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The exact steepest descent method — a new steepest descent method based on the exact WKB analysis

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Abstract.

Introducing a new notion of the exact steepest descent path, we develop a new steepest descent method applicable to general ordinary differential equations with polynomial coefficients. Its application to the connection problem for solutions is also discussed.

§1. Introduction

In [2] we proposed a new method called the "exact steepest descent method". It is designed to enlarge the scope of applicability of the steepest descent method by making use of the exact WKB analysis, i.e., WKB analysis based on the Borel resummation technique (cf. [15], [7], [9] and references cited there). It sheds a new light on some missing link between microlocal analysis (cf., e.g., [11], [8]) and exact WKB analysis and, at the same time, it provides us with a new tool in global analysis of ordinary differential equations with polynomial coefficients. In this paper we explain what the exact steepest descent method is and how it is related with microlocal analysis, and discuss its application to the connection problem of ordinary differential equations.

To help the reader's understanding of the theory, we first give an overview of the exact steepest descent method. Let us consider an ordinary differential equation with polynomial coefficients of the following

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form:

$$P\psi = \sum_{\substack{0\le j\le m\\ 0\le k\le n}} a_{jk}x^k\eta^{m-j}\frac{d^j\psi}{dx^j} = 0, \tag{1}$$

where a_{jk} is a complex constant and $\eta > 0$ is a large parameter. If all the coefficients are linear polynomials (i.e., $n = 1$), the Laplace transformation with respect to an independent variable x transforms (1) into a first order equation. Hence, by solving it explicitly, we can readily obtain an integral representation of solutions. In this case every Borel resummed WKB solution of (1) is represented as an integral along a steepest descent path passing through a saddle point and various connection problems for solutions (such as determination of the monodromy group, computation of Stokes multipliers, etc.) can be solved by tracing the configuration of such steepest descent paths ("steepest descent method"; cf., e.g., [12], [13], [14]). The exact steepest descent method allows us to apply this approach to more general equations. That is, to study (1) when $n \ge 2$, we consider the inverse Laplace transform

$$\int e^{\eta x\xi}\hat{\psi}_k(\xi,\eta)d\xi \tag{2}$$

of a WKB solution $\hat{\psi}_k$ $(1 \le k \le n)$ of the Laplace transformed equation $\hat{P}\hat{\psi} = 0$, using the idea of Berk et al. ([5]). We can then observe that the Borel transform $\hat{\psi}_{k,B}$ of $\hat{\psi}_k$ is related with the Borel transform of a WKB solution of (1) by the quantized Legendre transformation near a saddle point of the integral (2). Furthermore, if we introduce a sophisticated notion of the "exact steepest descent path" (which reflects the connection formula for Borel resummed WKB solutions of $\hat{P}\hat{\psi} = 0$; cf. §3 below for its precise definition), the Borel sum of a WKB solution of the original equation $P\psi = 0$ is represented as the integral (2) along an exact steepest descent path passing through a saddle point. Hence the global behavior of solutions of (1) can be analyzed by tracing the configuration of exact steepest descent paths; this is the "exact steepest descent method".

This paper is organized as follows: After reviewing the ordinary steepest descent method briefly in §2, we recall in §3 the notion of the exact steepest descent paths introduced in [2], emphasizing its relevance to microlocal analysis. In §4 we then show how to apply the exact steepest descent method to the computation of Stokes multipliers (which is a typical connection problem). Finally in §5 we give a summary and present some open problems.

§2. Review of the ordinary steepest descent method

As is mentioned in §1, when all the coefficients are linear polynomials (i.e., $n = 1$), an integral representation of solutions of (1) can be readily obtained by employing the Laplace transformation (with a large parameter η) $\psi(x) \mapsto \hat{\psi}(\xi)$, i.e.,

$$\psi(x) = \int e^{\eta x\xi}\hat{\psi}(\xi)d\xi. \tag{3}$$

Then the steepest descent method applied to the integral representation provides us with a powerful tool in global analysis of solutions of (1). Let us illustrate it by the following well-known example.

Example 1. Let us consider the Airy equation:

$$P\psi = \left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0. \tag{4}$$

For (4) the integral representation obtained through the Laplace transformation is given by the following:

$$\psi(x,\eta) = \int \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right)d\xi. \tag{5}$$

Let $f(x,\xi)$ denote the phase function $x\xi - \xi^3/3$ of (5). To study the analytic continuation of a solution of (4), we trace the configuration of steepest descent paths of $\operatorname{Re} f$ passing through saddle points of f. (Recall that, by definition, a saddle point of f is a point satisfying $\partial f/\partial\xi = 0$ and a steepest descent path of $\operatorname{Re} f$ is a level curve of $\operatorname{Im} f$ on which $\operatorname{Re} f$ decreases monotonically.) In this case there exist two saddle points $\xi = \xi_\pm = \pm\sqrt{x}$ and Fig. 1 describes the configuration of the steepest descent paths passing through these two saddle points for $\arg x = 0$, $\arg x = 2\pi/3$ and $\arg x = \pi$.

From Fig. 1 we can perceive that the integral (5) along a steepest descent path C_- for $\arg x = 0$ is analytically continued through the upper half plane to the sum of the integral along C_- and that along C_+ for $\arg x = \pi$. As scaling of the integration variable ξ ensures the equivalence between the asymptotics of (5) for $\eta \to \infty$ and that for $|x| \to \infty$ and further the asymptotics for $\eta \to \infty$ can be readily computed by the saddle point method, this implies that the asymptotic solution

$$\int_{C_-} \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right)d\xi \sim \frac{i\sqrt{\pi}}{\sqrt{\eta}}x^{-1/4}\exp\left(-\frac{2}{3}\eta x^{3/2}\right)(1+\cdots) \tag{6}$$

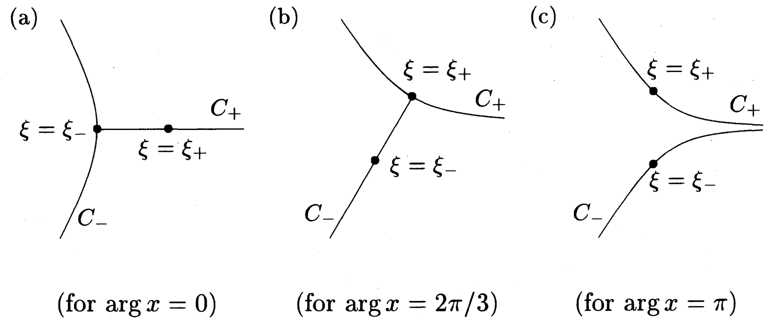


Fig. 1. Steepest descent paths of (5).

of (4) for $x \to +\infty$ is analytically continued to

$$(7) \qquad \int_{C_+} \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right) d\xi + \int_{C_-} \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right) d\xi$$
$$\sim \frac{2i\sqrt{\pi}}{\sqrt{\eta}}(-x)^{-1/4}\sin\left(\frac{2}{3}\eta(-x)^{3/2} + \frac{\pi}{4}\right)(1 + \cdots)$$

for $x \to -\infty$. This is the well-known "Stokes phenomenon" for the Airy equation. In this way, by the steepest descent method, that is, by tracing the configuration of steepest descent paths for the integral representation, we can solve connection problems for ordinary differential equations with linear coefficients.

In [13] and [14] some other interesting examples are discussed from this viewpoint. Note that the steepest descent method is, in a sense, equivalent to the exact WKB analysis for ordinary differential equations with linear coefficients. See [12] for the precise description of the relationship between these two methods. This approach is also related to the "hyperasymptotic analysis" of Berry and Howls ([6]).

Our goal is to generalize this method so that it may be applicable to ordinary differential equations with polynomial coefficients.

§3. Exact steepest descent method

In this section we explain the framework of the exact steepest descent method. For the details of the theory we refer the reader to [2].

Now, to generalize the steepest descent method so that it may be applied to an equation of the form (1), we again apply the Laplace transformation (3) to (1) and consider its Laplace transformed equation

$$\hat{P}\hat{\psi} = \sum_{\substack{0\le j\le m\\ 0\le k\le n}} a_{jk}\eta^{m-k}\left(-\frac{d}{d\xi}\right)^k (\xi^j\hat{\psi}) = 0. \tag{8}$$

In case $n \geq 2$ it is difficult to solve (8) explicitly. Instead we use a WKB solution

$$\hat{\psi}_k = \eta^{-1/2}\exp\left(\eta\int^{\xi}(-x_k(\xi))d\xi + \cdots\right) \tag{9}$$

of (8) and consider its inverse Laplace transform

$$\int e^{\eta x\xi}\hat{\psi}_k d\xi = \eta^{-1/2}\int \exp\left(\eta\left(x\xi - \int^{\xi} x_k(\xi)d\xi\right) + \cdots\right) d\xi, \tag{10}$$

where $x_k(\xi)$ $(k = 1, \ldots, n)$ denotes a root (with respect to x) of the characteristic equation

$$p(x,\xi) \stackrel{\text{def}}{=} \sum a_{jk}x^k\xi^j = 0, \tag{11}$$

and $\eta^{-1/2}$ is added to (9) for the sake of convenience in defining its Borel transform. (Throughout this paper we frequently use the terminologies in the exact WKB analysis. For their precise meaning see [9] or [1].)

Let $f_k(x,\xi)$ denote $x\xi - \int^{\xi} x_k(\xi)d\xi$. Roughly speaking, we apply the steepest descent method to the integral (10) with regarding $f_k(x,\xi)$ as its phase function. This idea was first presented by Berk et al. ([5]). In what follows we polish up their idea by examining it from the viewpoint of the exact WKB analysis.

Let us first fix the path of integration for (10). In parallel with the case of the Airy equation, we take a steepest descent path of $\operatorname{Re} f_k$ passing through a saddle point of f_k as the path of integration for (10). Since a saddle point $\tilde{\xi}$ of f_k satisfies $x = x_k(\tilde{\xi})$, $\tilde{\xi} = \xi_j(x)$ holds for some j $(j = 1, \ldots, m)$, where $\xi_j(x)$ denotes a root of (11) with respect to ξ. If we let $C_k^{(j)}$ denote the steepest descent path of $\operatorname{Re} f_k$ passing through $\xi_j(x)$, our task is then to relate the integral (10) along $C_k^{(j)}$ with a WKB solution of the original equation (1) of the form

$$\psi_j = \eta^{-1}\exp\left(\eta\int^{x}\xi_j(x)dx + \cdots\right), \tag{12}$$

where another normalization factor η^{-1} is used for later convenience.

Local correspondence of Borel transformed WKB solutions

In the exact WKB analysis a WKB solution is given its analytic meaning by the Borel resummation. Hence it follows from the definition of the Borel sum that the integral we are interested in is

$$\int_{C_k^{(j)}} e^{\eta x\xi}\left(\int e^{-\eta z}\hat{\psi}_{k,B}(\xi,z)dz\right)d\xi, \tag{13}$$

where $\hat{\psi}_{k,B}$ denotes the Borel transform of $\hat{\psi}_k$ and the integration in z-space is performed along the path $z=\int^{\xi}x_k(\xi)d\xi+v$, $v\geq 0$. Note that, if we write (9) as $(\exp(-\eta\int^{\xi}x_k(\xi)d\xi))\eta^{-1/2}(c_0+c_1\eta^{-1}+\cdots)$ after applying the Taylor expansion, the Borel transform $\hat{\psi}_{k,B}$ is, by definition, given by

$$\sum_{l=0}^{\infty}\frac{c_l}{\Gamma(l+1/2)}\left(z-\int^{\xi}x_k(\xi)d\xi\right)^{l-1/2}. \tag{14}$$

Furthermore, introducing a new integration variable $y=z-x\xi$, we find that the integral (13) can be rewritten as follows:

$$\iint \exp(-\eta y)\hat{\psi}_{k,B}(\xi,y+x\xi)d\xi dy. \tag{15}$$

Here the path of integration in y-space is described by $y=-\int^{x}\xi_j(x)dx+w$, $w\geq 0$, and the integration in ξ-space is performed on the portion $[\xi^{(-)},\xi^{(+)}]$ of the steepest descent path $C_k^{(j)}$, where $\xi^{(\pm)}$ is the two different points on $C_k^{(j)}$ satisfying $\operatorname{Re}f_k(x,\xi^{(\pm)})-\operatorname{Re}f_k(x,\xi_j(x))=-w$ for a fixed pair (x,w) $(w\geq 0)$. Therefore, the integral (13) can be written also as

$$\int_{y=-\int^{x}\xi_j(x)dx+w,w\geq 0} e^{-\eta y}\chi(x,y)dy \tag{16}$$

with

$$\chi(x,y)\overset{\text{def}}{=}\int_{[\xi^{(-)},\xi^{(+)}]}\hat{\psi}_{k,B}(\xi,y+x\xi)d\xi. \tag{17}$$

The form of the integral (16) is the same as that of the Borel sum of a WKB solution (12), provided that $\chi(x,y)$ is its Borel transform. To confirm that $\chi(x,y)$ is the Borel transform of (12), we should note that the corresondence (17) between χ and $\hat{\psi}_{k,B}$ is given by

$$(T\varphi)(x,y)\overset{\text{def}}{=}\int\varphi(\xi,y+x\xi)d\xi=\iint\delta(y-z+x\xi)\varphi(\xi,z)d\xi dz, \tag{18}$$

which is the so-called "quantized Legendre transformation", that is, a quantization of the canonical transformation from $T^*\mathbb{C}^2_{(\xi,z)}$ to $T^*\mathbb{C}^2_{(x,y)}$ with a generating function $\Omega(\xi,z,x,y) = y - z + x\xi$ (cf., e.g., [8, Example 4.2.5]). Through the transformation T operators in (ξ,z)-space and those in (x,y)-space correspond in the following manner:

$$\begin{aligned}&\xi \longmapsto \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{-1}, \quad z \longmapsto y + x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{-1}, \\ &\frac{\partial}{\partial \xi} \longmapsto -x\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \longmapsto \frac{\partial}{\partial y}.\end{aligned} \tag{19}$$

Having this correspondence in mind, we find

$$\begin{aligned}&\sum a_{jk}x^k\left(\frac{\partial}{\partial y}\right)^{m-j}\left(\frac{\partial}{\partial x}\right)^{j}\chi(x,y) \\ &= \sum a_{jk}\left(\frac{\partial}{\partial y}\right)^{m-k}\left(x\frac{\partial}{\partial y}\right)^{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{-1}\right)^{j} T(\hat{\psi}_{k,B}(\xi,z)) \\ &= T\left(\sum a_{jk}\left(\frac{\partial}{\partial z}\right)^{m-k}\left(-\frac{\partial}{\partial \xi}\right)^{k}\xi^j\hat{\psi}_{k,B}(\xi,z)\right) = 0.\end{aligned} \tag{20}$$

(The final equality follows from the definition of the Borel transform.) The differential equation (20) combined with the study of the local behavior of $\chi(x,y)$ near its singular point $y = -\int^x \xi_j(x)dx$ (which can be done by applying Prop. 4.2.4 in [11, p.422]) then entails that $\chi(x,y)$ is the Borel transform of (12).

Summing up, the quantized Legendre transformation relates $\hat{\psi}_{k,B}$ to $\psi_{j,B}$. This correspondence is valid near the saddle point $\xi_j(x)$, or as far as no extra singularities appear in the domain of integration of (15). In this manner the local aspect of the exact steepest descent method, i.e., local correspondence of Borel transformed WKB solutions, is governed by microlocal analysis. (Strictly speaking, the above proof of (20) is somewhat heuristic as we have not specified the meaning of $(\partial/\partial y)^{-1}$. See [2, Sect. III] for its rigorous proof based on the integration by parts.)

Global correspondence of Borel resummed WKB solutions

We have observed so far the local correspondence between $\hat{\psi}_{k,B}$ and $\psi_{j,B}$. However, this correspondence is violated when the steepest descent path $C_k^{(j)}$ crosses a Stokes curve of type $(k > k')$ for $\hat{P}$ given by

$$\operatorname{Im}\int_{\hat{a}}^{\xi}(x_k(\xi) - x_{k'}(\xi))d\xi = 0 \quad (k' \neq k) \tag{21}$$

at, say, $\xi = \xi_0$. (Cf. Fig. 2. Here $\hat{a}$ denotes a turning point for $\hat{P}$ from which the Stokes curve in question emanates. Note that "of type $(k > k')$" means that $\hat{\psi}_k$ is dominant over $\hat{\psi}_{k'}$ along the Stokes curve.) As a matter of fact, at $\xi = \xi_0$ the singularity of $\hat{\psi}_{k,B}(\xi, z)$ located at

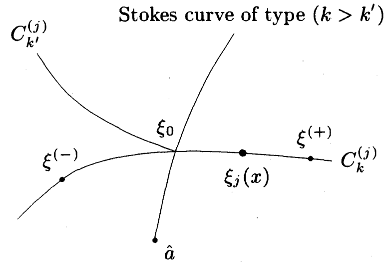


Fig. 2. Crossing of a steepest descent path and a Stokes curve.

$z = \int^{\xi} x_{k'}(\xi) d\xi$ hits the path of integration in z-space for the integral (13) by the definition of a Stokes curve in the exact WKB analysis (cf., e.g., [15]), and consequently a singular point $\xi = \xi_*$ of $\hat{\psi}_{k,B}(\xi, y + x\xi)$ corresponding to the above singularity hits the path of integration in ξ-space for the integral (15) (cf. Fig. 3). This observation implies that,

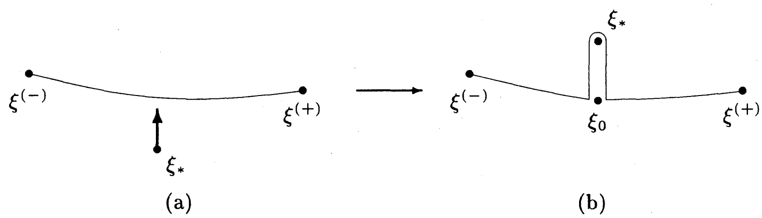


Fig. 3. Singular point ξ_* hitting the path of integration.

to get the analytic continuation of $\chi(x, y)$ defined by (17) beyond the crossing point, we have to take into account the effect of the integral I_* obtained by the integration from ξ_0 to ξ_* in Fig. 3 (b).

Then a natural question arises: Where does the integral I_* come from? The answer is quite simple: I_* is coincident with the integral

$$(22) \quad \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x\xi} \hat{\psi}_{k'} d\xi = \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x\xi} \left(\int e^{-\eta z} \hat{\psi}_{k',B}(\xi, z) dz \right) d\xi,$$

where $C_{k'}^{(j)}$ is a steepest descent path of $\operatorname{Re} f_{k'}$ emanating from the crossing point ξ_0 (cf. Fig. 2) and $\alpha_{k'}$ is a constant which appears in the connection formula

$$\hat{\psi}_k \longrightarrow \hat{\psi}_k + \alpha_{k'}\hat{\psi}_{k'} \tag{23}$$

that the dominant Borel resummed WKB solution $\hat{\psi}_k$ satisfies when crossing the Stokes curve in question. This leads to the conclusion that

$$\psi_j^\dagger \stackrel{\text{def}}{=} \int_{C_k^{(j)}} e^{\eta x\xi}\hat{\psi}_k d\xi + \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x\xi}\hat{\psi}_{k'} d\xi \tag{24}$$

gives the Borel sum of a WKB solution (12) of $P\psi = 0$ unless the steepest descent paths $C_k^{(j)}$ and $C_{k'}^{(j)}$ cross any other Stokes curves for $\hat{P}$. See [2, Sect. IV] for the proof of the coincidence of I_* and (22). (In [2] an additional assumption $n = 2$ is imposed. See also §5 below.)

We are thus forced to consider not only the steepest descent path $C_k^{(j)}$ of $\operatorname{Re} f_k$ passing through a saddle point $\xi_j(x)$ but also another steepest descent path $C_{k'}^{(j)}$ of $\operatorname{Re} f_{k'}$ bifurcated from $C_k^{(j)}$ at its crossing point with a Stokes curve for $\hat{P}$. In more general situations we should consider an "exact steepest descent path" which is defined as follows:

Definition. Let f_k denote $x\xi - \int^\xi x_k(\xi)d\xi$, i.e., the phase function of the inverse Laplace transform (10). An exact steepest descent path passing through a saddle point $\xi = \xi_j(x)$ is, by definition, the union of portions of steepest descent paths obtained by the following procedure:

Start with a steepest descent path $C_k^{(j)}$ of $\operatorname{Re} f_k$ for some k that passes through $\xi_j(x)$. If $C_k^{(j)}$ crosses a Stokes curve of type $(k > k')$ for $\hat{P}$, consider the steepest descent path $C_{k'}^{(j)}$ for $\operatorname{Re} f_{k'}$ which starts from the crossing point. If $C_{k'}^{(j)}$ (or $C_k^{(j)}$) crosses another Stokes curve of type $(k' > k'')$ (or $(k > k'')$), consider another steepest descent path $C_{k''}^{(j)}$ for $\operatorname{Re} f_{k''}$ in the same manner, and so on.

Letting $C^{(j)} = C_k^{(j)} \cup C_{k'}^{(j)} \cup C_{k''}^{(j)} \cup \cdots$ denote an exact steepest descent path in the above sense, we can then expect that

$$\psi_j^\dagger = \int_{C_k^{(j)}} e^{\eta x\xi}\hat{\psi}_k d\xi + \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x\xi}\hat{\psi}_{k'} d\xi + \alpha_{k''} \int_{C_{k''}^{(j)}} e^{\eta x\xi}\hat{\psi}_{k''} d\xi + \cdots \tag{25}$$

(where $\alpha_{k'}$ etc. are constants determined by the connection formula) should coincide with a Borel resummed WKB solution ψ_j of (1). In other words, (25) should give an integral representation of solutions of (1). Hence, it can be further expected that we can analyze global behavior of solutions of (1) by tracing the configuration of exact steepest descent paths. To use exact steepest descent paths instead of ordinary steepest descent paths is a key idea of the exact steepest descent method. The necessity of introducing bifurcated steepest descent paths is an effect of the connection formula for Borel resummed WKB solutions of $\hat{P}$. In this manner the global aspect of the method is governed by the exact WKB analysis.

Remark 1. As was observed by Berk et al. ([5]), the configuration of a steepest descent path abruptly changes when it hits a turning point for $\hat{P}$. But introduction of exact steepest descent paths resolves this trouble. For example, when a steepest descent path $C_k^{(j)}$ hits a simple turning point $\hat{a}$, no topological change occurs for the configuration of the exact steepest descent path $C_k^{(j)} \cup C_{k'}^{(j)}$ as is shown in Fig. 4. (In Fig. 4 a lightfaced line and a wiggly line respectively designate a Stokes curve and a cut defining the Riemann surface of $x_k(\xi)$.) Furthermore,

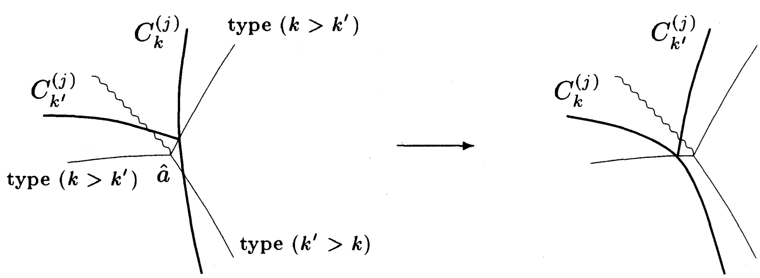


Fig. 4. Change of the configuration of an exact steepest descent path when it hits a simple turning point $\hat{a}$.

the integral

$$(26) \qquad \int_{C_k^{(j)}} e^{\eta x \xi} \hat{\psi}_k d\xi + \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x \xi} \hat{\psi}_{k'} d\xi$$

in Fig. 4 (a) is analytically continued to

$$(27) \qquad \int_{C_k^{(j)}} e^{\eta x \xi} \hat{\psi}_k d\xi + \tilde{\alpha}_{k'} \int_{C_{k'}^{(j)}} e^{\eta x \xi} \hat{\psi}_{k'} d\xi$$

in Fig. 4 (b), since the connection formula near a simple turning point guarantees that the analytic continuation of $\hat{\psi}_k$ on $C_k^{(j)}$ (resp. $\alpha_{k'}\hat{\psi}_{k'}$

on $C_{k'}^{(j)}$) in Fig. 4 (a) is equal to $\tilde{\alpha}_{k'}\hat{\psi}_{k'}$ on $C_{k'}^{(j)}$ (resp. $\hat{\psi}_k$ on $C_k^{(j)}$) in Fig. 4 (b) (cf. [15, p.245–p.246]). Hence, in general, the integral (25) is expected to be analytic even when a steepest descent path hits a turning point for $\hat{P}$.

Remark 2. At a crossing point ξ_0 of $C_k^{(j)}$ with a Stokes curve the steepest descent direction of $\operatorname{Re} f_k$ and that of $\operatorname{Re} f_{k'}$ always lie on the same side of the Stokes curve. To confirm this it suffices to note that the steepest descent direction of $\operatorname{Re} f_k$ and that of $\operatorname{Re} f_{k'}$ at $\xi = \xi_0$ are respectively given by $\vec{v}_k = -\operatorname{grad}_{(\operatorname{Re}\xi,\ \operatorname{Im}\xi)} \operatorname{Re} f_k = -\overline{(x - x_k(\xi_0))}$ and $\vec{v}_{k'} = -\overline{(x - x_{k'}(\xi_0))}$ and that they satisfy

$$i\,\overline{(x_k(\xi_0) - x_{k'}(\xi_0))}\,\overline{(\vec{v}_k - \vec{v}_{k'})} = i\,|x_k(\xi_0) - x_{k'}(\xi_0)|^2 \in i\mathbb{R}, \tag{28}$$

where $i\,\overline{(x_k(\xi_0) - x_{k'}(\xi_0))}$ is a normal vector of the Stokes curve. Thanks to this fact the orientation of the integral along $C_{k'}^{(j)}$ in (24) (or (25)) is naturally determined by that along $C_k^{(j)}$, that is, if the orientation along $C_k^{(j)}$ is the receding one from (resp. approaching one to) the saddle point, the orientation along $C_{k'}^{(j)}$ is also chosen to be receding (resp. approaching).

§4. An application to the computation of Stokes multipliers

In this section we examine the effectiveness of the exact steepest descent method by applying it to the computation of Stokes multipliers of a concrete example.

Example 2. Let us discuss the following equation

$$P\psi = \left(\frac{d^3}{dx^3} + \eta^2\frac{d}{dx} + x^2\eta^3\right)\psi = 0 \tag{29}$$

with its Laplace transform

$$\hat{P}\hat{\psi} = \eta\left(\frac{d^2}{d\xi^2} + (\xi^3 + \xi)\eta^2\right)\hat{\psi} = 0. \tag{30}$$

In this case the characteristic equation is given by

$$p(x, \xi) = \xi^3 + \xi + x^2 = 0 \tag{31}$$

and we label its roots $\xi = \xi_j(x)$ $(j = 0, 1, 2)$ and $x = x_\pm(\xi)$ as follows:

$$\tag{32} \begin{aligned} &\xi_j(x) \sim -\omega^j x^{2/3} \quad (\text{as } x \to +\infty, \text{ where } \omega = e^{2\pi i/3}), \\ &x_\pm(\xi) = \pm i\sqrt{\xi^3 + \xi} \quad (\text{where } \sqrt{\xi^3 + \xi} > 0 \text{ for } \xi > 0). \end{aligned}$$

Using the exact steepest descent method, we now discuss the connection problem for (29) with the aid of a computer. As a path of analytic continuation we take Γ obtained by slightly deforming the real axis (cf. Fig. 5, where the ordinary Stokes curves for (29) are also included for the reference of the reader familiar with the exact WKB analysis). Let

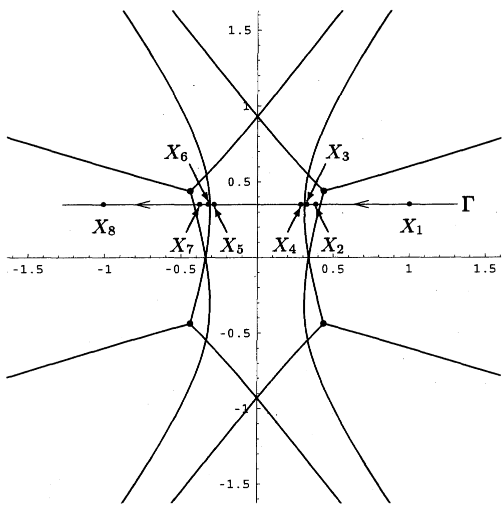
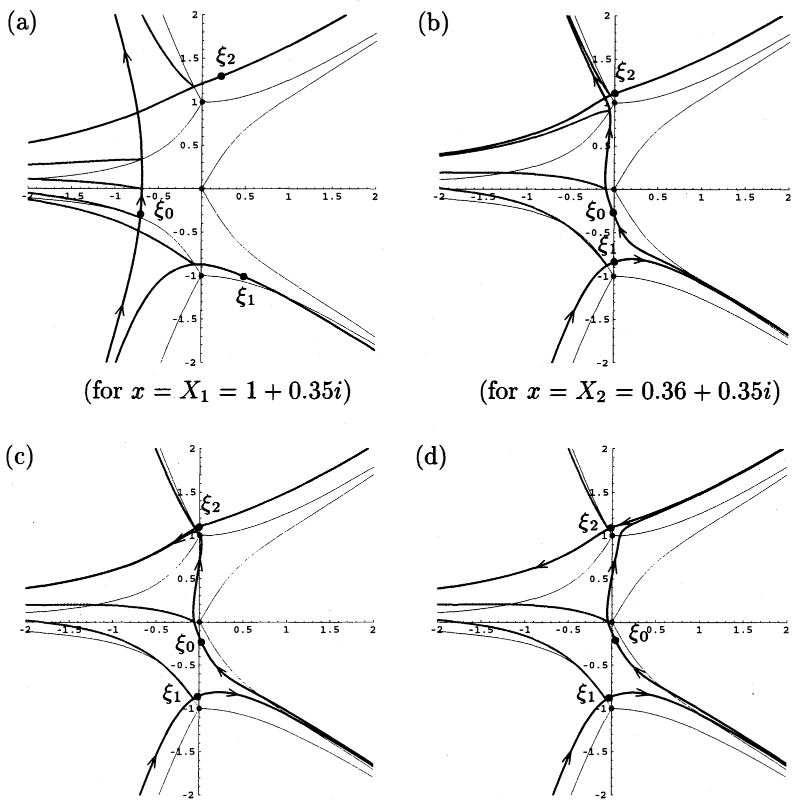


Fig. 5. The path Γ of analytic continuation.

us start with $\psi_0^\dagger$, a solution with integral representation (25) along an exact steepest descent path $C^{(0)}$ passing through a saddle point ξ_0, for a point $x = X_1$ on Γ (cf. Fig. 6 (a)). First, as is clear from the comparison between Fig. 6 (a) and (b), the exact steepest descent path $C^{(0)}$ hits another saddle point ξ_1 between $x = X_1$ and $x = X_2$. Hence the analytic continuation of $\psi_0^\dagger$ becomes the sum of $\psi_0^\dagger$ and $\psi_1^\dagger$ at $x = X_2$, where $\psi_1^\dagger$ is a solution with integral representation along $C^{(1)}$. Next, between $x = X_2$ and $x = X_3$ $C^{(0)}$ hits a turning point for $\hat{P}$ and consequently the role of the ordinary steepest descent path and that of a bifurcated one are interchanged. However, as is noted in Remark 1 in §3, the solution $\psi_0^\dagger$ is analytic and no abrupt change occurs with $\psi_0^\dagger$ there. Instead $\psi_0^\dagger$ acquires $\psi_2^\dagger$, a solution with integral representation along $C^{(2)}$, between



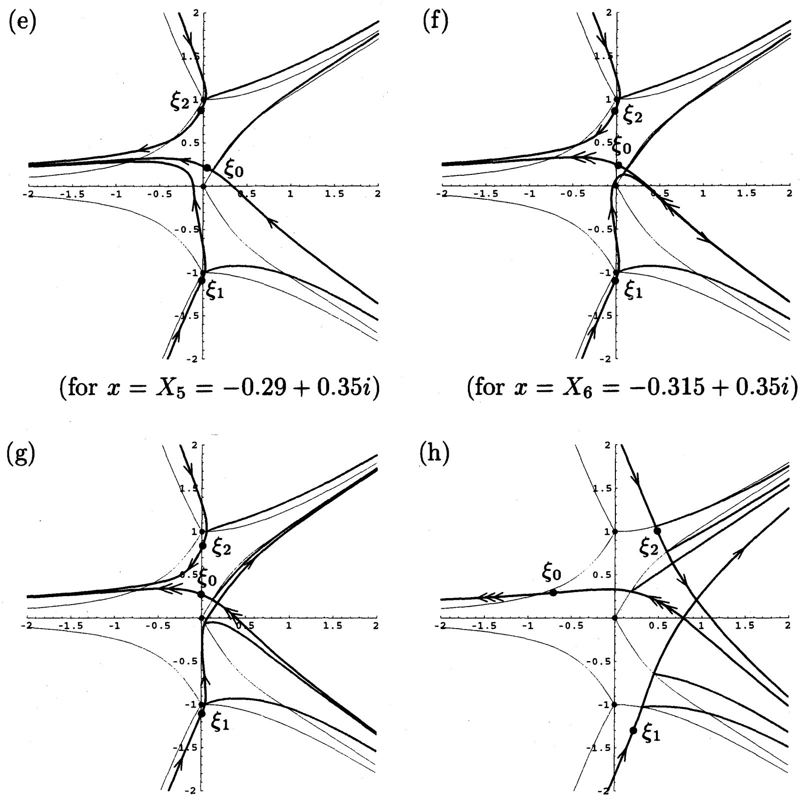
(for $x = X_1 = 1 + 0.35i$) (for $x = X_2 = 0.36 + 0.35i$)

(for $x = X_3 = 0.315 + 0.35i$) (for $x = X_4 = 0.29 + 0.35i$)

Fig. 6. Exact steepest descent paths (designated by boldfaced lines) for (29); lightfaced lines designate Stokes curves for $\hat{P}$.

$x = X_3$ and $x = X_4$ and the solution in question becomes $\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ at $x = X_4$. This procedure can be easily repeated until we reach the point $x = X_8$; our solution is changed into $2\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ at $x = X_6$ and finally into $3\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ at $x = X_8$. Note that an exact steepest descent path hits a saddle point exactly on a Stokes curve for P. We thus conclude that the analytic continuation of $\psi_0^\dagger$ along the real axis is given by $3\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ for $x \to -\infty$.

As the asymptotics of each $\psi_j^\dagger$ for $\eta \to \infty$ can be readily computed by the saddle point method (note that, except for exponentially small terms, the contribution to the $\eta \to \infty$ asymptotics comes from the saddle point



(e) (for $x = X_5 = -0.29 + 0.35i$)

(f) (for $x = X_6 = -0.315 + 0.35i$)

(g) (for $x = X_7 = -0.36 + 0.35i$)

(h) (for $x = X_8 = -1 + 0.35i$)

Fig. 6. (*Continued.*)

only) and, similarly to the case of the Airy equation, the asymptotics of (29) for $\eta \to \infty$ is consistent with that for $|x| \to \infty$, the above conclusion implies that the asymptotic solution

$$(33) \quad \psi_0^\dagger \sim \frac{2\sqrt{\pi}}{\sqrt{3\eta}} e^{3\pi i/4} x^{-2/3} \exp\left(\eta\left(-\frac{3}{5}x^{5/3} + x^{1/3} + \cdots\right)\right)(1 + \cdots)$$

of (29) for $x \to +\infty$ is analytically continued to

$$(34) \qquad 3\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$$

with

$$(35)\quad \psi_j^\dagger \sim \frac{2\sqrt{\pi}}{\sqrt{3}\eta} e^{(9-8j)\pi i/12}(-x)^{-2/3} \times \exp\left(\eta\left(\frac{3}{5}\omega^j(-x)^{5/3} - \omega^{2j}(-x)^{1/3} + \cdots\right)\right)(1+\cdots)$$

$(j = 0, 1, 2)$ for $x \to -\infty$. Thus by virtue of the exact steepest descent method we have succeeded in computing a Stokes multiplier of (29) explicitly.

§5. Summary and discussion

As we have observed so far, it is possible to develop a new steepest descent method applicable to ordinary differential equations with polynomial coefficients. A key point is the introduction of exact steepest descent paths; not only ordinary steepest descent paths passing through a saddle point but also bifurcated ones emanating from a crossing point of a steepest descent path and a Stokes curve for $\hat{P}$ should be taken into account. The theoretical background of the method is provided by microlocal analysis for its local aspect and by the exact WKB analysis for its global aspect.

In ending the paper, we present some open problems. The argument in [2] is based on the proviso that $\hat{P}$ is of the second order (i.e., $n = 2$). This proviso is imposed just because the exact WKB analysis is complete only for second-order operators; for higher-order operators we have to introduce new Stokes curves and virtual turning points (cf. [5], [1], [3]). To clarify the effect of new Stokes curves for $\hat{P}$ in the exact steepest descent method is the first step toward the complete understanding of the exact steepest descent method when $n > 2$. See [10] for some case study of this problem. As a related problem, we also note that finding out an algorithm of describing the complete Stokes geometry for a higher-order operator P is one of the most important open problems in the exact WKB analysis. Since the exact steepest descent method explained in this paper is useful to locate the Stokes curves of P as is emphasized in [2], it will turn out to be a powerful tool to attack this problem. See [4] also for some related problems.

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