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# On Spectra of Noises Associated with Harris Flows

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***Dedicated to Professor Kiyosi Itô on his 88th birthday***

**Abstract.**

A Harris flow is a stochastic flow on the real line given by SDE (2.1) below. We study the *noise* generated by Harris flows, particularly *spectra* of the noise. Our aim is to understand what lies beyond the finite order terms in the chaos expansion (the Wiener-Itô expansion) for *nonstrong* solutions of SDE (2.1).

## §1. Definitions and main results

The notion of noises in continuous time (i.e., the case of time $t \in \mathbf{R}$) has been introduced by Tsirelson (cf. [T 1], [T 2], [T 5]):

**Definition 1.1.** *A* noise $\mathbf{N} = [\{\mathcal{F}_{s,t}\}_{s\leq t}, \{T_h\}_{h\in\mathbf{R}}]$ *is a two parameter family of sub $\sigma$-fields $\mathcal{F}_{s,t}$, $s \leq t$, of events defined on a probability space $(\Omega, \mathcal{F}, P)$ which is stationary in time and possesses the following property:*

$$(1.1) \qquad \mathcal{F}_{s,u} = \mathcal{F}_{s,t} \otimes \mathcal{F}_{t,u}, \quad s \leq t \leq u,$$

*that is, $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent and generate $\mathcal{F}_{s,u}$, for every $s \leq t \leq u$. By the stationarity in time, we mean the existence of a measurable flow $\{T_h\}$, i.e., a measurable one-parameter group of automorphisms, on $(\Omega, \mathcal{F}_{-\infty,\infty} := \bigvee_{s\leq t} \mathcal{F}_{s,t})$, in which $\mathcal{F}_{s,t}$ is sent to $\mathcal{F}_{s+h,t+h}$ by $T_h$.*

In this article, it is always assumed that the probability space is complete and separable and that a sub $\sigma$-field contains all $P$-null sets.

In the discrete time case (i.e., the case of time $n \in \mathbf{Z}$), a noise can be defined similarly but it is essentially equivalent to giving an i.i.d. random sequence. In the continuous time case, noises generated by increments of a Wiener process (of finite or countably infinite dimension), a stationary Poisson point process, or an independent pair of them, are typical

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examples which we call *white*, *linearizable* or *classical noises*. There are many non-classical noises, however. Every noise $\mathbf{N} = \{\mathcal{F}_{s,t}\}$ contains a unique maximal (i.e., the largest) classical subnoise which is denoted by $\mathbf{N}^{lin} = \{\mathcal{F}^{lin}_{s,t}\}$.

A *Harris flow* (as will be defined precisely in Def.1.3 below) is a stochastic flow on the real line $\mathbf{R}$ determined uniquely by giving a real positive definite function $b(x)$ such that $b(0) = 1$, (cf. [H]). Note that $b(x) = b(-x)$. We assume that either $b(x) = \mathbf{1}_{\{0\}}(x)$ or $b(x)$ is continuous, $C^2$ on $\mathbf{R} \setminus \{0\}$ and *strictly positive-definite* in the sense that the matrix $\{b(x_i - x_j)\}$ is strictly positive-definite for any choice of finite different points $\{x_i\}$ in $\mathbf{R}$. The Harris flow in the discontinuous case of $b(x) = \mathbf{1}_{\{0\}}(x)$ is known as the *Arratia flow* ([A]).

Here is a formal definition of stochastic flows on the real line: Let $\mathcal{T}$ be the set of all non-decreasing right-continuous functions $\varphi : x \in \mathbf{R} \mapsto \varphi(x) \in \mathbf{R}$ with the metric defined by $\rho(\varphi, \psi) = \sum_{n=1}^{\infty} 2^{-n} (\rho_n(\varphi, \psi) \wedge 1)$ where

$$\rho_n(\varphi, \psi) = \inf\{ \varepsilon > 0 \mid \varphi(x - \varepsilon) - \varepsilon \le \psi(x) \le \varphi(x + \varepsilon) + \varepsilon \quad \text{for all } x \in [-n, n]\}.$$

Then $\mathcal{T}$ is a Polish space: The composite $(\varphi, \psi) \in \mathcal{T} \times \mathcal{T} \mapsto \psi \circ \varphi \in \mathcal{T}$, defined by $\psi \circ \varphi(x) = \psi(\varphi(x))$, and the evaluation map $\mathcal{T} \times \mathbf{R} \ni (\varphi, x) \mapsto \varphi(x) \in \mathbf{R}$ are all Borel measurable even though they are generally not continuous.

**Definition 1.2.** *By a* stochastic flow *on* $\mathbf{R}$, *we mean a family* $\mathbf{X} = \{X_{s,t};\ s \le t\}$ *of* $\mathcal{T}$*-valued random variables* $X_{s,t}$ *having the following properties:*

(1) *(Flow property),* $X_{s,u} = X_{t,u} \circ X_{s,t}$ *and* $X_{t,t} = \mathrm{id}$, *a.s. for every* $s \le t \le u$,

(2) *(Independence property), for any sequence* $t_0 \le t_1 \le \cdots \le t_n$, $\mathcal{T}$*-valued random variables* $X_{t_{k-1},t_k}$, $k = 1, \cdots, n$, *are independent,*

(3) *(Stationarity), for any* $h > 0$, $X_{s,t} \stackrel{d}{=} X_{s+h,t+h}$,

(4) *(Stochastic continuity),* $X_{0,h} \to \mathrm{id}$ *in probability as* $h \downarrow 0$.

Given a stochastic flow $\mathbf{X} = \{X_{s,t}\}$, it generates a noise $\mathbf{N}^X = [\{\mathcal{F}^X_{s,t}\}, \{T_h\}]$ by letting $\mathcal{F}^X_{s,t}$ to be the $\sigma$-field generated by $\mathcal{T}$-valued random variables $X_{u,v}$, $s \le u \le v \le t$, and $\{T_h\}$ to be a unique one-parameter family of automorphisms on $(\Omega, \mathcal{F}^X_{-\infty,\infty})$ such that $(T_h)_*(X_{u,v}(x)) = X_{u+h,v+h}(x)$, $u \le v$, $x \in \mathbf{R}$.

Now we give a formal definition of Harris flows. Generally, for a given filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \ge 0}$, we denote by $\mathcal{M}_2(\mathbf{F})$ the space of all locally square-integrable $\mathbf{F}$-martingales $M = (M_t)_{t\geq 0}$ with $M_0 = 0$ and by $\mathcal{M}_2^c(\mathbf{F})$ the subspace formed of all continuous elements in $\mathcal{M}_2(\mathbf{F})$.

**Definition 1.3.** *The* Harris flow $\mathbf{X} = \{X_{s,t}\}$ associated with the correlation function $b(x)$ *is a stochastic flow on* $\mathbf{R}$ *such that, for every* $x \in \mathbf{R}$, *if we define the process* $M(x) = (M_t(x))_{t\geq 0}$ *by setting* $M_t(x) = X_{0,t}(x) - x$ *and the filtration* $\mathbf{F}^X = \{\mathcal{F}_t^X\}$ *by setting* $\mathcal{F}_t^X = \mathcal{F}_{0,t}^X$, *then* $M(x) \in \mathcal{M}_2^c(\mathbf{F}^X)$ *and, for every* $x, y \in \mathbf{R}$, *we have*

$$(1.2) \qquad \langle M(x), M(y)\rangle_t = \int_0^t b\left(X_{0,s}(x) - X_{0,s}(y)\right) ds.$$

The law of a Harris flow is uniquely determined under our assumption on functions $b(x)$. The existence of Harris flows has been established in [H] (cf. also [LR 1]). A Harris flow is equivalently given by a stochastic differential equation (SDE) (2.1) in Section 2.

Let $\mathbf{X} = \{X_{s,t}\}$ be a Harris flow associated with the function $b(x)$ and $\mathbf{N}^X$ be the noise generated by it. *Suppose that* $b(x)$ *is continuous.* Then we can construct a centered Gaussian system $\mathbf{W} = \{W(t,x); t \in \mathbf{R}, x \in \mathbf{R}\}$ contained in $L_2(\mathcal{F}_{-\infty,\infty}^X)$ such that $(T_h)_*[W(t,x)-W(s,x)] = W(t+h,x) - W(s+h,x)$, $s \leq t$, $x \in \mathbf{R}$ and, if we set $w_t(x) = W(t,x) - W(0,x)$, then $w(x) = (w_t(x))_{t\geq 0} \in \mathcal{M}_2^c(\mathbf{F}^X)$ and, for every $x, y \in \mathbf{R}$, we have $\langle w(x), w(y)\rangle_t = tb(x-y)$. Indeed, $W(t,x) - W(s,x)$ is the $L_2$-limit of $M_\Delta^x(s,t)$ as $|\Delta| \to 0$. Here, for a sequence of times $\Delta : s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ and $x \in \mathbf{R}$, $M_\Delta^x(s,t) = \sum_{k=1}^n (X_{t_{k-1},t_k}(x) - x)$ and $|\Delta| = \max_k |t_k - t_{k-1}|$. $\mathbf{W}$ defines a Gaussian white noise $\mathbf{N}^W = \left[\{\mathcal{F}_{s,t}^W\}, \{T_h\}\right]$ where $\mathcal{F}_{s,t}^W = \sigma[W(v,x) - W(u,x); s \leq u \leq v \leq t, x \in \mathbf{R}]$. It is obvious that $\mathbf{N}^W$ is a subnoise of $\mathbf{N}^X$.

**Theorem 1.1.** *Suppose that the function* $b(x)$ *is continuous. Then, it holds that* $[\mathbf{N}^X]^{lin} = \mathbf{N}^W$. *Furthermore,* $\mathbf{N}^X = \mathbf{N}^W$ *holds, that is, the noise* $\mathbf{N}^X$ *generated by the Harris flow* $\mathbf{X}$ *is classical, if and only if*

$$(1.3) \qquad \int_{0+}^1 (1 - b(x))^{-1} dx = \infty.$$

Hence, the noise $\mathbf{N}^X$ is nonclassical if and only if

$$(1.4) \qquad \int_{0+}^1 (1 - b(x))^{-1} dx < \infty.$$

In the case of the Arratia flow, it generates a nonclassical noise: Tsirelson [T 3] (cf. also [LR 2]) showed that this noise is *black* in the sense that $(\mathcal{F}_{s,t}^X)^{lin} = \{\emptyset, \Omega\}$ for every $s \leq t$.

Tsirelson ([T 2], [T 5]) introduced the notion of *spectral measures* for noises which is an invariant under the isomorphism of noises and which can measure the degree of non-linearity (or sensitivity in the discrete-time approximation) of noises. Let $\mathcal{C}$ be the space formed of all compact sets in $\mathbf{R}$ endowed with the Hausdorff distance and $\mathcal{C}^f$ be its subclass formed of all finite sets: $\mathcal{C}^f = \{ S \in \mathcal{C} \mid |S| < \infty \}$. Here, $|S|$ denotes the number of elements in $S$.

**Definition 1.4.** *Let $\mathbf{N} = [\{\mathcal{F}_{s,t}\}, \{T_h\}]$ be a noise. To every $\Phi \in L_2(\mathcal{F}_{-\infty,\infty})$, there corresponds a unique finite Borel measure $\mu_\Phi$ on $\mathcal{C}$ such that*

$$(1.5) \qquad \mu_\Phi(\ \{S \in \mathcal{C} \mid S \subset J\}\ ) = E\left[E(\Phi|\mathcal{F}(J))^2\right]$$

*for every elementary set $J \subset \mathbf{R}$. Here, by an elementary set $J$, we mean a finite union $J = \bigcup_k [t_k, t_{k+1}]$ of non-overlapping intervals and we set $\mathcal{F}(J) = \bigvee_k \mathcal{F}_{t_k,t_{k+1}}$. $\mu_\Phi$ is called the* spectral measure *of the noise $\mathbf{N}$ associated with $\Phi \in L_2(\mathcal{F}_{-\infty,\infty})$.*

When $\Phi \in L_2(\mathcal{F}_{s,t})$, we have $\mu_\Phi(\mathcal{C} \setminus \mathcal{C}_{[s,t]}) = 0$ where $\mathcal{C}_{[s,t]} = \{S \in \mathcal{C} \mid S \subset [s,t]\}$, so that $\mu_\Phi$ is a measure on $\mathcal{C}_{[s,t]}$. The following is an important characterization of classical noises due to Tsirelson: *a noise is classical if and only if $\mu_\Phi(\mathcal{C} \setminus \mathcal{C}^f) = 0$ for every $\Phi \in L_2(\mathcal{F}_{-\infty,\infty})$.*

Set $L_2^{us}(\mathcal{F}_{s,t}) = \{ \Phi \in L_2(\mathcal{F}_{s,t}) \mid ||\Phi||_2 = 1 \}$; the unit sphere in $L_2(\mathcal{F}_{s,t})$. If $\Phi \in L_2^{us}(\mathcal{F}_{-\infty,\infty})$, then $\mu_\Phi$ is a Borel probability on $\mathcal{C}$ so that we can speak of a $\mathcal{C}$-value random variable with the distribution $\mu_\Phi$. We denote it by $S_\Phi$ and call it the *spectral set* of the noise associated with $\Phi$.

We wish to describe the spectral set $S_\Phi$ for the noise $\mathbf{N}^X$ generated by a Harris flow $\mathbf{X}$ when $\Phi = X_{0,1}(0) \in L_2^{us}(\mathcal{F}_{0,1}^X)$. The random set $S_\Phi$ in this case is denoted by $S_X$. We would also obtain some information on $S_\Phi$ for general $\Phi$. We consider naturally the case when the noise is nonclassical so that we assume (1.4). Furthermore, we assume that

$$(1.6) \quad b(x) \quad \textit{is non-increasing in } (0,\infty) \textit{ and satisfies } \lim_{x\to\infty} b(x) = 0.$$

Functions $b(x) = \exp(-c|x|^\alpha)$ for $c > 0$ and $0 < \alpha < 1$ are typical examples. Also, $b(x) = \mathbf{1}_{\{0\}}(x)$ (the case of the Arratia flow) is another typical example.

For $S \in \mathcal{C}$, let $S^{acc}$ be the the set of all accumulation points of $S$, so that $S^{acc} \neq \emptyset$ if and only if $S \notin \mathcal{C}^f$.

**Theorem 1.2.** *Let $\mathbf{X}$ be the Harris flow associated with the function $b(x)$ which satisfies (1.4) and (1.6) and let $S_X$ be the spectral set $S_\Phi$ of* *the noise* $\mathbf{N}^X$ *for* $\Phi = X_{0,1}(0)$. *Then the random set* $S_X^{acc}$ *has the same law as the random set* $\widetilde{S}$ *in* $[0,1]$ *defined by*

$$\widetilde{S} = \{\, t \mid 0 \le t \le \tau,\ \widehat{\xi}^+(\tau - t) = 0 \,\} \tag{1.7}$$

*where* $\widehat{\xi}^+ = \{\widehat{\xi}^+(t)\}_{t\ge 0}$ *is the reflecting diffusion process on* $[0,\infty)$ *with the generator*

$$\widehat{L} = \frac{d}{dx}(1 - b(x))\frac{d}{dx} \tag{1.8}$$

*and the initial distribution* $\mu(dx) := -db(x)$. *Here,* $\tau$ *is a* $[0,1]$*-valued and uniformly distributed random variable independent of* $\widehat{\xi}^+$.

In particular, we have

$$P(S_X^{acc} \ne \emptyset) = P(|S_X| = \infty) = P(\widetilde{S} \ne \emptyset) = P\left\{\exists t \in [0,\tau]; \widehat{\xi}^+(t) = 0\right\}$$

and this probability is also equal to $E\left[\int_0^1 (1 - b(\xi^+(t)))dt\right]$ where $\xi^+ = \{\xi^+(t)\}_{t\ge 0}$ is the reflecting diffusion process on $[0,\infty)$ with the generator

$$L = (1 - b(x))\frac{d^2}{dx^2} \tag{1.9}$$

which starts at 0. Still another expression of this probability is given by the expectation $\frac{1}{2}E\left[A^{-1}(1)\right]$, where $A(t)$ is an additive functional of the one-dimensional Wiener process $\beta(t)$ with $\beta(0) = 0$, defined by

$$A(t) = \frac{1}{2}\int_0^t (1 - b(\beta(s)))^{-1} ds, \tag{1.10}$$

and $t \to A^{-1}(t)$ is the inverse function of $t \to A(t)$.

In the case of the Arratia flow, $S_X^{acc} = S_X$ and it is a perfect set, a.s.. It is described as a zero points set of a (double speed) reflecting Brownian motion starting at 0 as in the theorem. This recovers a result of Tsirelson ([T 4]) who obtained it by an approximation by coalescing random walks.

In the following, we consider the class of Harris flows associated with the correlation functions $b(x)$ which satisfy (1.4), (1.6) and, for some $0 \le \alpha < 1$,

$$1 - b(x) \asymp |x|^\alpha \quad \text{as} \quad x \to 0. \tag{1.11}$$

Again, functions $b(x) = \exp(-c|x|^\alpha)$ for $c > 0$ and $0 < \alpha < 1$ are typical examples. Note also that the function $b(x) = \mathbf{1}_{\{0\}}(x)$ (the case of the Arratia flow) is a typical example of the case when $\alpha = 0$. From Theorem 1.2, we can obtain the following: Denoting by $\dim(S)$ the Hausdorff dimension of a subset $S$ in $\mathbf{R}$,

**Corollary 1.1.** $\dim(S_X^{acc}) = \frac{1-\alpha}{2-\alpha}$ a.s., *under the condition that it is not empty.*

**Theorem 1.3.** *Let* $\gamma = \inf\{\ \beta \mid \dim(S_\Phi) \le \beta,$ a.s. *for any* $\Phi \in L_2^{us}(\mathcal{F}_{-\infty,\infty}^X)\ \}$. *Then*

$$\gamma = \frac{1-\alpha}{2-\alpha}.$$

The proof of these theorems will be given in the subsequent sections by appealing to two main tools: *joinings of Harris flows* and certain *duality relations* between the reflecting (absorbing) $L$-diffusion and the absorbing (resp. reflecting) $\widehat{L}$-diffusion.

## §2. The joining of Harris flows: The proof of Th. 1.1.

Suppose that the correlation function $b(x)$ of a Harris flow $\mathbf{X}$ is continuous. Let $H(\subset \mathbf{C}_b(\mathbf{R} \to \mathbf{R}))$ be the (real) reproducing kernel Hilbert space associated with $b(x)$ so that, defining $f_x \in H$ by $f_x(y) = b(y-x)$, linear combinations $\sum c_i f_{x_i}$ are dense in $H$ and $(f_x, f_y)_H = b(x-y)$. The Gaussian system $\mathbf{W}$ introduced in Section 1 can be given equivalently by a Gaussian system $\{W(t,f); t \in \mathbf{R}, f \in H\}$ contained in $L^2(\mathcal{F}_{-\infty,\infty}^X)$ such that $(T_h)_*[W(t,f) - W(s,f)] = W(t+h,f) - W(s+h,f)$, $s \le t$, $f \in H$ and, if we set $w_t(f) = W(t,f) - W(0,f)$, then $w(f) = (w_t(f))_{t\ge 0} \in \mathcal{M}_2^c(\mathbf{F}^X)$ and, for every $f, g \in H$, we have $\langle w(f), w(g)\rangle_t = t(f,g)_H$. Indeed, we set $W(t,f) = \sum_i c_i W(t, x_i)$ when $f = \sum c_i f_{x_i}$ and extend this to general $f \in H$ by routine arguments.

We define an Itô-type stochastic integral $\int_0^t \psi_s \cdot W(ds, \varphi_s)$ for $\mathbf{F}^X$-predictable processes $\varphi$ and $\psi$ satisfying that $\int_0^t |\psi_s|^2 ds < \infty$, a.s., by

$$\int_0^t \psi_s \cdot W(ds, \varphi_s) = \sum_k \int_0^t \psi_s \cdot e_k(\varphi_s) db_k(s),$$

where $\{e_k\}$ is an orthonormal basis (ONB) in $H$ and $b_k(t) = W(t, e_k)$, so that $\{b_k(t)\}$ is an independent family of one-dimensional Wiener processes. As is easily seen, the definition is independent of a particular choice of ONB. Note that $\sum_k e_k(\varphi_s) e_k(\varphi'_s) = b(\varphi_s - \varphi'_s)$, so that, in particular, $\sum_k |e_k(\varphi_s)|^2 \equiv 1$. Now, (1.2) is equivalently given in the form of SDE for $X_t := X_{0,t}(x)$:

$$X_t = x + \int_0^t W(ds, X_s) = x + \sum_k \int_0^t e_k(X_s) db_k(s). \tag{2.1}$$

Since $\sum_k |e_k(x) - e_k(y)|^2 = 2(1 - b(x - y))$, the condition (1.3) implies the pathwise uniqueness of solutions for SDE (2.1) (cf. [IW], p.182). Hence, if the function $b$ satisfies the condition (1.3), then $X_t$ is a unique strong solution to SDE (2.1) so that $X_{0,t}(x)$ is $\mathcal{F}^W_{0,t}$-measurable for every $x$. By the stationarity, we see that $X_{s,t}(x)$ is $\mathcal{F}^W_{s,t}$-measurable for every $x$ and $s \leq t$. Therefore, $\mathbf{N}^X = \mathbf{N}^W$ holds. Thus, the *if part* of Th. 1.1 is proved.

To prove the *only if part*, we first remark the following martingale representation theorem for Harris flows.

**Proposition 2.1.** *Suppose the correlation function $b(x)$ of the Harris flow is continuous. Then, $M \in \mathcal{M}_2(\mathbf{F}^X)$ if and only if there exists a sequence $\varphi_k = (\varphi_k(t))$, $k = 1, 2, \dots$, of $\mathbf{F}^X$-predictable processes satisfying that $\sum_k \int_0^t \varphi_k^2(s)ds < \infty$, a.s., for each $t > 0$, and*

$$M(t) = \sum_k \int_0^t \varphi_k(s)db_k(s).$$

*In particular, it holds that $\mathcal{M}_2(\mathbf{F}^X) = \mathcal{M}_2^c(\mathbf{F}^X)$.*

*Proof.* Given distinct $x_1, x_2, \dots x_n \in \mathbf{R}$, any $\mathbf{R}^n$-valued process $(X_t^1, X_t^2, \dots X_t^n)$ of which each component $X_t^k$ solves the SDE (2.1) starting from $x_k$ and these components satisfy the coalescing property, has the same law as the $n$-point motion of the Harris flow $(X_{0,t}(x_1), X_{0,t}(x_2) \dots, X_{0,t}(x_n))$. From this uniqueness in law, it follows by the usual methods that any $M \in \mathcal{M}_2(\mathbf{F}^X)$ that is measurable with respect to this $n$-point motion is continuous and has the desired representation as a stochastic integral. The result can then be extended to an arbitrary $M \in \mathcal{M}_2(\mathbf{F}^X)$ using the fact that the set of representable martingales is closed in this space. ∎

From this proposition, we can easily deduce that $[\mathbf{N}^X]^{lin} = \mathbf{N}^W$, see also Lemma 6a5 of [T 5]. Indeed, if $\mathbf{N}^W$ is smaller than $[\mathbf{N}^X]^{lin}$, then there should exist some martingale in $\mathcal{M}_2(\mathbf{F}^X)$ which cannot be given by a sum of stochastic integrals by $b_k$. Hence, in order to prove the *only if part*, it is sufficient to show that *(1.4) implies that $\mathbf{N}^W$ is strictly smaller than $\mathbf{N}^X$*. For this, we introduce the following notion.

**Definition 2.1.** *By a* joining of a Harris flow, *we mean a pair $(\mathbf{X} = \{X_{s,t}\}, \mathbf{X}' = \{X'_{s,t}\})$ of copies of the Harris flow defined on a same probability space such that the joint process $\Xi = \{\Xi_{s,t} = (X_{s,t}, X'_{s,t}); s \leq t\}$ has the independence property (2) in Def.1.2. Given $0 \leq \rho \leq 1$, it is* *called a* $\rho$-joining *if it satisfies further the following:* $\mathbf{X}$ *and* $\mathbf{X}'$ *are stationarily correlated in the sense that the joint process* $\Xi$ *has the stationarity property (3) of Def.1.2 and, if filtrations* $\mathbf{F}^X = \{\mathcal{F}_t^X\}$, $\mathbf{F}^{X'} = \{\mathcal{F}_t^{X'}\}$ *and martingales* $M(x) = (M_t(x))$, $M'(x) = (M_t'(x))$ *are defined similarly as in Def.1.3 for* $\mathbf{X}$ *and* $\mathbf{X}'$, *respectively, then* $\mathbf{F}^X$ *and* $\mathbf{F}^{X'}$ *are jointly immersed, i.e.,* $\mathcal{M}_2(\mathbf{F}^X) \cup \mathcal{M}_2(\mathbf{F}^{X'}) \subset \mathcal{M}_2(\mathbf{F}^X \bigvee \mathbf{F}^{X'})$, *and, for every* $x, y \in \mathbf{R}$,

$$(2.2) \qquad \langle M(x), M'(y)\rangle_t = \int_0^t \rho \cdot b\left(X_{0,s}(x) - X'_{0,s}(y)\right) ds,$$

$b(x)$ *being the correlation function of the Harris flow.*

It is obvious that, for a $\rho$-joining, the corresponding Gaussian noises $\mathbf{W}$ and $\mathbf{W}'$ are jointly Gaussian and $\rho$-correlated.

**Lemma 2.1.** *For* $0 \le \rho < 1$, *a* $\rho$*-joining exists and is unique in law. If, in particular,* $\rho = 0$, *then it is a pair of independent copies.*

This lemma can be deduced from the fact that the following differential operator $\Lambda$ with variables $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $x' = (x_1', \cdots, x_m') \in \mathbf{R}^m$ is non degenerate at all such points $(x, x') \in \mathbf{R}^n \times \mathbf{R}^m$ as all coordinates in $x$ are different and also all coordinates in $x'$ are different:

$$\begin{aligned}\Lambda &= \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n b(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{2}\sum_{k=1}^m\sum_{l=1}^m b(x_k' - x_l')\frac{\partial^2}{\partial x_k' \partial x_l'} \\ &+ \rho\sum_{i=1}^n\sum_{k=1}^m b(x_i - x_k')\frac{\partial^2}{\partial x_i \partial x_k'}.\end{aligned}$$

Note that, for a $\rho$-joining $(\mathbf{X}, \mathbf{X}')$, the process

$$[0,\infty) \ni t \mapsto (X_{0,t}(x_1), \cdots, X_{0,t}(x_n), X'_{0,t}(x_1'), \cdots, X'_{0,t}(x_m'))$$

is a solution to the $\Lambda$-martingale problem.

We now assume (1.4) and prove that $\mathbf{N}^W$ is strictly smaller than $\mathbf{N}^X$. Take $\rho$-joinings $(\mathbf{X}^{(\rho)}, \mathbf{X}'^{(\rho)})$ for $\rho \in [0,1)$. By (2.2), the process $\xi^{(\rho)}(t) = X_{0,t}^{(\rho)}(0) - X_{0,t}'^{(\rho)}(0)$ is a Feller diffusion on $\mathbf{R}$ with the canonical scale $s(x) = x$ and the speed measure $m(dx) = (1 - \rho \cdot b(x))^{-1}dx$ which starts from the origin at time 0, (cf. [IM] for a general theory of Feller diffusions). As $\rho \nearrow 1$, the processes $\xi^{(\rho)}(t)$ converge to the Feller diffusion $\xi(t)$ with the canonical scale $s(x) = x$ and the speed measure $m(dx) = (1 - b(x))^{-1}dx$ which starts from the origin 0 at time 0. As is well-known, $\xi(t) = \beta(A^{-1}(t))$ for a one-dimensional Wiener process $\beta(t)$ and $A(t)$ is defined by (1.10). Then we have

$$\lim_{\rho \nearrow 1} E\left[|\xi^{(\rho)}(t)|^2\right] = E\left[|\xi(t)|^2\right] = \frac{1}{2}E[A^{-1}(t)] > 0$$

for $t > 0$. Suppose $\mathbf{N}^X \subset \mathbf{N}^W$ be true. Then $X_{0,t}^{(\rho)}(0) := \Phi \in L_2(\mathcal{F}_{0,t}^W)$ and $E\left[X_{0,t}'^{(\rho)}(0)|\mathbf{W}\right] = P_{-\log \rho}\Phi$ where $(P_s)_{s \geq 0}$ is the *Ornstein-Uhlenbeck semigroup* acing on $L_2(\mathcal{F}_{0,t}^W)$. Hence $E\left[|\xi^{(\rho)}(t)|^2\right] = 2\left(||\Phi||_2^2 - (\Phi, P_{-\log \rho}\Phi)_2\right)$. By the $L^2$-continuity of the Ornstein-Uhlenbeck semigroup, we have

$$\lim_{\rho \nearrow 1} E\left[|\xi^{(\rho)}(t)|^2\right] = \lim_{\rho \nearrow 1} 2\left(||\Phi||_2^2 - (\Phi, P_{-\log \rho}\Phi)_2\right) = 0.$$

Thus we have a contradiction and hence we cannot have $\mathbf{N}^X \subset \mathbf{N}^W$. This proves the *only if part* of Th.1.1 so that its proof now is completed.

In the following, we assume that (1.4) holds so that the noise generated by the Harris flow is nonclassical. In this case, 1-joinings are not unique. We specify two of them as the $1^+$*-joining* and the $1^-$*-joining.*

**Definition 2.2.** *The* $1^+$*-joining* $(\mathbf{X}, \mathbf{X}')$ *is the identity joining: i.e.,* $\mathbf{X} = \mathbf{X}'$. *The* $1^-$*-joining is the limit in law of the* $\rho$*-joinings* $(\mathbf{X}^{(\rho)}, \mathbf{X}'^{(\rho)})$ *as* $\rho \nearrow 1$ . *It is such that* $[0, \infty) \ni t \mapsto X_{0,t}(x) - X_{0,t}'(y)$, *for fixed* $x, y \in \mathbf{R}$, *is the Feller diffusion on* $\mathbf{R}$ *with the canonical scale* $s(x) = x$ *and the speed measure* $m(dx) = (1 - b(x))^{-1}dx$ *which starts at* $x - y$ *at time* 0.

For $\rho \in [0, 1)$, let $(\mathbf{X}, \mathbf{X}')$ be a $\rho$-joining with corresponding $\rho$-correlated Gaussian processes $\mathbf{W}$ and $\mathbf{W}'$. It is easy to see that the joint law $\Pi(d\mathcal{X}d\mathcal{X}'d\mathcal{W}d\mathcal{W}')$ of $(\mathbf{X}, \mathbf{X}', \mathbf{W}, \mathbf{W}')$ is given by

$$P(\mathbf{X} \in d\mathcal{X}|\mathbf{W} = \mathcal{W})P(\mathbf{X}' \in d\mathcal{X}'|\mathbf{W}' = \mathcal{W}')P(\mathbf{W} \in d\mathcal{W}, \mathbf{W}' \in d\mathcal{W}').$$

From this, we deduce that

$$\begin{aligned}
E[\Phi \cdot \pi_*(\Psi)] &= E\big[E[\Phi|\mathbf{W}] \cdot E[\pi_*(\Psi)|\mathbf{W}']\big] \\
&= E\big[E[\Phi|\mathbf{W}] \cdot E\big(E[\pi_*(\Psi)|\mathbf{W}']|\mathbf{W}\big)\big] \\
&= E\big[E[\Phi|\mathbf{W}] \cdot E\big[\pi_*(E(\Psi|\mathbf{W}))|\mathbf{W}\big]\big] \\
&= E\big[E[\Phi|\mathbf{W}] \cdot P_{-\log \rho}(E(\Psi|\mathbf{W}))\big]
\end{aligned}$$

whenever $\Phi, \Psi \in L_2(\mathcal{F}_{-\infty,\infty}^X)$. Here, $\pi_*$ is the unique isomorphism $\pi_* : L_0(\mathcal{F}_{-\infty,\infty}^X) \to L_0(\mathcal{F}_{-\infty,\infty}^{X'})$ such that $\pi_*(X_{s,t}(x)) = X_{s,t}'(x)$ for every $s, t$ and $x$, and $(P_s)$ is the Ornstein-Uhlenbeck semigroup acting on $L_2(\mathcal{F}^W_{-\infty,\infty})$. By the $L^2$-continuity of the Ornstein-Uhlenbeck semigroup, the above expectation converges to $E\big[E[\Phi|\mathbf{W}]\cdot E[\Psi|\mathbf{W}]\big]$ as $\rho \nearrow 1$. This proves existence of the $1^-$-joining as the limit of $\rho$-joinings. Moreover for a $1^-$-joining $(\mathbf{X}, \mathbf{X}')$ the corresponding Gaussian systems $\mathbf{W}$ and $\mathbf{W}'$ are equal and $\mathbf{X}$ and $\mathbf{X}'$ are conditionally independent given this common Gaussian process.

**Remark 2.1.** *For the Arratia flow, its $\rho$-joining for $\rho \in [0,1)$ is independent of $\rho$ and coincides with* 0*-joining, that is, a pair of independent copies of the Arratia flow. Hence, its* $1^-$*-joining is also a pair of independent copies of the Arratia flow.*

Let $F = \bigcup_{k=1}^n [t_{2k-2}, t_{2k-1}]$ be an elementary set in $\mathbf{R}$ defined for a sequence $t_0 < t_1 < \cdots < t_{2n-2} < t_{2n-1}$ of times. We would introduce the notion of $(\rho, F)$*-joining* $(\mathbf{X}, \mathbf{X}')$ of the Harris flow when $\rho \in [0,1)$, which is roughly the $\rho$-joining on $F$ and the identity joining outside $F$. To be more precise, set $t_{-1} = -\infty$ and $t_{2n} = \infty$ by convention. Take a $\rho$-joining $(\mathbf{Y}, \mathbf{Y}')$ and a $1^+$-joining $(\mathbf{Z}, \mathbf{Z}')$ which are mutually independent. Define $\mathbf{X} = [\{X_{s,t}\}_{s \le t}]$ as follows: First, set $X_{s,t} = Y_{s,t}$ if $t_{2k-2} \le s \le t \le t_{2k-1}$, $k = 1, \cdots, n$ and $X_{s,t} = Z_{s,t}$ if $t_{2k-1} \le s \le t \le t_{2k}$, $k = 0, \cdots, n$. Then, define $X_{s,t}$ for general $s \le t$, by

$$X_{s,t} = X_{t_l,t} \circ X_{t_{l-1},t_l} \circ \cdots \circ X_{t_k,t_{k+1}} \circ X_{s,t_k}$$

when $t_{k-1} < s \le t_k \le t_l \le t < t_{l+1}$, $0 \le k \le l \le 2n-1$. Define $\mathbf{X}' = [\{X'_{s,t}\}_{s \le t}]$ similarly from $\mathbf{Y}'$ and $\mathbf{Z}'$. Then $(\mathbf{X}, \mathbf{X}')$ defines a joining of the Harris flow in which, however, $\mathbf{X}$ and $\mathbf{X}'$ are not stationarily correlated.

**Definition 2.3.** *The pair $(\mathbf{X}, \mathbf{X}')$ defined above is called the $(\rho, F)$-*joining *of the Harris flow.*

Next, take mutually independent $1^-$-joining $(\mathbf{Y}, \mathbf{Y}')$ and $1^+$-joining $(\mathbf{Z}, \mathbf{Z}')$ and construct the pair $(\mathbf{X}, \mathbf{X}')$ in the same way.

**Definition 2.4.** *The pair $(\mathbf{X}, \mathbf{X}')$ defined above is called the $(1^-, F)$-*joining *of the Harris flow*

We turn now to the notion of the spectral measure $\mu_\Phi$ associated with some $\Phi \in L_2(\mathcal{F}^X_{-\infty,\infty})$ as defined in Def.1.4. This notion is intimately related to chaos expansions. The spectral measure of a random variable $\Phi \in L_2(\mathcal{F}^W_{-\infty,\infty})$, measurable with respect to $\mathbf{W}$, can be expressed by expanding $\Phi$ as a sum of multiple Wiener-Itô integrals with respect to the Brownian motions $b_k$. To be more precise, $\Phi = \sum_{m=1}^\infty I_m$

where $I_0$ is a constant and $I_m$, for $m = 1, 2. \cdots$, is given by an iterated Itô stochastic integral

$$I_m = \sum_{(k_1,\cdots,k_m)} \int \cdots \int_{-\infty<t_m<\cdots<t_1<\infty} f_\Phi^{(k_1,\cdots,k_m)}(t_1,\cdots \\ \cdots, t_m) db_{k_m}(t_m) \cdots db_{k_1}(t_1).$$

$\mu_\Phi$ is supported on $\mathcal{C}^f = \{S \in \mathcal{C} : |S| < \infty\}$ and

$$\mu_\Phi(\mathcal{C}^f) = E(\Phi^2) = \sum_{m=0}^{\infty} E(|I_m|^2)$$

$$= \sum_{m=0}^{\infty} \sum_{(k_1,\cdots,k_m)} \int \cdots \int_{-\infty<t_m<\cdots<t_1<\infty} |f_\Phi^{(k_1,\cdots,k_m)}(t_1,\cdots \\ \cdots, t_m)|^2 dt_m \cdots dt_1 < \infty.$$

The restriction of $\mu_\Phi$ to $\{S \in \mathcal{C} : |S| = m\}$ is determined (denoting $S = \{t_m, \cdots, t_1\}$, $-\infty < t_m < \cdots < t_1 < \infty$) by

$$\mu_\Phi(dS; |S| = m) = |f_\Phi^{(k_1,\cdots,k_m)}(t_1, \cdots, t_m)|^2 dt_m \cdots dt_1.$$

In particular, $\mu_\Phi(|S| = m) = E(|I_m|^2)$.

For a general $\Phi \in L_2(\mathcal{F}^X_{-\infty,\infty})$, the chaos expansion of $E[\Phi|\mathbf{W}]$ given by $E[\Phi|\mathbf{W}] = \sum_{m=0}^{\infty} I_m$, yields in the same fashion the restriction of $\mu_\Phi$ to $\mathcal{C}^f$ and in particular

$$E\left[E[\Phi|\mathbf{W}]^2\right] = \mu_\Phi(\mathcal{C}^f).$$

If $(\mathbf{X}, \mathbf{X}')$ is a $\rho$-joining for $\rho \in [0, 1)$ and $\Phi' = \pi_*(\Phi)$ as above, we have

$$E\big(E[\Phi'|\mathbf{W}']|\mathbf{W}\big) = P_{-log\rho}\big(E[\Phi|\mathbf{W}]\big) = \sum_{m=0}^{\infty} \rho^m I_m.$$

As was remarked above, the relation $E(\Phi\Phi') = E\big(E[\Phi|\mathbf{W}]E[\Phi'|\mathbf{W}']\big)$ holds. Hence,

$$E\big(\Phi\Phi'\big) = \sum_{m=0}^{\infty} \rho^m E(|I_m|^2) = \int_{\mathcal{C}} \rho^{|S|} \mu_\Phi(dS). \tag{2.3}$$

In the same way, we deduce for a $1^-$-joining $(\mathbf{X}, \mathbf{X}')$,

$$E\big(\Phi\Phi'\big) = \mu_\Phi(\mathcal{C}^f). \tag{2.4}$$

**Example 2.1.** *Consider the case $\Phi = g(X_{0,1}(x))$ for a bounded continuous function $g$ on $\mathbf{R}$. Note that $E(\Phi^2) = \int_{\mathbf{R}} p(1, x-y)g(y)^2 dy$ where*

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{x^2}{2t}\}, \quad t > 0, \ x \in \mathbf{R}.$$

*The chaos expansion of $E[\Phi|\mathbf{W}]$ was obtained explicitly by Veretennikov and Krylov (cf. [VK]): By setting*

$$T_t f(x) = \int_{\mathbf{R}} p(t, x-y) f(y) dy \quad \text{and} \quad Q_t^k f(x) = e_k(x) \frac{\partial}{\partial x} T_t f(x),$$

*we have*

$$g(X_{0,1}(x)) = \sum_{m=0}^{n} I_m + R_n, \quad I_0 = T_1 g(x) = E[\Phi],$$

*where $I_m$, $m = 1, \ldots, n$, and $R_n$ are given by the following iterated Itô stochastic integrals:*

$$\begin{aligned} I_m &= \sum_{(k_1, k_2, \cdots, k_m)} \int \cdots \int_{0<t_m<t_{m-1}<\cdots<t_2<t_1<1} \Big[ T_{t_m} Q^{k_m}_{t_{m-1}-t_m} \cdots \\ &\cdots Q^{k_2}_{t_1-t_2} Q^{k_1}_{1-t_1} g(x) \Big] db_{k_m}(t_m) db_{k_{m-1}}(t_{m-1}) \cdots db_{k_2}(t_2) db_{k_1}(t_1), \end{aligned}$$

$$\begin{aligned} R_n &= \sum_{(k_1, k_2, \cdots, k_n, k_{n+1})} \int \cdots \\ &\cdots \int_{0<t_{n+1}<t_n<\cdots<t_2<t_1<1} \Big[ Q^{k_{n+1}}_{t_n - t_{n+1}} Q^{k_n}_{t_{n-1}-t_n} \cdots \\ &\cdots Q^{k_2}_{t_1-t_2} Q^{k_1}_{1-t_1} g(X_{0,t_{n+1}}(x)) \Big] db_{k_{n+1}}(t_{n+1}) db_{k_n}(t_n) \cdots \\ &\qquad \cdots db_{k_2}(t_2) db_{k_1}(t_1). \end{aligned}$$

*From this, we obtain that*

$$E[\Phi|\mathbf{W}] = \sum_{m=0}^{\infty} I_m.$$

The following is a key lemma for the proof of Theorem 1.2 which records various generalizations of the identities (2.3) and (2.4). As above, we denote by $S_X$ the spectral set $S_\Phi$ when $\Phi = X_{0,1}(0)$ which is a $\mathcal{C}_{[0,1]}$-valued random variable.

**Lemma 2.2.** *(i) If* $(\mathbf{X}, \mathbf{X}')$ *is a* $(\rho, F)$*-joining of the Harris flow for* $\rho \in [0, 1)$*, then,*

$$E\left[\rho^{|S_X \cap F|}\right] = E\left[X_{0,1}(0)X'_{0,1}(0)\right], \tag{2.5}$$

*equivalently,*

$$E\left[1 - \rho^{|S_X \cap F|}\right] = \frac{1}{2}E\left[|X_{0,1}(0) - X'_{0,1}(0)|^2\right]. \tag{2.6}$$

*(ii) If* $(\mathbf{X}, \mathbf{X}')$ *is a* $(1^-, F)$*-joining of the Harris flow, then,*

$$P(\ |S_X \cap F| < \infty\ ) = E\left[X_{0,1}(0)X'_{0,1}(0)\right], \tag{2.7}$$

*equivalently,*

$$P(\ |S_X \cap F| = \infty\ ) = \frac{1}{2}E\left[|X_{0,1}(0) - X'_{0,1}(0)|^2\right]. \tag{2.8}$$

*(iii) More generally, let* $(\mathbf{X}, \mathbf{X}')$ *be a* $(\rho, F)$*-joining for* $0 \le \rho < 1$ *(a* $(1^-, F)$*-joining) and* $\Phi \in L_2^{us}(\mathcal{F}^X_{-\infty,\infty})$*. There is a unique isomorphism* $\pi_* : L_0(\mathcal{F}^X_{-\infty,\infty}) \to L_0(\mathcal{F}^{X'}_{-\infty,\infty})$ *such that* $\pi_*(X_{s,t}(x)) = X'_{s,t}(x)$ *for every* $s, t$ *and* $x$*. Set* $\Phi' = \pi_*(\Phi)$*. Then we have*

$$E\left[\rho^{|S_\Phi \cap F|}\right]\ \big(\text{ resp. } P(\ |S_\Phi \cap F| < \infty\ )\ \big) = E\left[\Phi\Phi'\right], \tag{2.9}$$

*equivalently,*

$$E\left[1 - \rho^{|S_\Phi \cap F|}\right]\ \big(\text{ resp. } P(\ |S_\Phi \cap F| = \infty\ )\ \big) = \frac{1}{2}E\left[|\Phi - \Phi'|^2\right] \tag{2.10}$$

*Proof.* In the case when $\Phi \in L_2^{us}(\mathcal{F}^X_{s,t})$ and $F = [s, t]$, (2.9) is nothing but (2.3) and (2.4). From this, we can deduce (2.9) in the general case of an elementary set $F = \bigcup_{k=1}^n [t_{2k-2}, t_{2k-1}]$, $t_{-1} = -\infty < t_0 < \cdots < t_{2n-1} < t_{2n} = \infty$, by considering the following $L^2$-space factorization:

$$L_2(\mathcal{F}^X_{-\infty,\infty}) = \bigotimes_{k=0}^{2n} L_2(\mathcal{F}^X_{t_{k-1},t_k}).$$

We omit the details. ■

## §3. Duality relations for $L$- and $\widehat{L}$-diffusions in the time reversal: The proof of Th. 1.2.

Let $\{\xi^+(t), P_x\}$ and $\{\widehat{\xi}^+(t), \widehat{P}_x\}$ be the reflecting $L$- and $\widehat{L}$-diffusion processes on $[0, \infty)$ introduced in Section 1. The associated Markovian semigroups of operators acting on the space $\mathbf{B}([0,\infty))$ of real bounded Borel functions are defined by

$$(3.1)\qquad T_t^+ f(x) = E_x[f(\xi^+(t))] \quad\text{and}\quad \widehat{T}_t^+ f(x) = \widehat{E}_x[f(\widehat{\xi}^+(t))].$$

Define also the semigroups for *absorbing processes* by

$$(3.2)\ T_t^- f(x) = E_x[f(\xi^+(t\wedge\sigma_0))] \quad\text{and}\quad \widehat{T}_t^- f(x) = \widehat{E}_x[f(\widehat{\xi}^+(t\wedge\widehat{\sigma}_0))],$$

where $\sigma_0$ and $\widehat{\sigma}_0$ are the first hitting time to 0 of $\xi^+(t)$ and $\widehat{\xi}^+(t)$, respectively. Introduce, further, the semigroups for *processes with the extinction at hitting to* 0 by

(3.3)

$$T_t^0 f(x) = E_x\left[f(\xi^+(t))\cdot \mathbf{1}_{[t<\sigma_0]}\right] \text{ and } \widehat{T}_t^0 f(x) = \widehat{E}_x\left[f(\widehat{\xi}^+(t))\cdot \mathbf{1}_{[t<\widehat{\sigma}_0]}\right].$$

$T_t^-$ and $\widehat{T}_t^-$ are Markovian semigroups and $T_t^0$ and $\widehat{T}_t^0$ are sub-Markovian semigroups. Note also that $T_t^+$ , $\widehat{T}_t^+$, $T_t^0$ and $\widehat{T}_t^0$ have the strong Feller property but $T_t^-$ and $\widehat{T}_t^-$ have the Feller property only. It holds that

$$(3.4)\qquad T_t^0 f = T_t^-(\mathbf{1}_{(0,\infty)}\cdot f) \quad\text{and}\quad \widehat{T}_t^0 f = \widehat{T}_t^-(\mathbf{1}_{(0,\infty)}\cdot f).$$

We have the following duality relations which form another key lemma in the proof of Th.1.2:

**Lemma 3.1.** *For $x, y \in [0,\infty)$ and $t>0$,*

$$(3.5)\quad T_t^+\mathbf{1}_{[0,y]}(x) = \widehat{T}_t^0\mathbf{1}_{[x,\infty)}(y) \quad\text{and}\quad T_t^-\mathbf{1}_{[0,y]}(x) = \widehat{T}_t^+\mathbf{1}_{[x,\infty)}(y).$$

*More generally, for $x, y \in [0,\infty)$ and $0 \le t_0 < t_1 < \ldots < t_{2n-1} < t_{2n} < t_{2n+1}$,*

$$\begin{aligned}(3.6)\qquad & T_{t_1-t_0}^+ T_{t_2-t_1}^- T_{t_3-t_2}^+ \cdots T_{t_{2n-1}-t_{2n-2}}^+ T_{t_{2n}-t_{2n-1}}^- \mathbf{1}_{[0,y]}(x) \\ = \ & \widehat{T}_{t_{2n}-t_{2n-1}}^+ \widehat{T}_{t_{2n-1}-t_{2n-2}}^0 \cdots \widehat{T}_{t_3-t_2}^0 \widehat{T}_{t_2-t_1}^+ \widehat{T}_{t_1-t_0}^0 \mathbf{1}_{[x,\infty)}(y),\end{aligned}$$

*and*

$$\begin{aligned}(3.7)\quad & T_{t_1-t_0}^+ T_{t_2-t_1}^- T_{t_3-t_2}^+ \cdots T_{t_{2n-1}-t_{2n-2}}^+ T_{t_{2n}-t_{2n-1}}^- T_{t_{2n+1}-t_{2n}}^+ \mathbf{1}_{[0,y]}(x) \\ = \ & \widehat{T}_{t_{2n+1}-t_{2n}}^0 \widehat{T}_{t_{2n}-t_{2n-1}}^+ \widehat{T}_{t_{2n-1}-t_{2n-2}}^0 \cdots \widehat{T}_{t_3-t_2}^0 \widehat{T}_{t_2-t_1}^+ \widehat{T}_{t_1-t_0}^0 \mathbf{1}_{[x,\infty)}(y).\end{aligned}$$

Admitting this lemma for a moment, we now proceed to prove Th. 1.2.

*Proof of Th. 1.2.* Let $F = [t_0,t_1]\cup[t_2,t_3]\ldots\cup[t_{2n-2},t_{2n-1}]$ be an elementary set in $[0,1]$ and $(\mathbf{X},\mathbf{X}')$ be a $(1^-,F)$-coupling of the Harris flow. Set $\xi(t) = X_{0,t}(0) - X'_{0,t}(0)$. Then $|\xi(t)|$ is a time-inhomogeneous diffusion process which behaves as a reflecting $L$-diffusion when $t \in F$ and as an absorbing $L$-diffusion (i.e., $L$-diffusion with 0 as a trap) when $t \in [0,1] \setminus F$. It is known that $P(S_X \ni t) = 0$ for every $t \in [0,1]$ (cf. [T 2]). Then (2.8), combined with this remark, yields that

$$P(|S_X \cap F| = \infty) = P(S_X^{acc} \cap F \neq \emptyset) = \frac{1}{2} E[|\xi(1)|^2].$$

By applying the Itô formula for $\xi(t)$ on each interval $[t_k, t_{k+1}]$, we have

$$\frac{1}{2} E[|\xi(1)|^2] = \int_0^1 E[(1-b)(\xi(t))]dt = 1 - \int_0^1 E[b(\xi(t))]dt,$$

and hence,

$$P(S_X^{acc} \cap F = \emptyset) = \int_0^1 E[b(\xi(t))]dt. \tag{3.8}$$

On the other hand,

$$E[b(\xi(t))] = \begin{cases} T^+_{t_1-t_0} T^-_{t_2-t_1} \cdots T^-_{t_{2k}-t_{2k-1}} T^+_{t-t_{2k}} b(0), & \text{if } t_{2k} \le t < t_{2k+1} \\ T^+_{t_1-t_0} T^-_{t_2-t_1} \cdots T^+_{t_{2k-1}-t_{2k-2}} T^-_{t-t_{2k-1}} b(0), & \text{if } t_{2k-1} \le t < t_{2k} \end{cases}.$$

Noting $b(x) = \int_{[0,\infty)} \mathbf{1}_{[0,y]}(x)\mu(dy)$, we have by Lemma 3.1 the following:

$$E[b(\xi(t))] = \begin{cases} \int_0^\infty \mu(dy) (\widehat{T}^0_{t-t_{2k}} \widehat{T}^+_{t_{2k}-t_{2k-1}} \cdots \widehat{T}^+_{t_2-t_1} \widehat{T}^0_{t_1-t_0} \mathbf{1}_{[0,\infty)})(y), \\ \qquad\qquad\qquad\qquad \text{if } t_{2k} \le t < t_{2k+1} \\ \int_0^\infty \mu(dy) (\widehat{T}^+_{t-t_{2k-1}} \widehat{T}^0_{t_{2k-1}-t_{2k-2}} \cdots \widehat{T}^+_{t_2-t_1} \widehat{T}^0_{t_1-t_0} \mathbf{1}_{[0,\infty)})(y), \\ \qquad\qquad\qquad\qquad \text{if } t_{2k-1} \le t < t_{2k} \end{cases}.$$

If the random set $\widetilde{S}$ is defined by (1.7), it is not difficult to deduce, from the last expression of $E[b(\xi(t))]$, that $\int_0^1 E[b(\xi(t))]dt$ coincides with $P(\widetilde{S} \cap F = \emptyset)$. Then $P(\widetilde{S} \cap F = \emptyset) = P(S_X^{acc} \cap F = \emptyset)$ by (3.8). Since this holds for every elementary set $F$, we can conclude that $S_X^{acc} \stackrel{d}{=} \widetilde{S}$. ■

*Proof of Lemma 3.1.* First, we prove (3.5). For $\lambda > 0$, let $U^+_\lambda$ and $\hat{U}^0_\lambda$ be the resolvent operators associated with the semigroups $T^+_t$

and $\hat{T}_t^0$ respectively. Let $f$ be continuous and compactly supported in $(0,\infty)$. Then $u = U_\lambda^+ f$ solves Poisson's equation

$$Lu - \lambda u = -f,$$

with the boundary conditions $u'(0+) = u(\infty) = 0$. Define functions $g$ and $v$ via

$$g(y) = \int_0^y \frac{f(x)}{a(x)} dx \quad \text{and} \quad v(y) = \int_0^y \frac{u(x)}{a(x)} dx,$$

where $a(x) = (1 - b(x))$. Dividing Poisson's equation through by $a(x)$ and integrating, we obtain

$$\hat{L}v - \lambda v = -g.$$

Moreover $v$ and $g$ are bounded and $v(0) = 0$. Thus we must have $v = \hat{U}_\lambda^0 g$. Letting $f$ approach a delta function we may write the relationship between $u$ and $v$ as:

$$\frac{1}{a(z)} \hat{U}_\lambda^0 \mathbf{1}_{[z,\infty)}(y) = \int_0^y \frac{u_\lambda^+(x,z)}{a(x)} dx,$$

where $u_\lambda^+$ is the continuous version of the resolvent density corresponding to $U_\lambda^+$. Recalling the symmetry relation,

$$\frac{1}{a(x)} u_\lambda^+(x,z) a(z) = u_\lambda^+(z,x),$$

we obtain

$$\hat{U}_\lambda^0 \mathbf{1}_{[z,\infty)}(y) = \hat{U}_\lambda^+ \mathbf{1}_{[0,y]}(z),$$

from which the first equality of (3.5) follows by uniqueness of Laplace transforms. The second equality may be proved by a similar method.

(3.6) and (3.7) can be proved by applying (3.5) successively: For example,

$$\begin{aligned} T^+_{t_1-t_0}\cdot T^-_{t_2-t_1}\mathbf{1}_{[0,y]}(x) &= \int_{[0,\infty)} T^+_{t_1-t_0}(x,du)T^-_{t_2-t_1}\mathbf{1}_{[0,y]}(u) \\ &= \int_{[0,\infty)} T^+_{t_1-t_0}(x,du)\widehat{T}^+_{t_2-t_1}\mathbf{1}_{[u,\infty)}(y) \\ &= \iint_{0\le u\le v<\infty} T^+_{t_1-t_0}(x,du)\widehat{T}^+_{t_2-t_1}(y,dv) \\ &= \int_{[0,\infty)} \widehat{T}^+_{t_2-t_1}(y,dv)T^+_{t_1-t_0}\mathbf{1}_{[0,v]}(x) \\ &= \int_{[0,\infty)} \widehat{T}^+_{t_2-t_1}(y,dv)\widehat{T}^0_{t_1-t_0}\mathbf{1}_{[x,\infty)}(v) \\ &= \widehat{T}^+_{t_2-t_1}\cdot\widehat{T}^0_{t_1-t_0}\mathbf{1}_{[x,\infty)}(y). \end{aligned}$$

This proves a particular case of (3.6). In the same way, the general case can be proved easily by induction. ■

**Remark 3.1.** *We remark that an alternative proof of (3.5) is possible by means of the time reversal of stochastic flows on the half line. A stochastic flow on the half line* $[0,\infty)$ *is defined similarly by replacing the whole line* $\mathbf{R}$ *by* $[0,\infty)$ *in Def.1.2. A key idea in the proof is to construct a stochastic flow* $\mathbf{X}=(X_{s,t})$ *on* $[0,\infty)$ *whose one-point motion* $t\mapsto X_{0,t}(x), x\in\mathbf{R}$, *is given by the absorbing L-diffusion* $\xi^-(t)$, *i.e., the diffusion with the semigroup* $T^-_t$, *and then show that its* time reversed flow $\widehat{\mathbf{X}}=(\widehat{X}_{s,t})$, *defined by* $\widehat{X}_{s,t}=(X_{-t,-s})^{-1}$, *has the one-point motion given by the reflecting* $\widehat{L}$*-diffusion* $\widehat{\xi}^+(t)$, *i.e., the diffusion with the semigroup* $\widehat{T}^+_t$. *Here, for a right-continuous and non-decreasing* $\varphi:[0,\infty)\to[0,\infty)$ *such that* $\lim_{x\nearrow\infty}\varphi(x)=\infty$, $\varphi^{-1}$ *is the right-continuous inverse of* $\varphi$: $\varphi^{-1}(x)=\inf\{y|\varphi(y)>x\}$. *This is connected to the fact that L and* $\widehat{L}$, *when written in Hörmander form, differ only in the sign of the drift term. The corresponding fact in the case of stochastic flows of homeomorphisms is well-known (cf. [K] p.131, [IW] p.265).*

## §4. Proof of Th. 1.3.

Consider a Harris flow $\mathbf{X}$ satisfying (1.4), (1.6) and (1.11).

*Proof of Cor. 1.1.* It is sufficient to show that the set of zeros of $\widehat{L}$-diffusion $\widehat{\xi}(t)$ has the Hausdorff dimension $(1-\alpha)/(2-\alpha)$, $\widehat{P}_0$-almosy surely. The set of zeros of $\widehat{\xi}(t)$ is the range of the inverse local time $l^{-1}(t)$ at 0 of $\widehat{\xi}(t)$, which is a subordinator with exponent $\Psi(\lambda) = g_\lambda(0,0)^{-1}$:

$$E\left(e^{-\lambda l^{-1}(t)}\right) = e^{-t\Psi(\lambda)} = e^{-t/g_\lambda(0,0)}.$$

Here, $g_\lambda(x,y)$ is the Green function (resolvent density) with respect to the speed measure $dx$ of reflecting $\widehat{L}$-diffusion where $\widehat{L} = \frac{d}{dx}(1-b(x))\frac{d}{dx}$. If we introduce the scale $\xi = \int_0^x (1-b(y))^{-1}dy$ as the coordinate of $[0,\infty)$, then $\widehat{L} = (1-\tilde{b}(\xi))^{-1}\frac{d^2}{d^2\xi}$ where $\tilde{b}(\xi) = b(x(\xi))$, so that the speed measure in the new coordinate is given by $d\tilde{m}(\xi) = a(\xi)d\xi$ with $a(\xi) = 1-\tilde{b}(\xi)$. It is easy to deduce from (1.11) that $a(\xi) \asymp \xi^{\alpha/(1-\alpha)}$ as $\xi \to 0$. Let $\tilde{g}_\lambda(\xi,\eta)$ be the Green function for $\widehat{L}$-diffusion with respect to the speed measure so that $\tilde{g}_\lambda(0,0) = g_\lambda(0,0)$. By Th.2.3 in p.243 of [KW], we have

$$\Psi(\lambda) = \tilde{g}_\lambda(0,0)^{-1} \asymp \lambda^{1/(2+\frac{\alpha}{1-\alpha})} = \lambda^{\frac{1-\alpha}{2-\alpha}} \quad \text{as } \lambda \to \infty.$$

Then we can conclude that the range of the subordinator $l^{-1}(t)$ has the Hausdorff dimension $\frac{1-\alpha}{2-\alpha}$ almost surely, by a result of Blumenthal and Getoor (cf. [B], p. 94, Th. 16). ■

Now we proceed to prove Th. 1.3. We need several lemmas.

**Lemma 4.1.** *(i) Let $\Phi_1, \Phi_2 \in L_2^{us}(\mathcal{F}^X_{-\infty,\infty})$ and consider their linear combination $\Phi = \alpha\Phi_1 + \beta\Phi_2 \in L_2^{us}(\mathcal{F}^X_{-\infty,\infty})$. If $A \in \mathcal{B}(\mathcal{C})$ satisfies $P(S_{\Phi_1} \in A) = P(S_{\Phi_2} \in A) = 1$, then it holds that $P(S_\Phi \in A) = 1$.*
*(ii) Let $\Phi_n \in L_2^{us}(\mathcal{F}^X_{-\infty,\infty})$, $n = 1,2,\ldots$, constitute a dense family in $L_2^{us}(\mathcal{F}^X_{-\infty,\infty})$. If $A \in \mathcal{B}(\mathcal{C})$ satisfies $P(S_{\Phi_n} \in A) = 1$ for all $n$, then it holds that $P(S_\Phi \in A) = 1$ for all $\Phi \in L_2^{us}(\mathcal{F}^X_{-\infty,\infty})$.*

*Proof.* According to Theorem 3d12 of [T 5], every $A \in \mathcal{B}(\mathcal{C})$ is associated with a closed subspace $\mathcal{H}_A$ of $L_2(\mathcal{F}^X_{-\infty,\infty})$ such that the spectral measure $\mu_\Phi$ of any $\Phi$ satisfies

$$||P_A\Phi||^2 = \mu_\Phi(A),$$

where $P_A$ denotes the orthogonal projection onto $\mathcal{H}_A$. Both parts of this lemma are immediate consequences. ■

**Lemma 4.2.** *Let $t_1 < t_2 < t_3$ and $\Phi = \Phi_1\Phi_2 \in L_2^{us}(\mathcal{F}^X_{t_1,t_3})$ such that $\Phi_1 \in L_2^{us}(\mathcal{F}^X_{t_1,t_2})$ and $\Phi_2 \in L_2^{us}(\mathcal{F}^X_{t_2,t_3})$. Then,*

$$S_\Phi \cap [t_1,t_2] \stackrel{d}{=} S_{\Phi_1}, \quad S_\Phi \cap [t_2,t_3] \stackrel{d}{=} S_{\Phi_2}.$$

*Furthermore, $S_\Phi \cap [t_1,t_2]$ and $S_\Phi \cap [t_2,t_3]$ are mutually independent.*

The proof is easy and omitted.

**Lemma 4.3.** *Let $S$ be a $\mathcal{C}_{[0,1]}$-valued random variable and assume, for $0<\beta<1$ and $K>0$, that*

$$P(S\cap[t,t+\epsilon]\neq\emptyset)\leq K\epsilon^{\beta}\quad \textit{for all}\ \ 0<\epsilon<1\quad \textit{and}\ \ t\in[0,1].$$

*Then, $P(\dim S\leq 1-\beta)=1$.*

*Proof.* For every $a>1-\beta$, we have

$$\begin{aligned}
&E\left(\sum_{k=1}^{n}\mathbf{1}_{\{S\cap[\frac{k-1}{n},\frac{k}{n}]\neq\emptyset\}}\cdot\left(\frac{1}{n}\right)^{a}\right)\\
=&\ \sum_{k=1}^{n}P\left(S\cap\left[\frac{k-1}{n},\frac{k}{n}\right]\neq\emptyset\right)\cdot\left(\frac{1}{n}\right)^{a}\\
\leq&\ nK\left(\frac{1}{n}\right)^{\beta}\cdot\left(\frac{1}{n}\right)^{a}=K\cdot n^{1-(\beta+a)}\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Hence, there exists a subsequence $n_\nu\to\infty$ such that, almost surely,

$$\sum_{k=1}^{n_\nu}\mathbf{1}_{\{S\cap[\frac{k-1}{n_\nu},\frac{k}{n_\nu}]\neq\emptyset\}}\cdot\left(\frac{1}{n_\nu}\right)^{a}\to 0\quad\text{as}\quad \nu\to\infty.$$

Let $\mathcal{C}_\nu$ be the collection of intervals $E_k=[\frac{k-1}{n_\nu},\frac{k}{n_\nu}]$, $k=1,\dots,n_\nu$, which have nonempty intersections with the set $S$. Then $\mathcal{C}_\nu$ is a covering of $S$ and

$$\sum_{E_k\in\mathcal{C}_\nu}(\text{diam } E_k)^{a}\to 0\quad\text{a.s.,}\ \ \text{as}\quad \nu\to\infty.$$

Hence, $\dim S\leq a$, a.s., implying that $\dim S\leq 1-\beta$, a.s. ∎

*Proof of Th. 1.3.* It is sufficient to show that

$$\dim S_\Phi\leq\frac{1-\alpha}{2-\alpha}\quad\text{a.s.}\tag{4.1}$$

for $\Phi\in L_2^{us}(\mathcal{F}_{0,1}^X)$. Indeed, if (4.1) is true for $\Phi\in L_2^{us}(\mathcal{F}_{0,1}^X)$, then by the stationarity of the flow, it is also true for $\Phi\in L_2^{us}(\mathcal{F}_{n,n+1}^X)$. By Lemma 4.2, (4.1) is true for a finite product of such $\Phi$'s. Since linear combinations of such products are dense in $L_2(\mathcal{F}_{-\infty,\infty}^X)$, we can conclude by Lemma 4.1 that (4.1) is true for any $\Phi\in L_2^{us}(\mathcal{F}_{-\infty,\infty}^X)$.

First, we consider the case when $\Phi\in L_2^{us}(\mathcal{F}_{0,1}^X)$ is given by

$$\Phi=f(X_{0,1}(x_1),\dots,X_{0,1}(x_n)),\quad x_1,\dots,x_n\in\mathbf{R},$$

and a function $f$ is uniformly Lipschitz-continuous on $\mathbf{R}^n$.

Let $F = [t, t+\epsilon]$, $0 \le t < t+\epsilon \le 1$, and let $(\mathbf{X}, \mathbf{X}')$ be a $(1^-, F)$-joining. Then we know by Lemma 2.2 that $2P(S_X^{acc} \cap F \neq \emptyset) = E(|X_{0,1}(0) - X'_{0,1}(0)|^2)$ and similarly, we have $2P(S_\Phi^{acc} \cap F \neq \emptyset) = E(|\Phi - \Phi'|^2)$ where $\Phi' = f(X'_{0,1}(x_1), \ldots, X'_{0,1}(x_n))$. Therefore, noting that $E(|X_{0,1}(x) - X'_{0,1}(x)|^2)$ is independent of $x$, we have

$$
\begin{aligned}
P(S_\Phi^{acc} \cap F \neq \emptyset) &= \frac{1}{2} E(|\Phi - \Phi'|^2) \\
&\le KE(|X_{0,1}(0) - X'_{0,1}(0)|^2) = 2KP(S_X^{acc} \cap F \neq \emptyset)
\end{aligned}
\tag{4.2}
$$

where a constant $K$ depends on $n$ and the Lipschitz constant of $f$.

Let $\{\widehat{\xi}^+(t), \widehat{P}_\xi\}$ be the reflecting $\widehat{L}$-diffusion on $[0, \infty)$. As in the proof of Cor.1.1, *take a canonical scale* $\xi$ *as the coordinate* so that $\widehat{L} = \frac{d^2}{a(\xi) d\xi^2}$ and we have $a(\xi) \asymp \xi^{\alpha/(1-\alpha)}$ as $\xi \to 0$ and $a(\xi) \to 1$ as $\xi \to \infty$. Let $\mu(d\xi) = da(\xi)$. By what we have shown above,

$$
\begin{aligned}
& P(S_X^{acc} \cap [t, t+\epsilon] \neq \emptyset) = P(\widetilde{S} \cap [t, t+\epsilon] \neq \emptyset) \\
= & \int_0^1 \widehat{P}_\mu \left( \widehat{\xi}^+(u-s) = 0 \text{ for some } s \in [0, u] \cap [t, t+\epsilon] \right) du \\
= & \int_t^1 \widehat{P}_\mu \left( \widehat{\xi}^+(\theta) = 0 \text{ for some } \theta \in [(u-t-\epsilon)_+, u-t] \right) du \\
= & \ O(\epsilon) + \int_t^1 \widehat{P}_\mu \left( \widehat{\xi}^+(\theta) = 0 \text{ for some } \theta \in [u-t, u-t+\epsilon] \right) du.
\end{aligned}
$$

We would show

(4.3)

$$
I(t) := \int_t^1 \widehat{P}_\mu \left( \widehat{\xi}^+(\theta) = 0 \text{ for some } \theta \in [u-t, u-t+\epsilon] \right) du = O\left( \epsilon^{\frac{1}{2-\alpha}} \right)
$$

as $\epsilon \to 0$ uniformly in $t \in [0,1]$. If we can show this, then

$$
P(S_X^{acc} \cap [t, t+\epsilon] \neq \emptyset) = O(\epsilon^{1/(2-\alpha)})
$$

as $\epsilon \to 0$ uniformly in $t \in [0,1]$ and, combining this with (4.2), we see that $P(S_\Phi^{acc} \cap [t, t+\epsilon] \neq \emptyset) = O(\epsilon^{1/(2-\alpha)})$, so that, by Lemma 4.3, we can conclude that the estimate (4.1) holds for $\Phi$ because $1 - 1/(2-\alpha) = (1-\alpha)/(2-\alpha)$.

To obtain (4.3), we estimate

$$
\begin{aligned}
I(t) &\le \int_0^1 \widehat{P}_\mu\left(\widehat{\xi}^+(\theta)=0 \text{ for some } \theta\in[u,u+\epsilon]\right)du \\
&= \int_0^1 \widehat{E}_\mu\left(\widehat{P}_{\widehat{\xi}^+(u)}[\widehat{\sigma}_0\le\epsilon]\right)du \le e\int_0^1 e^{-u}\widehat{E}_\mu\left(\widehat{P}_{\widehat{\xi}^+(u)}[\widehat{\sigma}_0\le\epsilon]\right)du \\
&\le e\int_0^\infty e^{-u}\widehat{E}_\mu\left(\widehat{P}_{\widehat{\xi}^+(u)}[\widehat{\sigma}_0\le\epsilon]\right)du \\
&= e\int_{[0,\infty)}\mu(d\xi)\int_{[0,\infty)}\tilde{g}_1(\xi,\eta)\widehat{P}_\eta[\widehat{\sigma}_0\le\epsilon]a(\eta)d\eta,
\end{aligned}
$$

where $\widehat{\sigma}_0$ is the first hitting time of $\widehat{\xi}^+(t)$ to 0. Since the resolvent density $\tilde{g}_1(\xi,\eta)$ is bounded, we have, for some $C>0$,

$$I(t)\le C\int_{[0,\infty)}\widehat{P}_\eta[\widehat{\sigma}_0\le\epsilon]a(\eta)d\eta.$$

The process $\widehat{\xi}^+(t)$ under $\widehat{P}_\eta$, $\eta>0$, and in the coordinate $\xi$, is obtained from a one-dimensional Brownian motion $B(t)$ with $B(0)=0$ by

$$\widehat{\xi}^+(t)=|\eta+B(A^{-1}(t))| \quad \text{where} \quad A(t)=\int_0^t a(|\eta+B(s)|)ds.$$

Hence,

$$\widehat{P}_\eta(\widehat{\sigma}_0\le\epsilon)=P\left(\int_0^{\sigma_0}a(|\eta+B(s)|)ds\le\epsilon\right)$$
$$\text{where} \quad \sigma_0=\min\{s|\eta+B(s)=0\}.$$

and, noting $a(\xi)\ge K^{-1}\cdot\xi^{\alpha/(1-\alpha)}\wedge 1$ for some $K>0$,

$$\widehat{P}_\eta(\widehat{\sigma}_0\le\epsilon)\le P\left(\int_0^{\sigma_0}\left(|\eta+B(s)|^{\alpha/(1-\alpha)}\wedge 1\right)ds\le K\epsilon\right).$$

The scaling property of $B(t)$ combined with an easy inequality $(\epsilon a)\wedge 1\ge \epsilon(a\wedge 1)$ for $a>0$ and $1\ge\epsilon>0$ yields that the RHS is dominated by $\phi(\epsilon^{-(1-\alpha)/(2-\alpha)}\eta)$, where

$$\phi(\eta)=P\left(\int_0^{\sigma_0}\left(|\eta+B(s)|^{\alpha/(1-\alpha)}\wedge 1\right)ds\le K\right).$$

Then,

$$\begin{aligned} I(t) &\leq C \int_{[0,\infty)} \phi(\epsilon^{-(1-\alpha)/(2-\alpha)}\eta) a(\eta) d\eta \\ &\leq K' \int_{[0,\infty)} \phi(\epsilon^{-(1-\alpha)/(2-\alpha)}\eta)\eta^{\alpha/(1-\alpha)} d\eta \\ &= K'\epsilon^{1/(2-\alpha)} \int_{[0,\infty)} \phi(\eta)\eta^{\alpha/(1-\alpha)} d\eta \end{aligned}$$

and we have otained (4.3).

In the same way, we have the estimate (4.1) for $\Phi = f(X_{s,t}(x_1), \ldots, X_{s,t}(x_n))$, $x_1, \ldots, x_n \in \mathbf{R}$, $s < t$, where $f$ is uniformly Lipschitz continuous on $\mathbf{R}^n$. Then, by Lemma 4.2, we have the estimate (4.1) for $\Phi = \Phi_1\Phi_2\cdots\Phi_m \in L_2^{us}(\mathcal{F}_{0,1}^X)$ if $t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$, and $\Phi_k \in \text{ub}[L^2(\mathcal{F}_{t_{k-1},t_k}^X)]$, $k = 1, 2, \ldots, m$, is given in the form $\Phi_k = f_k\left(X_{t_{k-1},t_k}(x_1^{(k)}), \ldots, X_{t_{k-1},t_k}(x_{n_k}^{(k)})\right)$, $x_1^{(k)}, \ldots, x_{n_k}^{(k)} \in \mathbf{R}$, where $f_k$ is uniformly Lipschitz continuous on $\mathbf{R}^{n_k}$. By Lemma 4.1 (i), the estimate (4.1) still holds for a finite linear combination of such functionals and this class of functionals is dense in $L_2^{us}(\mathcal{F}_{0,1}^X)$. ■

## References


[A] R. A. Arratia, Coalescing Brownian motions, and voter model on Z, Unpublished manuscript, Univ. of Southern California.

[B] J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996

[H] T. E. Harris, Coalescing and noncoalescing stochastic flows in $\mathbf{R}_1$, Stochastic Processes and their Applications, **17**(1984), 187-210

[IW] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Amsterdam/Tokyo, 1988

[IM] K. Itô and H. P. McKean, Jr., *Diffusion Processes and their Sample Paths*, Springer, Berlin, 1965, Second Printing 1974, in *Classics in Mathematics*, 1996

[K] H. Kunita, *Stochastic flows and stochastic differential equations,* Cambridge University Press, Cambridge, 1990

[KW] S. Kotani and S. Watanabe, Krein's spectral theory of strings and generalized diffusion processes, *in Functional Analysis in Markov Processes, ed. by M. Fukushima*, **LNM 923**, Springer(1982), 235-259

[LR 1] Y. Le Jan and O. Raimond, Flow, coalescence and noise, PR/0203221(2002)

[LR 2] Y. Le Jan and O. Raimond, The noise of Brownian sticky flow is black, PR/0212269(2003)

[T 1] B. Tsirelson, Within and beyond the reach of Brownian innovation, "Documenta Mathematica", Extra Volume ICM 1998, III, 311-320

[T 2] B. Tsirelson, Unitary Brownian motions are linearizable, PR/9806112(1998)

[T 3] B. Tsirelson, Brownian coalescence as a black noise. I, preprint 1998

[T 4] B. Tsirelson, Fourier-Walsh coefficients for a coalescing flow (discrete time), PR/9903068(1999)

[T 5] B. Tsirelson, Scaling Limit, Noise, Stability, In: Saint-Flour Summer School for Probability 2002, Lect. Notes in Math. to appear.

[VK] A. Ju. Veretennikov and N. V. Krylov, On explicit formulas for solutions of stochastic equations, Math. USSR Sbornik, **29**(1976), 239-256

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