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Mathematics of Professor Oka
– a landscape in his mind –

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Preface

As a mathematician working in several complex variables, I would like to share with you the pleasure of being here to attend this marvelous symposium, held in honor of late Professor Kiyoshi Oka's 100th birthday. As one of his students, I would like to express my sincere thanks to you all for its success.

He was born on 1901/04/19 in Osaka and passed away on 1978/03/01 in Nara. The day of his last breath was, to my memory, a calm day of early spring. I remember that a thin veil of mist was wandering on the hills of Saki that day. More than two decades have slipped by since then.

In his carrier Oka published only ten papers, except for three summaries. However, in contrast to this apparent scarceness, influence of his work is immense. His idea does not stay within the realm of several complex variables, but extends far beyond, contributing to the development of whole mathematics. Such a state of art will surely be credited by many of the talks that are going to be given here.

On my side, I was first allowed to call him as my teacher in 1956. Thanks to this opportunity, from the next year I could spend a few years with him in Nara Women's University. Actually we sat together desk to desk in a humble office and I could hear many things then from horse's mouth.

Further, by courtesy of his bereaved family, I could read his many unpublished papers and materials prepared for research. These are mainly gathered in the volume 'Posthumous Papers', which is exhibited in the homepage of the library of Nara Women's University (http://www.lib.nara-wu.ac.jp/oka/). So there are materials enough for drawing up a complete picture of Oka's research style, namely we can see how he approached each question and from where he viewed the whole landscape of the research field.

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To my great pleasure I was asked to give the first talk in this meeting. What is anticipated for in the talk is, I guess, to describe a sort of landscape which Oka had in mind. Therefore the purpose of my talk will be to give you a suggestion how a theory of mathematics grew up in his mind. I will do it by pursuing the process how he established "the lifting principle" which I regard, without so much prejudice I hope, his ultimate achievement.

Such a trial may be justified because, as I have already mentioned, Oka left many things behind about this topic. For instance, a paper titled with

"Rappelées du printemps"

written around 1949 presents his research activity historically as a reminiscence. (The word 'printemps' ought to mean here 'the beginning'.) In the following, many materials are without mentioning attributed to this article. Nevertheless my apology is for letting some speculation to creep in, which is, admittedly I hope, inevitable to accomplish my task.

Oka's main research purposes

Let me first show you, just as it is, how Oka began the introduction of Paper I.

> Malgré le progrès récént de la théorie des fonctions analytiques de plusieurs variables complexes, diverses choses importantes restent plus ou moins obscures, notamment : le type de domaines dans lesquels le théorème de Runge ou ceux de M. P. Cousin subsistent, la relation entre la convexité de M. F. Hartogs et celle de MM. H. Cartan et P. Thullen; parmi eux il y a des relations intimes. C'est à traiter ces problèmes que le présent mémoire et ceux qui suivront, sont destinés.

This is supplemented in the footnote:

> Voir l'Ouvrage de MM. H. Behnke et P. Thullen : *Theorie der Funktionen mehrerer komplexer Veränderlichen*, spécialement aux pages 54, 68, 79.

To solve the problems that had been raised in this book, Oka devoted most of his life.

Keeping this in mind, let me briefly describe some of the steps made during his youth, when he had not yet confronted with this task.

§1. Steps to the principal questions

1. In 1919, Oka entered the Third High School in Kyoto, which was one of the most prestigious high school in Japan at that time. This is today a part of Kyoto University. I was told that he was then fascinated by Poincaré's essays 'La Science et l'Hypothèse' and 'Science et Méthode' which were both widely read in those days. Later he recalls Poincaré's influence on his thought as follows.

> "I was moved very much by Poincaré's question 'How do mathematical discoveries come up?' Since then I have intended to solve it to the best of my ability."

It seems that he considered this question not to be pursued just from curiosity, but to be exactly solved by exploiting all his research activity as basic data. The same thing can be said for another question 'How do mathematical researches start?'

The point is that Oka, as a mathematician, sincerely asked how mathematical researches should be done.

Probably I must admit then that he regarded me as one of the marmots to test his method. Of course you cannot conclude from this miserable example that his method is useless, for the results of experiments usually depend on the materials.

2. Let me come back to Oka's mathematics. It was around 1927 that he started his own research. Really nothing had been done before. He wrote down the result in French under the title

Fonctions algébriques permutables avec une fonction rationnelle non-linéaire

and left it as a typewritten manuscript. This paper was never published although it was once submitted for publication through G. Julia. Let me show you only the statement of the result:

> Let $R(x)$ be a rational function of degree at least 2, and let $A(x)$ be an algebraic function. If one substitutes $A(x)$ into $R(x)$, the composite $R[A(x)]$ is well-defined as an algebraic function. Conversely, if one substitutes $R(x)$ into $A(x)$, the composite $A[R(x)]$ splits in general into several different algebraic functions corresponding to the branches of A. If one of these algebraic functions happens to coincide with $R[A(x)]$, we say that $R(x)$ and $A(x)$ are permutable.

In this situation the following holds true.

> ≪If a given algebraic function (of a reduced form in a suitable sense) is permutable with a rational function of degree at least 2, it must satisfy an algebraic equation arising from the multiplication formulae of e^z, $\cos z$ or elliptic functions.≫

Clearly this is a continuation of Julia's work[1].

As a paper of Oka, this article is, to my impression, most involved and most intriguing. Probably he enjoyed himself by writing a puzzling paper.

Later he wrote that this article might be published as well because of the following reasons.

1. This is the very 'melody' of the current of our mathematical research (indicating something more permanent than the mathematical result itself).
2. Without this paper, mutual consistency (or harmony) of later works would not be clear enough (because it shows the original form).

It had, however, never been published before his death.

3. To say some words about his surroundings at that time, Oka got a sabbatical in 1929 to visit France and stayed there for three years. The work on iteration just mentioned above was carried over in France for one year, more or less, and written down around 1930.

The reason why Oka wanted to stay in France was probably because he wanted to get acquainted with Julia, but the biggest motivation for going abroad was, according to his own word, because he thought "I will never be able to see a new ground of mathematics that deserves to be cultivated, as long as I stick to a life in Japan".

As he expected, soon after he moved to France he found the desired new ground, the field of several complex variables.

He used to talk about the feeling when he first saw Goursat's brief introduction to this field which was written in the middle part of the second volume of his 'Cours d'Analyse'. He explained it to me by quoting a haiku:

Kiri nagara Ookina machini Idenikeri (Ichiku)

(The meaning is; I have just arrived at a place, where I see a huge city in a dense fog.)

[1] G. Julia, Mémoire sur la permutabilité des fractions rationnelles, Annale de l'Ecole Normale Superieure, 1922.

After that he set forth to work in the field of several complex variables, by throwing away at first the affection to the subject of iteration, on which he had been working for almost 3 years.

There was, according to him, another difficulty in changing the research field. He told me that in several variables, compared to the case of one variable, things were much more complicated. However, he could overcome this difficulty by reading Julia's famous paper[2] again and again. Soon he could obtain several results before going back to Japan in 1932. These results were published as a summary without proofs under the title

Note sur les familles de fonctions multiformes etc., 1934

which became accordingly his first published paper. Its detail is left as an unfinished paper, whose manuscript is handwritten in French, and is approximately 150 pages long. Nevertheless, when it is combined with another manuscript, which was not totally translated into French, the paper is more or less in a complete form.

Let me show you main results of this Note. I have chosen two simple ones among three theorems.

A. Normal family of analytic sets. Let $\mathcal{F} = \{S\}$ be a family of analytic surfaces in a domain of two complex variables x, y. If the area of S is uniformly bounded, then $\mathcal{F}$ is a normal family, (normality of $\mathcal{F}$ is defined in terms of the defining functions of S). In general, the hull of normality for $\mathcal{F}$ is pseudoconvex.

B. Generalization of Hartogs theorem. Let E be a closed subset of a domain $\mathcal{D}$ of (x, y)-variables. E is called an H-set if E is locally the complement of a pseudoconvex domain. Let $\mathcal{D}$ be the product of a domain in the x-plane and the y-plane, and let E be an H-set in $\mathcal{D}$. Suppose that $E(x') := E \cap \{x = x'\}$ is uniformly bounded in x'. In this situation, if the set of points x for which $E(x)$ consists of finitely many points has positive logarithmic capacity, then $E(x)$ consists of finitely many points for all x, and E is then an analytic surface of $\mathcal{D}$. A similar statement is true when $E(x)$ consists of countably many points.

Some explanation seems to be needed about the motivation of this research. As the title shows, the purpose of this paper was to study the convergence of multivalued analytic functions from the viewpoint of function theory of two variables. As in the first work on iteration,

[2]G. Julia, Sur les familles de fonctions analytiques de plusieurs variables, Acta Math., **47**, 1927.

the problem had been raised by Julia. However, this time, the work is related to Oka's later accomplishment in many aspects.

4. Before describing how Oka did his full-scale works, I would like to draw your attention, as well as aforementioned Pincaré's problem, to the following episode.

Oka says "When I need a method to solve a problem, and cannot get anything in mathematics, I try to find a key to the riddle in other places". To carry out this somewhat paradoxical way, he started to study Basho's poetry deeply, already when he was staying in France, because he thought "I must absorb the time-honored culture of Japan that may help mathematical researches", according to his words. If I am allowed to say it in other words, Oka's style of doing mathematics is first to bring things in mind closer to his inmost emotion, so that he can recognize the spiritual melody they play, and next to give some forms to it, which is the stage of creation.

§2. The rise of problems

1. In 1932, soon after the return from Paris, Oka got a position as an associate professor of Hiroshima Bun-Rika Daigaku (= Hiroshima University).

Although he was quite eager to work in the field of several complex variables, it was still necessary for him to grasp its whole picture before getting down to a full-scale research.

However, the library of that university, which was still in its infancy, was useless for that purpose. The lack of information should have disappointed him very much. It was just at that moment that the book of Behnke-Thullen

Theorie der Funktionen mehrerer komplexer Veränderlichen

was published and brought to his desk.

As he wrote in the introduction of Paper I, Oka could grasp the central current questions of this field in virtue of Behnke-Thullen's book. Later he repeatedly wrote about the benefit of this book, showing his sincere gratitude.

2. That book, which Oka kept long at hand, finds now its place in the library of Nara Women's University. From the notes in the blanks one knows that he began to read it on January 2 in 1935. When he had read it through, he said, the problems written in the introduction of Paper I emerged as 'mountains that separate the future and the past of several complex variables'.

By the words that the problems 'separate the future and the past', Oka meant that it is impossible to go ahead without conquering them. From the beginning Oka bore a plan "to cross this mountain pass and to open a flower garden beyond it". Exploiting this opportunity I would like to add that Oka's Paper X was written as "an example of flower garden that would be opened beyond the pass". Once he told me, "Contrary to my expectation, it took me very long before crossing it."

§3. The lifting principle

1. Soon after the problems were clearly caught in his eyes, Oka set forth to work on them. What he did was, however, not to attack Cousin's first problem on analytic polyhedra. I can say it because, this 'goal'was almost achieved in the footnote of Oka's Paper I. Namely Cousin's first problem was solved for rationally convex domains by using Weil's integral formula similarly as Cousin's integral. For Oka, this discovery must not have been too small. In fact, an integral of this kind is used in the integral equation which plays the key role in solving the inverse problem of Hartogs.

In Oka's mind, the problems were connected to each other into one piece. For instance, in order to see the relation between the convexity notions of Hartogs and Cartan-Thullen, it is necessary to prove how the validity of Runge's theorem and Cousin's theorem depend on the types of the domains. In addition, if a solution of the latter question is useless to solve the former, that solution is meaningless, to put it extremely.

2. Oka read through Behnke-Thullen's book in a couple of months, and soon afterwards began to "look for the first move". In three months from then, however, he was stuck. He says "After such a period of no progress, no plan came into my mind any more, no matter how ridiculous it is". It is after this point that Oka showed marvelous originality.

Let me explain more concretely how he continued. Oka's method was to pursue the problem to a single point. Moreover he did it with a "pin-point accuracy". In the above case the point may be described as follows.

Inside the space of two complex variables x, y, a general domain D can be visualized in the following way.

First, on a sheet of paper you draw the x-plane and, to its right hand side, the y-plane. Then you take one point x' in the x-plane, and draw on the y-plane the slice $D(x')$ of D by $x = x'$. Since $D(x')$ changes as x' moves around, you may imagine a family of $D(x')$ when x' runs through a part of the x-plane.

The point is then

≪To provide a general circumstance for the domains of this kind, in which one can solve some questions, Cousin's first problem for instance.≫

(To my impression, there was in Oka's mind influence of Riemann's idea, of introducing Riemann surfaces to study multivalued functions.)

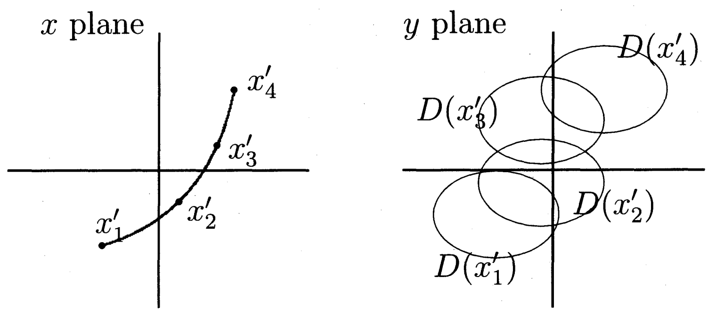

Fig. 1

Anyway, if you take several points on the x-plane and draw the corresponding domains in the y-plane, the picture looks, as above, like the leaves of trees reflected on the surface of water. Oka called such a figure "Ryoku-in-zu (projection of green leaves)" and looked them carefully day after day. He once told me that he had felt like easily walking on thin ice, which might mean that these days were for him a period of delightful devotion to a heavenly mission. This 'delight' is the melody of his heartstring. (Nothing can be understood by knowledge without ethos.) In such a situation, where he felt that his idea was exhausted, it seems that he was grasping something in such a way.

3. This question was settled in the following form:

Let $R_j(x)\,(j = 1, \dots, m)$ be rational functions in n complex variables $x_1, \dots, x_n$, and let (Δ) be a bounded closed region in the (x)-space defined by

$$(\Delta) \qquad |x_i| \leq r_i \ (i = 1, 2, \dots, n), \quad |R_j(x)| \leq 1 \ (j = 1, 2, \dots, m).$$

Then, by adding m complex variables $y_1, \dots, y_m$ to (x), we consider in the product of the (x)-space and the (y)-space a closed polycylinder

$$(C) \qquad |x_i| \leq r_i \,(i = 1, 2, \dots, n), \quad |y_j| \leq 1 \,(j = 1, 2, \dots, m)$$

and an analytic subset

$$(\Sigma) \qquad y_j = R_j(x)\,(j = 1, 2, \ldots, m)$$

of (C).

The following expresses the miniatuarized relationship between (Δ), (Σ) and (C).

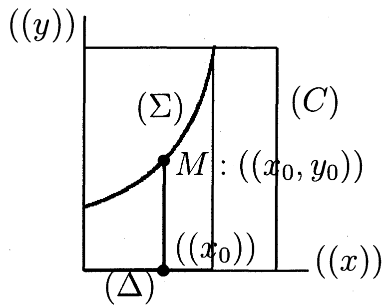

Fig. 2

In such a geometric situation the result is:

Theorem . *Given any holomorphic function $f(x)$ on (Δ), one can find a holomorphic function $F(x, y)$ on (C) satisfying*

$$f(x) = F[x_1, x_2, \ldots, x_n, R_1(x), \ldots, R_m(x)].$$

Oka used to call this theorem 'Jôku Ikô no Genri' (the lifting principle or, more literally, the hovering principle). A foundation was laid by this principle to study the types of domains on which the theorems of Cousin and Runge hold true.

August of that year was nearly going when Oka discovered this theorem. There is a widespread episode that someone close to Oka nicknamed him "encephalitis lethargica (= sleeping sickness)" because he nearly continued to doze every day during the end of that period.

As for the proof of this theorem, it was done by double induction on the dimensions, and, as a result, Cousin's first problem was settled at the same time. Oka wrote about this method in a footnote of Paper I as

≪ Je dois l'idée à M. H. Cartan pour ce mode d'application du théorème de M. Cousin.≫

and quoted Cartan's paper

Sur les fonctions de deux variables complexes. Bull. Sci. math. 1930.

When you look at this solution, you will see that the overlapping leaves in Fig. 1 are lifted to Σ, and holomorphic functions on Δ are extended holomorphically to the polycylinder above. Probably the lifting principle was already emotionally grasped when he was drawing the overlapping leaves.

§4. Subsequent questions

After 'the first move' was followed by the discovery of the lifting principle for rationally convex domains, several questions naturally arose.

1. In Paper I, the problem was solved for rationally convex domains, i.e. for those domains which are convex with respect to the family of rational functions. The gap between the rationally convex domains and the domains of holomorphy was filled by Paper II:

When $R_j(x)$ are holomorphic functions defined only on some neighbourhood of (Δ) , the proof for the rational case does not extend, although the geometric relation between (Δ), (C) and (Σ) is preserved. However, you draw now an analytic polyhedron, defined by polynomials, in any neighbourhood of (Σ). Then, by lifting it again you can solve the problem.

Paper II, which contains this, has many subtle points for reading. According to Oka, however, not so much difficulty was left before this work was finished, because he had already a stock of researches in the former Note.

2. It became a problem to find a distinction, if any, between the domains of holomorphy and rationally convex ones. As is well known, there is no such problem in the one variable case. This was also an important question on which Oka said "It's impossible to proceed further without knowing its solution".

Oka started from Gronwall's example which shows that Cousin's second problem is not necessarily solvable even on a domain of holomorphy.

Like Gronwall's example, take a domain of holomorphy, say $\mathcal{D}$, such that there exists an analytic hypersurface S for which there is no holomorphic function on $\mathcal{D}$ whose zero locus coincides with S, but one has a holomorphic function f on a neighbourhood of S satisfying $S = f^{-1}(0)$. In this situation, there exists a meromorphic function on $\mathcal{D}$ whose principal part is $1/f$. It is expected then, that this meromorphic function cannot be approximated by a sequence of rational functions uniformly on compact subsets of $\mathcal{D} \setminus S$, because if it were not the case the denominators of those rational functions would define analytic hypersurfaces that 'converge' to S.

To clarify this situation, Oka constructed another simple example of a domain of holomorphy on which Cousin's second problem is not necessarily solvable. This shows in particular that a domain of holomorphy is not necessarily rationally convex.

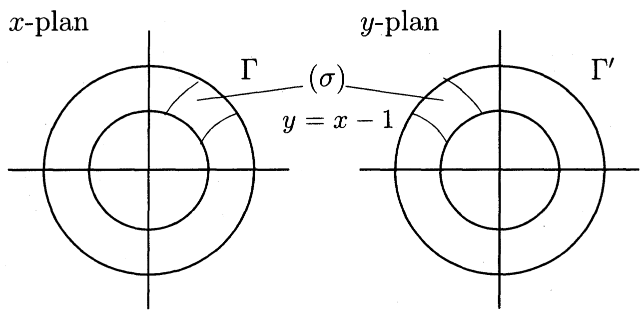

Fig. 3

III-Example, in Japanese, was written only to describe this example. The published Paper IV amounts to it.

There he classifies the domains into four species:

polycylinders, rationally convex domains,
domains of holomorphy, pseudoconvex domains.

According to this classification, things are stated more in order;

1. Paper I established a method of reducing rationally convex domains to polycylinders.
2. Paper II established a method of reducing domains of holomorphy to rationally convex ones.

From this viewpoint, it is clearly seen that the fruit of Oka's researches was establishment of a method of reducing the domains of arbitrary shape to standard ones as in the case of Riemann's mapping theorem.

3. It must be noted here that, in advance of IV, Cousin's second problem had been solved in Paper III. Oka had an opinion that Cousin's second problem itself should have been worried about much later if the above mentioned classification results had nothing to do with it. The reason is that it was not directly related to the principal questions for Oka. Nevertheless, solving Cousin's second problem after the first one may well be regarded as a quite natural procedure.

4. Once it was known that the domains of holomorphy are not necessarily rationally convex, the condition for the validity of Weil's integral formula was to be examined next. It was done as follows, to put it concisely.

Let $\mathcal{D}$ be a domain in the space of two variables x, y, and let $X_1, X_2, \ldots, X_N$ be N holomorphic functions on $\mathcal{D}$ such that the (closed) domain

$$\Delta \quad : \quad |X_i(x,y)| \leq 1 \quad (i = 1, 2, \ldots, N)$$

is a compact subset of $\mathcal{D}$. Then, letting S_i be the set $|X_i| = 1$ on the boundary of $\mathcal{D}$, we put $\sigma_{ij} = S_i \cap S_j$.

Suppose that one can associate, to each $X_i(x,y)$, two holomorphic functions $P_i(x, y; x_0, y_0)$, $Q_i(x, y; x_0, y_0)$ in $(x, y) \in \mathcal{D}$ and $(x_0, y_0) \in \mathcal{D}$, in such a way that

$$\text{(W)} \qquad X_i(x,y) - X_i(x_0, y_0) = (x - x_0)P_i + (y - y_0)Q_i$$

holds true.

Then, if we put

$$K_{ij}(x, y, x_0, y_0) = \frac{(P_i Q_j - P_j Q_i)}{[X_i(x,y) - X_i(x_0,y_0)][X_j(x,y) - X_j(x_0,y_0)]}$$

any holomorphic function $f(x, y)$ has an integral representation

$$f(x_0, y_0) = \frac{-1}{4\pi^2} \sum_{(i,j)} \int_{\sigma_{ij}} K_{ij}(x, y, x_0, y_0) f(x, y)\, dx dy,$$

which is the celebrated Weil's integral formula.

At that time, however, the condition (W) was known to hold only when $X_i(x, y)$ are rational functions. Concerning this question, Oka made a breakthrough by first discovering a fact that

≪Every holomorphic function on a domain of holomorphy can be uniformly approximated on the compact subsets by the branches of algebraic functions≫

and deduced from it that Weil's integral formula holds true on the domains of holomorphy without any essential changes. This is Paper V. Later it was simplified by H. Hefer and was generalized further after the introduction of the theory of systems of ideals of undetermined domains. They are stuffs that arose subsequently after the establishment of the lifting principle.

He has told me that these things came out very smoothly after the discovery of the lifting principle. In fact, around October of that year, the manuscript of Paper I (written in Japanese) was written up almost in the final form. As for the solvability of Cousin's second problem, the so called Oka principle, the discovery, belongs to later periods, to my speculation.

§5. Hartogs inverse problem

The inverse problem of Hartogs was a really difficult question even after such a preparation, so that Oka had to wait until some point around 1940 to get a solution. This problem was pursued to the following point:

Let $\mathcal{D}$ be a bounded domain in the space of two complex variables x, y, and let $\mathcal{D}_1$, $\mathcal{D}_2$ be respectively the subsets of $\mathcal{D}$ defined by Im $x > a_1$ and Im $x < a_2$. We assume here that $\mathcal{D}_1$ and $\mathcal{D}_2$ are both holomorphically convex.

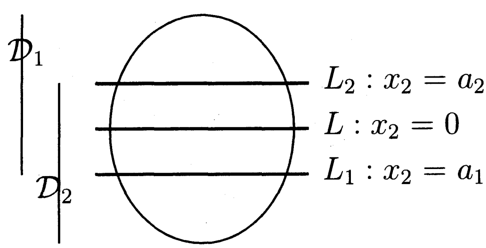

Fig. 4

In this geometric setting, the point is

≪to solve Cousin's first problem on $\mathcal{D}$≫.

This was settled in Paper VI by solving an integral equation including the Weil integral. He told me that when he was consulting Goursat's book, Cours d'Analyse, Volume III, certain integral equation caught his eyes, which finally led him to the solution of the problem.

Paper VI is written by restricting the situation to the space of two complex variables. This is probably because Oka planned to replace the Weil integral by the Cauchy integral, by using the lifting principle as before.

Oka wanted to do this not just because it is troublesome to apply Weil's formula in higher dimensions. He intended, from the beginning, to generalize the principal questions to the many-sheeted domains. This plan was completely realized after two years in 1943.

Concluding remarks

Oka called his work through Paper VI 'examination of the shore'. By this word one may understand that he had encouraged himself further to cross the river. As you know, a splendid bridge was built later, after the invention of the theory of systems of ideals of undetermined domains.

The lifting principle, together with its generalization remained Oka's lifelong research subject, whereas I could just describe how its most naïve form was created. However I must be contented with it if I succeeded in describing how a mathematical part of nature had grown in Oka's mind.

Thank you very much for your attention.

(Originally written in Japanese and translated by T. Ohsawa.)

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