligned}$$

It follows that the sequence $\frac{\log m_\ell}{d^\ell}$ will eventually decrease by at least $\frac{\log a_1}{2}$ each step. This implies that eventually $\frac{\log m_\ell}{d^\ell} < 0$ which implies that $m_\ell < 1$, contradicting that $m_\ell \geq R$. This proves the Lemma.

■

Theorem 3.4. *The set $U = \{z \in \mathbb{C}^k; \psi(z) > \log a_\infty\}$ is open and ψ is pluriharmonic on U.*

Lemma 3.5. *If $\psi(z) > \log a_\infty$ then there exists n arbitrarily large so that $|z_1^n| > 2\tau \max\{|z_2^n|, \ldots, |z_k^n|, \eta_n\}$.*

Proof of the Lemma: Pick two constants α, β, $\min\{\psi(z), 0\} > \log \alpha > \log \beta > \log a_\infty$. There exists a large integer n_1 so that if $n \geq n_1$ then $a_n < \beta$. Suppose that for some $n_2 \geq n_1$,

$$|z_1^n| \leq 2\tau \max\{|z_2^n|, \ldots, |z_k^n|, \eta_n\}, \ \forall\, n \geq n_2.$$

Set $m_n := \max\{|z_2^n|, \ldots, |z_k^n|\}$. Suppose that for some $n \geq n_2$, we have that $m_n \geq 1$. Then $|P_j(z^n)| \leq \tau \eta_{n+1} (2\tau m_n)^{d-1}$. Hence $m_{n+1} \leq \tau^d \beta^{d^{n+1}} m_n^d 2^{d-1}$. Therefore,

$$\begin{aligned}
\frac{\log m_{n+1}}{d^{n+1}} &\leq \frac{\log \tau}{d^n} + \log \beta + \frac{\log m_n}{d^n} + (d-1)\frac{\log 2}{d^n} \\
&\leq \frac{1}{2} \log \beta + \frac{\log m_n}{d^n} \text{ if } n_2 \text{ is large enough}
\end{aligned}$$

This easily implies that for some large n, $\log m_n < 0$ so $m_n < 1$. Now suppose that $n \geq n_2$ and that $m_n < 1$. Then $|P_j(z^n)| \leq \tau \eta_{n+1}$. Hence

$$\begin{aligned}
\phi_{n+1} &= \max\{|z_1^{n+1}|, \ldots, |z_k^{n+1}|, \eta_{n+1}\} \\
&\leq 2\tau \max\{|z_2^{n+1}|, \ldots, |z_k^{n+1}|, \eta_{n+1}\} \\
&\leq 2\tau \max\{\tau \eta_{n+1}, \eta_{n+1}\} \\
&= 2\tau^2 \eta_{n+1}
\end{aligned}$$

But then $\frac{\log \phi_{n+1}}{d^{n+1}} \leq \frac{\log(2\tau^2)}{d^{n+1}} + \log a_{n+1} \leq \frac{\log(2\tau^2)}{d^{n+1}} + \log \beta$. This contradicts that $\psi(z) > \log \alpha$ if n_2 is chosen even larger.

■

Proof of the Theorem: Suppose that $\psi(z) > \log a_\infty$. Then by Lemma 3.5 there exists an arbitrarily large integer n_1 so that

$$|z_1^{n_1}| > 2\tau \max\{|z_2^{n_1}|, \ldots, |z_k^{n_1}|, \eta_{n_1}\}.$$

By continuity this inequality holds for all w in some neighborhood V of z. But then by Lemma 3.2(iv) this inequality is still true for all $n \geq n_2$ on V. Hence $\psi_n \equiv \log|f_1^n|$ on V. Therefore the ψ_n are pluriharmonic on V. Moreover, they converge (almost) monotonically to a limit ψ which has a finite value at z. Hence the limit is pluriharmonic on V. In particular, ψ is continuous on V so $\{\zeta \in \mathbb{C}^k; \psi(\zeta) > \log a_\infty\}$ contains an open neighborhood of z.

■

Lemma 3.6. *Let* $K^{compact} \subset \{\psi < c_1\}, \log a_\infty < c_1 < c_2$. *Then there exists for any* $\epsilon > 0$ *an open set* $U \subset \{\psi < c_2\}$ *and an automorphism* Φ *of* $\mathbb{C}^k$ *so that* $\Phi(U) = B^k(0,1), \Phi(K) \subset B^k(0,\epsilon)$.

Proof: Since $\psi < c_1$ on K, there exists an integer N so that $\psi_n < c_1$ for all $n \geq N$. Hence $\frac{\log \phi_n}{d^n} < c_1$ on K $\forall$ $n \geq N$. This implies that $|f_j^n| < e^{c_1 d^n}$ on $K, n \geq N, j = 1, \ldots, k$. Suppose next that $w \in \mathbb{C}^k$ and $|f_j^n(w)| < Re^{c_1 d^n}$ for some $n \geq N$, $j = 1, \ldots, k$. Then $\phi_n(w) = \max\{|f_1^n|(w), \ldots, |f_k^n|(w), \eta_n\}$. Hence $\psi_n = \frac{\log \phi_n(w)}{d^n} < \max\{\frac{\log Re^{c_1 d^n}}{d^n}, \frac{\log a_n^{d^n}}{d^n}\}$. We can assume that $\log a_n < c_1$. Hence $\psi_n(w) < \max\{\frac{\log R}{d^n} + c_1, c_1\} < c_2$, n large. This completes the proof of the Lemma.

■

Theorem 3.7. *For any* $c > \log a_\infty$ *the sublevel set* $\{\psi < c\}$ *is connected and is a short* $\mathbb{C}^k$.

Proof: We can write $\{\psi < c\} = \cup_{n=1}^\infty K_n^{\text{compact}}, K_n \subset \text{int}(K_{n+1})$. We next find a sequence of open sets $U_n, K_n \subset U_n \subset\subset U_{n+1} \subset\subset \{\psi < c\}$ and biholomorphic maps $\Phi_n : \mathbb{C}^k \to \mathbb{C}^k$ so that $\Phi_n(U_n) = B(0,1), \Phi_{n+1}(U_n) \subset B(0, \frac{1}{n})$. If we have found U_n, set $K = \overline{U_n} \cup K_{n+1}$. Then there exists $\log a_\infty < c_1 < c_2 < c$ so that $K \subset \{\psi < c_1\}$. We apply Lemma 3.6 to find $U_{n+1} \subset \{\psi < c_2\}$.

■

In the same way we get:

Theorem 3.8. *Let H be a Hénon map, and let G^+ ([H]) be the pluricomplex Green function, $G^+(z) = \lim_{n\to\infty} \frac{\log^+ \|H^n(z)\|}{d^n}$, $d =$ degree H. Then for every $c > 0$, $\{G^+ < c\}$ is a "short" $\mathbb{C}^2$.*

Next we discuss the nature of the set $\psi = \log a_\infty$ when $a_\infty = 0$. Notice that any $\mathbb{C}^k$ contained in a sublevel set $\{\psi < c\}$ must be contained in $\{\psi = \log a_\infty\}$. Since $\{\psi = -\infty\}$ is a pluripolar set, there is no $\mathbb{C}^k$ contained in any sublevel set of ψ in this case.

Lemma 3.9. *Let $\delta, \epsilon, R > 0$ be given. If $a \in \mathbb{C}$, $0 < |a|$ small enough, then $\left|\frac{\log|z^2+aw|}{2} - \log|z|\right| < \epsilon$ if $|z|, |w| \leq R, |z| \geq \delta$. Moreover $\frac{\log|z^2+aw|}{2} < \log\delta + 1$ on $\{|z| \leq \delta, |w| \leq R\}$.*

Proof: Let $|a| < \frac{\delta^2}{R}$. Then if $(z,w) \in K := \{\delta \leq |z| \leq R, |w| \leq R\}$, $|z^2 + aw| \geq |z|^2 - |aw| > \delta^2 - |a|R > 0$. We get, for $(z,w) \in K$, $\frac{\log|z^2+aw|}{2} - \log|z| = \frac{1}{2}\log\left|1 + a\frac{w}{z^2}\right|$. Since $\left|\frac{aw}{z^2}\right| \leq |a|\frac{R}{\delta^2}$ we can clearly choose $|a|$ small enough that $|\log|1 + \frac{aw}{z^2}|| < \epsilon$. The last part is obvious.

■

Theorem 3.10. *Let $F_n(z,w) = (z^2 + a_n w, a_n z)$. Suppose that $|a_n| \searrow 0$ sufficiently rapidly. Then $\{\psi = -\infty\} =: P$ has the following shape: $P \setminus (0)$ is closed in $\mathbb{C}^2 \setminus (0)$ and is foliated by Riemann surfaces.*

Proof: Suppose that $\{a_j\}_{j\leq n}$ have been chosen. Set $F(n) = (f_1^n, f_2^n)$. Let $X_n = \{f_1^n = 0\}$. Then X_n is the pole set of $\tilde{\psi}_n = \frac{\log|f_1^n|}{2^n}$. Set $U_n = \{(z,w); \frac{1}{n} \leq \max\{|z|,|w|\} \leq n\}$. Set $V_n = \{(z,w); \tilde{\psi}_n < -n\}$. Let $\hat{\psi} = \max\{\tilde{\psi}_n, -n\}$. Then $X_n \subset V_n$. $F(n)(X_n) = \{z = 0\}$. If $\delta > 0$ is small enough, then $F(n)(\{\max\{|z|,|w|\} \leq n\}) \subset \Delta^2(0,R)$ for some $R > 0$ and $F(n)(V_n) \supset \{|z| \leq \delta, |w| \leq R\}$. Set $\epsilon = 1$. We apply Lemma 3.9 to find a constant $a = a_{n+1}$, $0 < |a_{n+1}| << |a_n|^2$ so that if $F_{n+1}(z,w) = (z^2 + a_{n+1}w, a_{n+1}z)$ then

$$\left|\frac{\log|z^2 + a_{n+1}w|}{2} - \log|z|\right| < 1, \delta \leq |z| \leq R, |w| \leq R.$$

Moreover, $\log|z^2 + a_{n+1}w| < \log\delta + 1$ on $\{|z| \leq \delta, |w| \leq R\}$. It follows that

$$\left|\frac{\log|f_1^{n+1}|}{2^{n+1}} - \frac{\log|f_1^n|}{2^n}\right| < \frac{1}{2^n} \text{ if } \delta \leq |f_1^n| \leq R, |f_2^n| \leq R.$$

and

$$\tilde{\psi}_{n+1} < \frac{\log\delta + 1}{2^{n+1}} \text{ on } \{|f_1^n| \leq \delta, |f_2^n| \leq R\}.$$

Choosing δ even smaller, we may assume that $\frac{\log\delta+1}{2^{n+1}} < -n-1$.

Suppose that $|z|, |w| \leq n$. Then $|f_1^n(z,w)|, |f_2^n(z,w)| \leq R$.

(i) $|f_1^n(z,w)| \leq \delta$. Then $(z,w) \in V_n$ and hence $\hat{\psi}_n(z,w) = -n$. Moreover, $\hat{\psi}_{n+1}(z,w) = \max\{\tilde{\psi}_{n+1}(z,w), -n-1\} = -n-1$.

(ii) $|f_1^n(z,w)| \geq \delta$. Then $|\tilde{\psi}_n(z,w) - \tilde{\psi}_{n+1}(z,w)| \leq \frac{1}{2^n}$. Hence,

$$|\max\{\tilde{\psi}_n, -n-1\} - \max\{\tilde{\psi}_{n+1}, -n-1\}| \leq \frac{1}{2^n}.$$

Suppose $\hat{\psi}_n > -n$. Then $\tilde{\psi}_n > -n$ so $\tilde{\psi}_{n+1} > -n-1$, so $|\hat{\psi}_n - \hat{\psi}_{n+1}| \leq \frac{1}{2^n}$ whenever $\hat{\psi}_n > -n$.

Next observe that $\{z^2 + a_{n+1}w = 0\}$ is a parabola of the form $w = -\frac{z^2}{a_{n+1}}$. Hence on U_n, X_{n+1} consists locally of two graphs over X_n and these can be chosen arbitrarily close to X_n.

The above shows that $\psi = -\infty$ is the limit in the Hausdorff metric of $\{X_n\}$ and this has the desired laminar structure.

■

Theorem 3.11. *Let* $F_n = (z^2 + a_n w, a_n z)$. *Suppose that* $0 < |a_n| < c < 1$ *and* $|a_{n+1}| \geq |a_n|^t$ *for some* $1 < t < 2$. *Then the basin of attraction of* 0 *is biholomorphic to* $\mathbb{C}^2$.

Proof: We first estimate the rate of convergence towards the origin. So assume that $(z_0,w_0) \in K^{\text{compact}} \subset \Omega$. Set $(z_n,w_n) = F(n)(z_0,w_0)$, $\delta_n = \sup_{(z_0,w_0)\in K} \max\{|z_n|,|w_n|\}$.

Lemma 3.12. *There exists an* $n_0 = n_0(z_0, z_0) > 0$ *and a constant* $\alpha > 0$ *so that if* $n > n_0$, $\delta_{n+k} \leq |a_{n+k+1}|c^{\alpha k}$ *for all* $k \geq 0$.

We omit the details of the proof.

Set $A_n := F_n'(0)$. Then $A_n(z,w) = (a_n w, a_n z)$ and $A_n^{-1}(z,w) = (w/a_n, z/a_n)$. Hence, $A_n^{-1} \circ F_n - Id = \left(\frac{a_n z}{a_n}, \frac{z^2+a_n w}{a_n}\right) - (z,w) = (0, z^2/a_n)$. Next estimate $A_1^{-1} \circ \cdots A_{n+1}^{-1} \circ F_{n+1} \circ \cdots F_1$ for large n.

We get $\|A_1^{-1} \circ \cdots A_{n+1}^{-1} \circ F_{n+1} \circ \cdots F_1 - A_1^{-1} \circ \cdots A_n^{-1} \circ F_n \circ \cdots F_1\| = \|A_1^{-1} \circ \cdots A_n^{-1}(A_{n+1}^{-1} \circ F_{n+1} - Id) \circ F_n \cdots F_1\| \leq \frac{1}{|a_1|\cdots|a_n|} \frac{|\delta_n|^2}{|a_{n+1}|}$

Now observe that for $k \geq 0$ we have $\delta_{n+k+1} \leq \delta_{n+k}^2 + |a_{n+k+1}|\delta_{n+k} \leq \delta_{n+k}|a_{n+k+1}|(c^{\alpha k}+1)$. Hence, inductively, we have

$$\delta_{n+k+1} \leq \delta_n |a_{n+k+1}||a_{n+k}| \cdots |a_{n+1}| \pi_{j=0}^k (1 + c^{\alpha j}).$$

We can also rewrite this estimate as $\delta_\ell \leq C_1 |a_1 a_2 \cdots a_\ell|$ for a large constant C_1 and for all ℓ. Notice that by Lemma 3.12 we also have $\delta_\ell \leq C_2 |a_{\ell+1}| e^{\alpha \ell}$ for all $\ell \geq 0$.

$$\begin{aligned}
& \|A_1^{-1} \circ \cdots A_{n+1}^{-1} \circ F_{n+1} \circ \cdots F_1 - A_1^{-1} \circ \cdots A_n^{-1} \circ F_n \circ \cdots F_1\| \\
\leq \; & \frac{\delta_n^2}{|a_1 \cdots a_{n+1}|} \\
\leq \; & \frac{\delta_n}{|a_1 \cdots a_n|} \frac{\delta_n}{|a_{n+1}|} \\
\leq \; & C_1 C_2 e^{\alpha n}
\end{aligned}$$

This implies uniform convergence on compact subsets of Ω. Next we show that the limit map is a biholomorphic map from Ω onto $\mathbb{C}^2$.

First it is clear, since the Jacobian determinant at the origin is always equal to one and never vanishes for any $\Phi_n := A_1^{-1} \circ \cdots A_n^{-1} \circ F_n \circ \cdots F_1$, that the limit map $\Phi : \Omega \to \mathbb{C}^2$ has a Jacobian which never vanishes. Hence Φ is locally one-to-one. To show that Φ is globally one-to-one assume to the contrary that $\Phi(p) = \Phi(q), q \neq p$. Then two small neighborhoods of p, q are mapped onto the same neighborhood of $\Phi(p)$. By the open mapping theorem it follows that the same holds for small perturbations of Φ and hence for Φ_n for large n. This contradicts that each Φ_n is one-to-one.

It remains to show that Φ is onto $\mathbb{C}^2$.

Lemma 3.13. *Let $R_0 > 0$. Then there exists a number n_0 large enough so that if $0 < R \leq R_0$ and $n \geq n_0$, then*

$$F_{n+1}(\Delta^2(0, R|a_1 \cdots a_n|)) \supset \Delta^2(0, R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{n/2}}).$$

Proof: Let $j \geq 0$ be the integer for which $c^{t^{j+1}} < |a_{n+1}| \leq c^{t^j}$. One shows at first with a short calculation that $|a_1 \cdots a_n| \leq c^{n/2}|a_{n+1}|$ if $n \geq n_0$.

To complete the proof of the Lemma, we show that if

$$(z, w) \in \partial\Delta^2(0, R|a_1 \cdots a_n|)$$

then $(z', w') \in \mathbb{C}^2 \setminus \Delta^2(0, R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{n/2}})$.

Assume at first that $|z| = R|a_1 \cdots a_n|$. Then

$$|w'| = |a_{n+1}||z| \geq R|a_1 \cdots a_{n+1}| > R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{n/2}}$$

so we are done. Assume next that $|w| = R|a_1 \cdots a_n|, |z| \leq R|a_1 \cdots a_n|$. Then

$$\begin{aligned}
|z'| &= |z^2 + a_{n+1}w| \\
&\geq |a_{n+1}||w| - (R|a_1 \cdots a_n|)^2 \\
&\geq R|a_1 \cdots a_{n+1}| - R|a_1 \cdots a_n|Rc^{\frac{n}{2}}|a_{n+1}| \\
&\geq R|a_1 \cdots a_{n+1}|(1 - Rc^{\frac{n}{2}}) \\
&\geq R|a_1 \cdots a_{n+1}|e^{-2Rc^{\frac{n}{2}}}, n \geq n_0 \\
&\geq R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{\frac{n}{2}}}.
\end{aligned}$$

■

Next, fix a number $R_0 > 0$. We want to prove that $\Phi(\Omega) \supset \Delta^2(0, \frac{R_0}{2})$. Fix n_0 large as in the above Lemma and define $U := \{(z, w) \in \mathbb{C}^2; F(n_0)(z, w) \in \Delta^2(0, R_0|a_1 \cdots a_{n_0}|)$. Then $\overline{U}$ is compact in Ω. Using the above Lemma, it follows for any $n \geq n_0$ that $F(n)(U) \supset \Delta^2(0, R_0|a_1 \cdots a_n|e^{-2R_0 \sum_{n \geq n_0} c^{\frac{n}{2}}}) \supset \Delta^2(0, \frac{R_0}{2}|a_1 \cdots |a_n|)$ Hence it follows that $\Phi_n(U) \supset \Delta^2(0, \frac{R_0}{2})$ for all $n \geq n_0$. Hence $\Phi(\Omega) \supset \Phi(\overline{U}) \supset \Delta^2(0, \frac{R_0}{2})$. Since R_0 was arbitrary it follows that $\Phi(\Omega) = \mathbb{C}^2$.

■

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Mathematics Department
The University of Michigan
East Hall, Ann Arbor, Mi 48109
USA
fornaess@umich.edu