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Uniqueness problem for meromorphic mappings under conditions on the preimages of divisors

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Abstract.

We first give a finiteness theorem for meromorphic mappings. Next, we give conditions under which two holomorphic mappings from a finite analytic covering space over the complex m-space into a smooth elliptic curve are algebraically related.

Introduction.

The uniqueness problem of meromorphic mappings under condition on the preimages of divisors was first studied by G. Pólya and R. Nevanlinna. They proved the following famous five point theorem: Let f and g be nonconstant meromorphic functions on $\mathbb{C}$. If $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct five points $a_1, \cdots, a_5$ in $\mathbb{P}_1(\mathbb{C})$, then f and g are identical. So far, many researchers have studied unicity theorems for meromorphic functions on $\mathbb{C}$, as well in the multidimensional case. Among these, H. Fujimoto has proved a number of remarkable unicity theorems. For example, he proved the following excellent theorem ([4]):

Theorem (Fujimoto). *Let f, $g : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ be nonconstant meromorphic mappings with the same inverse images of q hyperplanes in general position.*

(1) *If $q = 3n+1$, then there exists an automorphism L of $\mathbb{P}_n(\mathbb{C})$ such that $f = L \cdot g$.*

(2) *If $q = 3n + 2$ and either f or g is linearly nondegenerate, then f and g are identical.*

The finiteness theorem for meromorphic mappings was also studied by H. Cartan and R. Nevanlinna in 1920's. The finiteness theorem of Cartan-Nevanlinna states that there exist at most two meromorphic functions on $\mathbb{C}$ that have the same inverse images with multiplicities

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for distinct three values in $\mathbb{P}_1(\mathbb{C})$. In 1981, H. Fujimoto generalized the theorem of Cartan-Nevanlinna to the case of meromorphic mappings of $\mathbb{C}^m$ into complex projective spaces $\mathbb{P}_n(\mathbb{C})$ by making use of Borel's identity ([5]). He proved the finiteness of families of linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}_n(\mathbb{C})$ with the same inverse images for some hyperplanes. In his results, the number of hyperplanes in general position is essential and must be larger than a certain number depending on the dimension of the projective spaces. Note that an essential problem in the multidimensional case exists in this point. Namely, in the case where a given divisor is irreducible, what kind of condition yields the finiteness of families of meromorphic mappings ? In this paper, we first give a finiteness theorem for meromorphic mappings f of $\mathbb{C}^m$ into a compact complex manifold M and for an irreducible divisor D on M. Next, we give some theorems on uniqueness problems of holomorphic mappings into smooth elliptic curves.

§1. Finiteness theorem for meromorphic mappings.

In this section, we give a finiteness theorem. For details, see [1]. To state our results, we give some definitions. Let $L \to M$ be a fixed line bundle over M, and let $\sigma_1, \cdots, \sigma_s$ be linearly independent holomorphic sections of $L \to M$ with $s \geq 2$. Throughout this paper, we assume that $(\sigma_j) = dD_j$ $(1 \leq j \leq s)$ for some positive integer d, where D_j are effective divisors on M. Set

$$\varpi = c_1\sigma_1 + \cdots + c_s\sigma_s,$$

where $c_j \in \mathbb{C}^*$. Let D be a divisor defined by $\varpi = 0$. We define a meromorphic mapping $\Psi : M \to \mathbb{P}_{s-1}(\mathbb{C})$ by $\Psi = (\sigma_1, \cdots, \sigma_s)$.

Definition 1.1. Let p be a nonnegative integer. For divisors Z_1 and Z_2 on $\mathbb{C}^m$, we write

$$Z_1 \equiv Z_2 \pmod{p}$$

if there exists a divisor Z' on $\mathbf{C}^m$ such that $Z_1 - Z_2 = pZ'$; in the special case of $p = 0$, $Z_1 \equiv Z_2 \pmod{0}$ if and only if $Z_1 = Z_2$.

Let Z be a nonzero effective divisor on $\mathbb{C}^m$. We denote by

$$\mathcal{F}(p; (\mathbb{C}^m, Z), (M, D))$$

the set of all meromorphic mappings $f : \mathbb{C}^m \to M$ such that

$$f^*D \equiv Z \pmod{p}.$$

Definition 1.2. We say that a meromorphic mapping $f : \mathbb{C}^m \to M$ has *the Zariski dense image* if $f(\mathbb{C}^m)$ is not included in any proper analytic subset of M.

Let

$$\mathcal{F}^*(p; (\mathbb{C}^m, Z), (M, D))$$

denote the subset of all $f \in \mathcal{F}(p; (\mathbb{C}^m, Z), (M, D))$ with *the Zariski dense image*. The main result of the present article is as follows ([1, Theorem 2.1]):

Theorem 1.3. *If* rank $\Psi = \dim\ M$ *and* $d > (s+1)!\ \{(s+1)! - 2\}$, *then the number of mappings in* $\mathcal{F}^*(d; (\mathbb{C}^m, Z), (M, D))$ *is bounded by a constant depending only on* D.

§2. Holomorphic curves into smooth elliptic curves.

In this section, we give some theorems on the uniqueness of holomorphic mappings into smooth elliptic curves E. In particular, we consider the problem to determine the condition which yields $f = \varphi(g)$ for an endomorphism φ of the abelian group E. For details, see [2]. The uniqueness problem of holomorphic mappings into elliptic curves was first studied by E. M. Schmid (Math. Z. **23** (1971)). Schmid's unicity theorem is the following: Let $f,\ g : R \to E$ be nonconstant holomorphic mappings, where R is an open Riemann surface of a certain type. Then there exists a nonnegative integer d depending only on R such that, if $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct $d+5$ points $a_1, \cdots, a_{d+5}$ in E, then f and g are identical. In the special case $R = \mathbb{C}$, we have $d = 0$. However, there have been only few studies on the uniqueness problem of holomorphic mappings into elliptic curves (cf. [3]).

Let $\pi : X \to \mathbb{C}^m$ be a finite analytic covering space and s_0 its sheet number. We denote by $[p]$ the point bundle determined by $p \in E$ and set $\tilde{F} = \pi_1^*[p] \otimes \pi_2^*[p]$, where $\pi_j : E \times E \to E$ are the natural projections. Let $f,\ g : X \to E$ be nonconstant holomorphic mappings. We denote by $\mathrm{End}(E)$ the ring of endomorphisms of E. If E has no complex multiplication, it is well-known that $\mathrm{End}(E) \cong \mathbb{Z}$. Hence $\varphi(x) = nx$ for some integer n. We now seek conditions which yield $g = \varphi(f)$ for some $\varphi \in \mathrm{End}(E)$. Let $\varphi \in \mathrm{End}(E)$ and consider a curve

$$\tilde{S} = \{(x,\ y) \in E \times E;\ y = \varphi(x)\}$$

in $E \times E$. Let $[\tilde{S}]$ be the line bundle determined by $\tilde{S}$. Denote by $\tilde{\gamma}$ the infimum of rational numbers γ such that $\gamma\tilde{F} \otimes [\tilde{S}]^{-1}$ is ample.

Then we have $\tilde{\gamma} = \deg \varphi + 1$ which is proved by T. Katsura (see [2]). Hence, if $\varphi \in \mathrm{End}(E)$ is an endomorphism defined by $\varphi(x) = nx$, then $\tilde{\gamma} = n^2 + 1$. Let Z be an effective divisor on X, and let k be either a positive integer or $+\infty$. If $Z = \sum_j \nu_j Z_j$ for distinct irreducible hypersurfaces Z_j in X and for nonnegative integers ν_j, then we define the support of Z with order at most k by $\mathrm{Supp}_k Z = \bigcup_{0<\nu_j\le k} Z_j$. We now have the following:

Theorem 2.1. *Let f and g be as above. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and φ a endomorphism of E. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is also d. Suppose that $\mathrm{Supp}_k\ f^*D_1 = \mathrm{Supp}_k\ g^*D_2$ for some k. If $d > 2(\deg\varphi + 1) + 8(s_0 - 1)(1 + k^{-1})$, then $g = \varphi(f)$.*

In the above theorem, we assume that the cardinality $\sharp D_2$ of the point set D_2 equals d. However, it may happen that $\sharp D_2 < d$. For example, if $\varphi(x) = nx$ ($n \in \mathbb{Z}$) and there exists at least one pair $(i,\ j)$ such that $a_i - a_j$ is n-torsion point, then $\sharp D_2 < d$. In this case, we have the following:

Theorem 2.2. *Let $f,\ g : \mathbb{C}^m \to E$ be nonconstant holomorphic mappings. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and $\varphi \in \mathrm{End}(E)$. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is d'. Suppose that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. If $dd' > (d + d')(\deg\varphi + 1)$, then $g = \varphi(f)$.*

Corollary 2.3. *Let f and g be as in Theorem 2.2. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and set $D_2 = \{na_1, \cdots, na_d\}$ for some integer n. Assume that the number of points in D_2 is d'. Suppose that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. If $dd' > (d+d')(n^2+1)$, then $g = nf$.*

We do not know whether Theorem 2.2 is sharp or not. However, if the condition $dd' > (d + d')(\deg\varphi + 1)$ is not satisfied, then it is not necessarily true that $g = \varphi(f)$.

Example 2.4. Let φ be an endomorphism defined by $\varphi(x) = 2x$. Define $f, g : \mathbb{C} \to E$ by $f(z) = \overline{\pi}(x)$ and $g(z) = -2\overline{\pi}(x)$, where $\overline{\pi} : \mathbb{C} \to E$ be the universal covering mapping. Let $D_1 = \{x \in E; 4x = 0\}$. Then $D_2 = \varphi(D_1) = 2D_1$. It is clear that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. In this case, $d = 16$, $d' = 4$ and $\deg\varphi + 1 = 5$. Thus we have

$$dd' - (d + d')(\deg\varphi + 1) = -36 < 0$$

and $g \neq \varphi(f)$.

For nonconstant holomorphic mappings $f,\ g : X \to E$, we have the following unicity theorem, which is a direct conclusion of Theorem 2.1:

Theorem 2.5. *Let $a_1, \cdots, a_d$ be distinct points in E. Suppose that* $\mathrm{Supp}_k\ f^*a_j = \mathrm{Supp}_k\ g^*a_j$ *for all j, where $1 \leq k \leq +\infty$. If $d > 8s_0 - 4 + 8k^{-1}(s_0 - 1)$, then f and g are identical.*

In the case of $X = \mathbb{C}^m$, we have the following:

Theorem 2.6. *Let $a_1, \cdots, a_d$ be distinct points in E. Suppose that $X = \mathbb{C}^m$ and* $\mathrm{Supp}_1\ f^*a_j = \mathrm{Supp}_1\ g^*a_j$ *for all j. If $d \geq 5$, then f and g are identical.*

We give here the concluding remark. If we choose special points of E, we obtain an example which yields that Theorem 2.6 is sharp. Indeed, let $a_1, \cdots, a_4$ be two-torsion points in E and let $\wp$ be the Weierstrass $\wp$ function. If $f_1^* a_j = f_2^* a_j$ for $j = 1, \cdots, 4$, it is easy to see that $\wp \circ f_1 = \wp \circ f_2$ by Nevanlinna's four points theorem. Hence $f_1 = f_2$ or $f_1 = -f_2$. Since $p \mapsto -p$ $(p \in E)$ is an automorphism of E, it is acceptable that f_1 and f_2 are essentially identical. In this example, it seems that the structure of the function field of E affects strongly the uniqueness problem for holomorphic mappings.

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