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Representation of Martingales with Jumps and Applications to Mathematical Finance

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Dedicated to Professor Kiyosi Itô on his 88-th birthday

Abstract.

We study representations of martingales with jumps based on the filtration generated by a Lévy process. Two types of representation theorem are obtained. The first formula is valid for any martingale and written as the sum of the stochastic integral based on the Brownian motion and that based on the compensated Poisson random measure. See (0.1). The second formula is valid only for a process which is a martingale for any equivalent martingale measure. See (0.2). The latter representation formula is then applied to a problem in mathematical finance. The upper hedging strategy and the lower hedging strategy of a contingent claim is obtained through the representation kernel.

§0. Introduction

It is a well known fact that any martingale with respect to the filtration generated by a Brownian motion can be represented as Itô's stochastic integral based on the Brownian motion. On the other hand, martingales with respect to the filtration generated by a Lévy process are not always represented by Itô's stochastic integrals based on the Lévy process, even the latter is a martingale. What is known is that any square integrable martingale with respect to the filtration generated by a Lévy process is represented by stochastic integrals based on the Brownian motion and the compensated Poisson random measure.

In the first half of this paper, we recall these representation theorems following Kunita-Watanabe [6] (Section 1). Let $(\mathcal{F}_t)$ be the filtration generated by a m-dimensional Lévy process. Then every (local)martingale $M(t)$ with respect to the filtration $(\mathcal{F}_t)$ is represented

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by

$$(0.1)\quad M(t) = M(0) + \sum_{i=1}^{m} \int_0^t \phi_i(s)dW^i(s) + \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}(dsdz),$$

where $W(t) = (W^1(t), ..., W^m(t))$ is a standard Brownian motion and $\tilde{N}(dsdz)$ is the compensated Poisson random measure, which appear in the Lévy-Itô decomposition of the Lévy process. The pair $((\phi_1(s), ..., \phi_m(s)), \psi(s,z))$ is a predictable process with parameter z satisfying certain integrability conditions (See Theorem 1.1). We are particularly interested in the exponential representation of positive martingales (Theorem 2.1). We apply it to the study of Radon Nikodym density of equivalent probability measure and extend Girsanov's theorem to jump processes (Theorem 2.3).

In the second half of the paper, we apply these representation theorems to some problems in mathematical finance. Suppose that we are given a stochastic process $\xi_t = (\xi_t^1, ..., \xi_t^d)$ (e.g., a price process or its return process in mathematical finance) governed by a Lévy process. If the process ξ_t has jumps, there are infinitely many equivalent probability measures with respect to which ξ_t is a localmartingale (called equivalent martingale measures). Now suppose that $M(t)$ is a localmartingale for any equivalent martingale measure. We will show that under some conditions for ξ_t, $M(t)$ is represented by a stochastic integral based on ξ_t, i.e., it is written as

$$(0.2)\qquad M(t) = M(0) + \sum_{i=1}^{d} \int_0^t \varphi_i(s)d\xi_s^i.$$

The difference of these two representations (0.1) and (0.2) are big. We show further that an adapted process $X(t)$ is a supermartingale for any equivalent martingale measure if and only if it admits the unique Doob-Meyer decomposition (not depending on each equivalent martingale measure) and the localmartingle part $M(t)$ is represented as (0.2). See Theorem 3.4 in Section 3.

At the end of Section 3, we apply the above representation theorem to determine the upper hedging price and the lower hedging price of a given contingent claim (Theorem 3.5).

Finally, we mention that there are several works on determing the upper or the lower hedging prices of contingent claims in the case where the price processes have jumps. See e.g. Kabanov-Stricker [3] and references therein. In these works, more general price processes are studied in an abstract manner.

§1. Representation of localmartingales

Let T be a positive number and let $Z(t), t\in[0,T]$ be an m-dimensional Lévy process such that $Z(0) = 0$. Then it admits the Lévy-Itô decomposition:

$$(1.1)\qquad Z(t) = \sigma W(t) + bt + \int_{(0,t]}\int_{|z|>1} zN(dsdz) + \int_{(0,t]}\int_{|z|\le 1} z\{N(dsdz) - \hat{N}(dsdz)\},$$

where σ is an $m \times m$ matrix, $W(t) = (W^1(t), ..., W^m(t))$ is an m-dimensional standard Brownian motion and $N(dsdz)$ is a Poisson counting measure on $[0,T]\times\mathbf{R}^m$ with intensity measure $\hat{N}(dsdz) = ds\nu(dz)$, which is independent of $W(t)$. In the following, we denote

$$(1.2)\qquad \tilde{N}(dsdz) = N(dsdz) - \hat{N}(dsdz).$$

Let $(\mathcal{F}_t), t \in [0,T]$ be the filtration generated by the Brownian motion $W(t)$ and the Poisson random measure $N(dtdz)$. Then both $W(t)$ and $\int_{|z|\le 1} z\tilde{N}(dsdz)$ are martingales adapted to the filtration. Let $M(t), t \in [0,T]$ be an $(\mathcal{F}_t)$-adapted cadlag (right continuous with the left hand limits) process. It is called a localmartingale if there exists a non-decreasing sequence of stopping times $\tau_n, n = 1, 2, ...$ with values in $[0,T]$ such that $P(\tau_n < T) \to 0$ as $n \to \infty$ and the stopped process $M(t\wedge\tau_n)$ is a martingale for any n. In particular if we can choose the sequence such that the stopped process $M(t \wedge \tau_n)$ is a square integrable martingale for any n, $M(t)$ is called a locally square integrable martingale. Any continuous localmartingale is a locally square integrable martingale, but it is not always the case for a localmartingale with jumps. An $(\mathcal{F}_t)$ adapted cadlag process $X(t)$ is called a semimartingale if it is written as a sum of a localmartingale and a process of bounded variation. In particular if the corresponding process of bounded variation is locally integrable, $X(t)$ is called a special semimartingale. A special semimartingale is decomposed uniquely to the sum of a localmartingale and a predictable process of bounded variation.

We denote by Φ the set of all m dimensional predictable processes $\phi(t) = (\phi_1(t), ..., \phi_m(t))$ such that $\int_0^T |\phi(s)|^2 dt < \infty$ a.s. Then the stochastic integral based on the m-dimensional Brownian motion $W(t) = (W^1(t), ..., W^m(t))$ is well defined for $\phi \in \Phi$. We use the notation:

$$(1.3)\qquad \int_0^t (\phi(s), dW(s)) = \sum_{i=1}^m \int_0^t \phi_i(s) dW^i(s)$$

It is a continuous locally square integrable martingale.

Let $\mathcal{P}$ be the predictable σ-algebra on $[0,T] \times \Omega$ and let $\mathcal{B}$ be the Borel algebra on $\mathbf{R}^m$. A functional $\psi(s,z,\omega), (s,z,\omega) \in [0,T] \times \mathbf{R}^m \times \Omega$ is called a predictable process if it is $\mathcal{P} \times \mathcal{B}$-measurable.

We will recall the definition of the stochastic integral of the predictable process $\psi(s,z)$ based on the compensated Poisson random measure $\tilde{N}(dsdz)$ following Kunita-Watanabe [6]. Note first that if $E_1, ..., E_n$ are disjoint Borel subsets of $[0,T] \times \mathbf{R}^m$ such that $\hat{N}(E_1) < \infty, ..., \hat{N}(E_n) < \infty$, then $\tilde{N}(E_1), ..., \tilde{N}(E_n)$ are independent random variables with mean 0 and variance $\hat{N}(E_1), ..., \hat{N}(E_n)$, respectively. Now, let $\psi(t,z)$ be a step process of the form $\sum_{i,j} a_{ij} 1_{(t_i,t_{i+1}]}(t) 1_{F_{ij}}(z)$, where $0 = t_0 < \cdots < t_N = T$ and for each i $F_{i1}, ..., F_{in}$ are disjoint subsets of $\mathbf{R}^m$ satisfying $\nu(F_{ij}) < \infty, j = 1, ..., n$ and a_{ij} are bounded $\mathcal{F}_{t_i}$-adapted random variables. We define the stochastic integral of ψ based on $\tilde{N}$ by

$$(1.4) \qquad \int_0^T \int_{\mathbf{R}^m} \psi(t,z) \tilde{N}(dtdz) = \sum_{i,j} a_{ij} \tilde{N}((t_i, t_{i+1}] \times F_{ij}).$$

Then we have

(1.5)

$$E\left[\left(\int_0^T \int_{\mathbf{R}^m} \psi(t,z) \tilde{N}(dtdz)\right)^2\right] = \sum_i E\left[(\sum_j a_{ij} \hat{N}((t_i, t_{i+1}] \times F_{ij}))^2\right]$$
$$+ \sum_{i<k} E\left[\{\sum_j a_{kj} \hat{N}((t_k, t_{k+1}] \times F_{kj})\}\{\sum_j a_{ij} \hat{N}((t_i, t_{i+1}] \times F_{ij})\}\right].$$

Since $\{a_{ij}, j = 1, 2, ...\}$ and $\tilde{N}((t_i, t_{i+1}] \times F_{ij}), j = 1, 2, ...$ are independent and the latters are of mean 0, the first term of the left hand side is computed as

$$(1.6) \quad \sum_{ij} E\left[a_{ij}^2 \tilde{N}((t_i, t_{i+1}] \times F_{ij})^2\right] = \sum_{ij} E\left[a_{ij}^2 \hat{N}((t_i, t_{i+1}] \times F_{ij})\right]$$

The last term of (1.5) is 0, since $E[\sum_j a_{kj} \tilde{N}((t_j, t_{j+1}] \times F_{kj}) | \mathcal{F}_{t_{i+1}}] = 0$. Therefore we have

(1.7)

$$E\left[\left(\int_0^T \int_{\mathbf{R}^m} \psi(t,z) \tilde{N}(dtdz)\right)^2\right] = E\left[\int_0^T \int_{\mathbf{R}^m} |\psi(t,z)|^2 \hat{N}(dtdz)\right].$$

Now suppose that $\psi(t,z)$ is a predictable process satisfying the condition $E[\int_0^T \int_{\mathbf{R}^m} |\psi(t,z)|^2 dt\nu(dz)] < \infty$. Then we can choose a sequence $\{\psi_n(t,z)\}$ of step processes such that $E[\int_0^T\int_{\mathbf{R}^m}|\psi(t,z)-\psi_n(t,z)|^2 dt\nu(dz)] \to 0$, as $n \to \infty$. Denote the stochastic integral of ψ_n (formula (1.4)) by M_n. Then M_n converges in L^2. We denote the limit by $M = \int_0^T \int_{\mathbf{R}^m} \psi(t,z)\tilde{N}(dtdz)$. Then it satisfies (1.7) again.

The stochastic integral $\int_0^T 1_{(0,t]}(s)\psi(s,z)\tilde{N}(dsdz)$ is denoted by $\int_0^t \psi(s,z)\tilde{N}(dsdz)$. It is a cadlag process with time t and in fact is a square integrable martingale. This fact can be shown directly in the case where $\psi(t,x)$ is a step process defined above. Then the martingale property is extended to any ψ such that (1.7) is finite.

We denote by $\Psi_2(\hat{N})$ $(\Psi_1(\hat{N}))$ the set of all predictable processes $\psi(t,z)$ which are square integrable (resp. integrable) with respect to the measure $\hat{N}(dtdz)$ a.s. Then we can define the stochastic integral $\int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}(dsdz)$ for $\psi \in \Psi_2(\hat{N})$ as a locally square integrable martingale. For $\psi \in \Psi_1(\hat{N})$, we define the stochastic integral by

$$(1.8)\quad \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}(dsdz) := \int_0^t \int_{\mathbf{R}^m} \psi(s,z)N(dsdz) - \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\hat{N}(dsdz).$$

It is a localmartingale. For a general $\psi(t,z)$, we set $\psi_1(t,z) = \psi(t,z)1_{\{|\psi|>1\}}(t,z), \psi_2(t,z) = \psi(t,z)1_{\{\{|\psi|\le 1\}}(t,z)$, and we denote by $\Psi_{1,2}(\hat{N})$ the set of all predictable process $\psi(t,z)$ such that $\psi_1 \in \Psi_1(\hat{N})$ and $\psi_2 \in \Psi_2(\hat{N})$. Then, for any $\psi \in \Psi_{1,2}(\hat{N})$, the stochastic integral is defined as the sum of stochastic integrals of ψ_1 and ψ_2. It is a localmartingale.

The following notations will be used

(1.9)

$$N_t(\psi) = \int_0^t \int_{\mathbf{R}^m} \psi(s,z)N(dsdz), \quad \tilde{N}_t(\psi) = \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}(dsdz),$$

$$\hat{N}_t(\psi) = \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\hat{N}(dsdz),$$

if these are well defined.

Now, we give a representation theorem of localmartingales.

Theorem 1.1. *([6], Example at p.227 and Proposition 5.2) Let $M(t)$ be a localmartingale. Then there exist $\phi(s) \in \Phi$, $\psi(s,z) \in \Psi_{1,2}(\hat{N})$,* *and $M(t)$ is represented by*

$$M(t) = M(0) + \int_0^t (\phi(s), dW(s)) + \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}(dsdz). \tag{1.10}$$

The representation kernel $(\phi(s), \psi(s,z))$ is uniquely determined from $M(t)$, i.e., if $M(t)$ is represented by (1.10) with another $(\phi'(s), \psi'(s,z))$, then we have $\phi(s) = \phi'(s)$ a.e. $\lambda \otimes P$ and $\psi(s,z) = \psi'(s,z)$ a.e. $\hat{N} \otimes P$, where λ is the Lebesgue measure on $[0,T]$.

Proof. In the paper [6], the above theorem is proved for square integrable martingale by using the theory of additive functionals of Markov processes. Here we give a direct and simpler proof by applying Itô [2]. For simplicity we prove the theorem in the case $m = 1$ only. Let $\mathbf{Z} = (Z(t))$ be a one dimensional Lévy process and let (1.1) be the Lévy-Itô decomposition. We introduce a random measure $M(E)$ on $[0,T] \times \mathbf{R}$ by

$$M(E) = \int_{E(0)} dW(t) + \int_{E-E(0)} \frac{z}{1+|z|}\tilde{N}(dtdz), \tag{1.11}$$

where $E(0) = \{(t,0);\ (t,0) \in E\}$. Then we have $E[M(E_1)M(E_2)] = \mu(E_1 \cap E_2)$, where

$$\mu(E) = |E(0)| + \int_{E-E(0)} \left(\frac{z}{1+|z|}\right)^2 dt\nu(dz).$$

For each positive integer p, we define the multiple Wiener integral by

$$I_p(f) = \int \cdots \int f(\xi_1, ..., \xi_p) dM(\xi_1) \cdots dM(\xi_p). \tag{1.12}$$

Let $\mathbf{H_Z}$ be the L^2 space over $(\Omega, \mathcal{F}_T, P)$ and let ${\mathbf{H_Z}}^{(p)}$ be the closed linear manifold of $\{I_p(f); f \in L_p^2\}$, where L_p^2 is the L^2 space on $\mathbf{R}^p$ with the product measure of μ. Then it is shown in [2] that one has the direct sum expansion: $\mathbf{H_Z} = \sum_{p \geq 0} \oplus {\mathbf{H_Z}}^{(p)}$. Note that each $I_p(f)$ is written as the sum of the following terms

(1.13)

$$\int \cdot \int_{0 \leq t_1 < \cdots < t_p \leq T, (z_1, ..., z_p) \in \mathbf{R}^p} f((t_1, z_1), ..., (t_p, z_p)) dM(t_1 z_1) \cdots dM(t_p z_p)$$

$$= \int_0^T \int_{\mathbf{R}} \varphi(t_p, z_p) dM(t_p z_p),$$

where

$$\varphi(t_p, z_p) = \int \cdot \int_{\Lambda(t_p,z_p)} f((t_1, z_1), ..., (t_p, z_p))dM(t_1z_1) \cdots dM(t_{p-1}z_{p-1})$$

and $\Lambda(t_p, z_p) = \{0 < t_1 < \cdots < t_{p-1} < t_p, (z_1, ..., z_{p-1}, z_p) \in \mathbf{R}^{p-1}\}$. Setting $\phi(t) = \varphi(t, 0)$ and $\psi(t, z) = \varphi(t, z)\frac{1+|z|}{z}$ ($|z| > 0$), we find that the above is written as

$$(1.14) \qquad \int_0^T \phi(s)dW(s) + \int_0^T \int_{\mathbf{R}} \psi(s, z)\tilde{N}(dsdz).$$

Therefore any element of ${\mathbf{H_Z}}^{(p)}$ and hence any element X of $\mathbf{H_Z}$ with mean 0 is written as the above. Now taking the conditional expectation of (1.14), we obtain the representation (1.10) for square integrable martingale $M(t) = E[X|\mathcal{F}_t]$.

The extension to locally square integrable martingales will be obvious. The extension to localmartingales will be discussed after Theorem 2.1 in the next section.

Let $M(t)$ and $N(t)$ be two locally square integrable martingales such that $M(0) = N(0) = 0$. Then by the Doob-Meyer decomposition of a supermartingale, there exist adapted continuous increasing processes $\langle M\rangle_t$, $\langle N\rangle_t$ and an adapted continuous process of bounded variations $\langle M, N\rangle_t$ such that $\langle M\rangle_0 = \langle N\rangle_0 = \langle M, N\rangle_0 = 0$ and $M(t)^2 - \langle M\rangle_t$, $N(t)^2 - \langle N\rangle_t$ and $M(t)N(t) - \langle M, N\rangle_t$ are localmartingales. Such bracket processes are uniquely determined. Note that $\langle M, M\rangle_t = \langle M\rangle_t$ by the definition. If $M(t)$ is represented by (1.10) with $M(0) = 0$ and $N(t)$ is represented with the kernel $(\tilde{\phi}, \tilde{\psi})$, then we have the formula

$$(1.15) \quad \langle M, N\rangle_t = \int_0^t (\phi(s), \tilde{\phi}(s))ds + \int_0^t \int_{\mathbf{R}^m} \psi(s, z)\tilde{\psi}(s, z)\hat{N}(dsdz).$$

We can define the quadratic co-variation of two semimartingales $X(t)$ and $Y(t)$ by

$$(1.16) \qquad [X, Y]_t = \exists \lim_{|\Delta|\to 0} \sum_{k=1}^{n} (X(t_k) - X(t_{k-1}))(Y(t_k) - Y(t_{k-1})),$$

where Δ are partitions of the time interval $[0, t]$ such that $0 = t_0 < t_1 < \cdots t_n = t$ and $|\Delta| = \max_{1\le k\le n} |t_k - t_{k-1}|$. We set $[X]_t = [X, X]_t$. If $M(t)$ and $N(t)$ are continuous localmartingales, it is known that the bracket process $\langle M, N\rangle_t$ and the quadratic co-variation coincides, i.e., $[M, N]_t = \langle M, N\rangle_t$. However if both $M(t), N(t)$ have jumps, the bracket process is not equal to the quadratic variation. In the case where the representation kernels of $M(t)$ and $N(t)$ are (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$, respectively, we have

$$(1.17)\quad [M,N]_t = \int_0^t (\phi(s), \tilde{\phi}(s))ds + \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{\psi}(s,z)N(dsdz).$$

Two locally square integrable martingales $M(t)$ and $N(t)$ are called *orthogonal* if $M(t)N(t)$ is a localmartingale or equivalently, the bracket process $\langle M, N\rangle_t$ is identically 0. By the formula (1.15) we see that the continuous local martingale $\sum_{i=1}^m \int_0^t \phi_i(s)dW^i(s)$ and the discontinuous one $\int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}(dsdz)$ are orthogonal.

Suppose that $M(t)$ and $N(t)$ are not orthogonal. There exists a predictable process $\varphi(t)$ such that $\langle M, N\rangle_t = \int_0^t \varphi(s)d\langle N\rangle_s$. We define new locally square integrable martingales by $M^1(t) = \int_0^t \varphi(s)dN(s)$ and $M^2(t) = M(t) - M^1(t)$. Then $M^1(t)$ and $M^2(t)$ are orthogonal each other because of the equality

$$\langle M^1, M^2\rangle_t = \langle M^1, M\rangle_t - \langle M^1, M^1\rangle_t = \int_0^t \varphi(s)^2 d\langle N\rangle_s - \int_0^t \varphi(s)^2 d\langle N\rangle_s = 0.$$

The locally square integrable martingale $M^1(t)$ is called the *orthogonal projection* of $M(t)$ to $N(t)$. In the case where $M(t)$ and $N(t)$ are represented with kernels (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$, respectively, the kernel φ of the orthogonal projection is given by

$$(1.18)\qquad \varphi(t) = \frac{(\phi(t), \tilde{\phi}(t)) + \int_{\mathbf{R}^m} \psi(t,z)\tilde{\psi}(t,z)\nu(dz)}{|\tilde{\phi}(t)|^2 + \int_{\mathbf{R}^m} \tilde{\psi}(t,z)^2\nu(dz)}.$$

We denote by $\mathcal{M}^2_{loc}$ (resp. $\mathcal{M}^1_{loc}$) the set of all locally square integrable martingales (resp. localmartingales) $M(t)$ with $M(0) = 0$. It is a vector space. A sequence $\{M_k(t), k = 1, 2, ..\}$ of $\mathcal{M}^2_{loc}$ is said to converge to $M(t)$ if there exists a nondecreasing sequence of stopping times $\tau_n, n = 1, ..$ such that $P(\tau_n < T) \to 0$ as $n \to \infty$ and each sequence of stopped processes $\{M_k^{\tau_n}(t) := M_k(t \wedge \tau_n), k = 1, 2, ...\}$ converges to $M^{\tau_n}(t)$ in L^2. Then $\mathcal{M}^2_{loc}$ is a complete space by this topology.

For a given $M(t) \in \mathcal{M}^2_{loc}$, we set

$$(1.19)\qquad \mathcal{L}(M) = \left\{ \int_0^t \varphi(s)dM(s);\ \varphi(s) \in \Phi(\langle M\rangle) \right\},$$

where $\Phi(\langle M\rangle)$ is the set of all predictable processes φ such that $\int_0^T |\varphi(t)|^2 d\langle M\rangle_t < \infty$ a.s. It is a subset of $\mathcal{M}^2_{loc}$. Let $\mathcal{N}$ be a subset of $\mathcal{M}^2_{loc}$. It is called a subspace of $\mathcal{M}^2_{loc}$ if it is a closed vector space including $\mathcal{L}(N)$ whenever $N \in \mathcal{N}$.

Given a subset $\mathcal{N}$ of $\mathcal{M}^2_{loc}$, we denote by $\mathcal{L}(N)$ the smallest closed subspace containing the set $\mathcal{N}$. We denote by $\mathcal{N}^\perp$ the set of all $M(t) \in \mathcal{M}^2_{loc}$ which is orthogonal to any $N \in \mathcal{N}$. Then $\mathcal{N}^\perp$ is a closed subspace of $\mathcal{M}^2_{loc}$. Further, if $\mathcal{N}$ is a closed subspace of $\mathcal{M}^2_{loc}$, every $M(t) \in \mathcal{M}^2_{loc}$ is decomposed uniquely to the sum of $M^1(t) \in \mathcal{N}$ and $M^2(t) \in \mathcal{N}^\perp$. We have thus the orthogonal decomposition

$$\mathcal{M}^2_{loc} = \mathcal{N} \oplus \mathcal{N}^\perp. \tag{1.20}$$

§2. Exponential representation of positive martingales and extension of Girsanov's theorem

We shall consider the exponential representation of a positive localmartingale. Here a localmartingale α_t is called positive if $\alpha_t > 0$ holds for all $t \in [0,T]$ a.s. For a predictable process $g(t,z)$, we set $g_1 = g1_{|g|>1}$ and $g_2 = g1_{|g|\le 1}$ as before. $g(s,z)$ is said to belong to $\Psi_{e,2}(\hat{N})$ if $e^{g_1(t,z)} - 1 \in \Psi_1(\hat{N})$ and $g_2 \in \Psi_2(\hat{N})$. Then it holds that $g \in \Psi_{e,2}(\hat{N})$ if and only if $\psi \equiv e^g - 1 \in \Psi_{1,2}(\hat{N})$.

Theorem 2.1. *(c.f. [6], Theorem 6.1) Let α_t be a positive localmartingale such that $\alpha_0 = 1$. Then there exists a pair of predictable process $f(t) = (f_1(t), ..., f_m(t))$ of Φ and $g(s,z)$ of $\Psi_{e,2}$ such that the localmartingale α_t is represented by*

(2.1)
$$\alpha_t = \exp\Big\{ \Big(\int_0^t (f(s), dW(s)) - \frac{1}{2}\int_0^t |f(s)|^2 ds \Big) + \Big(N_t(g_1) - \hat{N}_t(e^{g_1} - 1) \Big) + \Big(\tilde{N}_t(g_2) - \hat{N}_t(e^{g_2} - 1 - g_2) \Big) \Big\}.$$

Further, the pair (f, g) is uniquely determined from α_t.

Conversely let $(f(t), g(t,z))$ be a pair of predictable processes belonging to Φ and $\Psi_{e,2}(\hat{N})$, respectively. Define α_t by (2.1). Then it is a positive localmartingale.

The above α_t is characterized as the solution of the following Itô's stochastic differential equation starting from 1 at time 0:

$$d\alpha_t = \alpha_{t-}(f(t), dW(t)) + \alpha_{t-} \int_{\mathbf{R}^m} \Big(e^{g(t,z)} - 1 \Big) \tilde{N}(dtdz). \tag{2.2}$$

In fact, apply Itô's formula ([6], Theorem 5.1) to the function $F(x) = e^x$ and $X(t) = \log \alpha_t$, where α_t is given by (2.1). Note the obvious formula $e^g - 1 = (e^{g_1} - 1) + (e^{g_2} - 1)$. Then we find that α_t satisfies the above SDE. It is determined by two integrands $f(t)$ and $g(t, z)$. We denote the positive localmartingale by $\alpha_t = \alpha_t(f, g)$.

The above theorem is proved in [6] in the case where α_t is a multiplicative functional of a Markov process. We give here a direct and simpler proof.

Lemma 2.2. *(cf [6], Lemma 6.1.) Let $f(t)=(f_1(t), ..., f_m(t)), g(t,z)$, $h(t, z)$ be predictable processes such that $f \in \Phi$, h is bounded belonging to $\Psi_2(\hat{N})$, $gh = 0$ and $A(t)$ is a right continuous predictable process of bounded variation. Set*

$$\beta_t = \exp\left\{\int_0^t (f(s), dW(s)) + N_t(g) + \tilde{N}_t(h) - A(t)\right\}. \tag{2.3}$$

Then β_t is a localmartingale if and only if the following two conditions are satisfied.

$$e^g - 1 \in \Psi_1(\hat{N}), \tag{2.4}$$

$$A(t) = \frac{1}{2}\int_0^t |f(s)|^2 ds + \hat{N}_t(e^g - 1) + \hat{N}_t(e^h - 1 - h). \tag{2.5}$$

Proof. By Itô's formula, we have

$$\begin{aligned}\beta_t - 1 = \int_0^t \beta_{s-}(f(s), dW(s)) + \int_0^t \int_{\mathbf{R}^m} \beta_{s-}(e^g - 1)dN \\ + \int_0^t \int_{\mathbf{R}^m} \beta_{s-}(e^h - 1)d\tilde{N} \\ + \frac{1}{2}\int_0^t \beta_{s-}|f(s)|^2 ds + \int_0^t \int_{\mathbf{R}^m} \beta_{s-}(e^h - 1 - h)d\hat{N} - \int_0^t \beta_{s-}dA(s).\end{aligned}$$

Therefore if (2.4) and (2.5) are satisfied, then

$$\begin{aligned}\beta_t - 1 = \int_0^t \beta_{s-}(f(s), dW(s)) + \int_0^t \int_{\mathbf{R}^m} \beta_{s-}(e^g - 1)d\tilde{N} \\ + \int_0^t \int_{\mathbf{R}^m} \beta_{s-}(e^h - 1)d\tilde{N}.\end{aligned}$$

Therefore β_t is a localmartingale.

Conversely suppose that β_t is a localmartingale. We want to prove (2.4). Set $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$. Then $g = g^+ - g^-$. We shall prove first $e^{-g^-} - 1 \in \Psi_1(\hat{N})$. It holds by Itô's formula

$$e^{-N_t(g^-)} - 1 = -\int_0^t (1 - e^{-g^-})e^{-N_{s-}(g^-)} dN.$$

Since $-N_t(g^-) \leq 0$, the expectation of the above is finite and is equal to $-E[\int_0^t (1 - e^{-g^-})e^{-N_{s-}(g^-)}\hat{N}(dsdz)]$. Therefore, $(1 - e^{-g^-})e^{-N_{s-}(g^-)} \in \Phi_1(\hat{N})$ and this implies $(1 - e^{-g^-}) \in \Psi_1(\hat{N})$. Next, we have by Itô's formula,

$$\begin{aligned}\int_0^t \frac{d\beta_s}{\beta_{s-}} &= \int_0^t (f(s), dW(s)) + N_t(e^g - 1) + \tilde{N}_t(e^h - 1) \\ &+ \frac{1}{2}\int_0^t |f(s)|^2 ds + \hat{N}_t(e^h - 1 - h) - A(t).\end{aligned}$$

The left hand side is a localmartingale. All terms except $N_t(e^g - 1)$ of the right hand side are locally integrable. Further we have $N_t(e^g - 1) = N_t(e^{g^+} - 1) + N_t(e^{-g^-} - 1)$ and the last term is locally integrable. Then $N_t(e^{g+} - 1)$ should be locally integrable, which shows that $\int_0^t (e^{g^+} - 1)d\hat{N}$ is also locally integrable, proving that $e^{g^+} - 1 \in \Psi_1(\hat{N})$. We have thus proved (2.4).

Now since (2.4) holds, the bounded variation part of $\beta_t - 1$ can be written as

$$\frac{1}{2}\int_0^t \beta_{s-}|f(s)|^2 ds + \int_0^t \beta_{s-}(e^g - 1)d\hat{N} + \int_0^t \beta_{s-}(e^h - 1 - h))d\hat{N} - \int_0^t \beta_{s-}dA(s).$$

It should be 0 since β_t is a localmartingale. This implies (2.5).

Proof of Theorem 2.1. Suppose that α_t is a positive localmartingale. Set $X(t) = \log \alpha_t$. It is a semimartingale. Consider

$$P^n(t) = \sum_{s \leq t, 1 \leq |\Delta X(s)| \leq n} \Delta X(s), \quad \text{(finite sum)}.$$

It is a locally integrable process of bounded variation. There exists a continuous process of boundecd variation $C^n(t)$ such that $M^n(t) = P^n(t) - C^n(t)$ is a locally square integrable martingale by Doob-Meyer decomposition. Then there exists $\psi_n \in \Psi_2(\hat{N})$ such that $M^n(t) = \tilde{N}(\psi_n)$ by Theorem 1.1 (for locally square integrable martingales). Jump parts of $P^n(t)$ and $N_t(\psi_n)$ coincide. Therefore we have $P^n(t) = N_t(\psi_n)$. It holds

$\psi_m = \psi_n 1_{|\psi_n|\le m}$ a.e. $\hat{N} \times P$ for any $m < n$. Then there exists ψ such that $\psi_n = \psi 1_{|\psi|\le n}$ and we have

$$N_t(\psi) = \sum_{s\le t, 1\le|\Delta X(s)|<\infty} \Delta X(s), \quad \text{(finite sum)}.$$

Now set $Y(t) = X(t) - N_t(\psi)$. It is a semimartingale such that $|\Delta Y(s)| \le 1$. Therefore it is a special semimartingale. Then it is decomposed uniquely to the sum of a martingale $M(t)$ and a predictable process of bounded variation, denoted by $B(t)$. Further $M(t)$ is locally square integrable so that it is written as $M(t) = \int_0^t (f(s), dW(s)) + \tilde{N}_t(\eta)$, where $f \in \Phi$ and $\eta \in \Psi_2(\hat{N})$. It holds $\psi\eta = 0$ since $N_t(\psi)$ and $Y(t)$ do not have common jumps. Then we get the decomposition:

$$\alpha_t = \exp\left\{\int_0^t (f(s), dW(s)) + N_t(\psi) + \tilde{N}_t(\eta) + B(t)\right\}. \tag{2.6}$$

Since α_t is a localmartingale, we have $e^{\psi} - 1 \in \Psi(\hat{N})$ and

$$-B(t) = \frac{1}{2}\int_0^t |f(s)|^2 ds + \hat{N}_t(e^{\psi} - 1) + \hat{N}_t(e^{\eta} - 1 - \eta)$$

by the previous lemma. Now set $g = \psi + \eta$. Then we have $g_1 = \psi$ and $g_2 = \eta$. Therefore we get the formula (2.1). The proof is complete.

Proof of Theorem 1.1 (continued). Let $M(t)$ be a martingale. We set $M^+ = M(T)\vee 0$ and $M^- = (-M)\vee 0$ and define $M_1(t) = E[M^+|\mathcal{F}(t)]$ and $M_2(t) = E[M^-|\mathcal{F}(t)]$. Then both are nonnegative martingales and $M(t) = M_1(t) - M_2(t)$. We consider positive martingales $M_{i,\epsilon}(t) = M_i(t) + \epsilon$ ($\epsilon > 0$). These are represented by $M_{i,\epsilon}(t) = M_{i,\epsilon}(0)\alpha_t^i$, where $\alpha_t^i = \alpha_t(f_i', g_i')$ are exponential martingales. These satisfy SDE (2.2). Now set $\phi_i(t) = \alpha_{t-}^i f_i'(t)$ and $\psi_i(t,z) = \alpha_{t-}^i(e^{g_i'(t,z)} - 1)$. Then, since $\sup_t \alpha_t < \infty$ a.s., $\phi_i \in \Phi$ and $\psi_i \in \Psi_{1,2}(\hat{N})$. Further, we get the representation (1.10) for $M_{i,\epsilon}(t), i = 1, 2$. Thus we get the representation (1.10) where $\phi \in \Phi$ and $\psi \in \Psi_{1,2}(\hat{N})$.

Let α_t be a positive martingale with mean 1. We can define a probability measure Q by the formula

$$Q(A) = \int_A \alpha_T dP, \quad A \in \mathcal{F}. \tag{2.7}$$

Then $((\mathcal{F}_t), Q)$ and $((\mathcal{F}_t), P)$ are equivalent (mutually absolutely continuous). Conversely let $((\mathcal{F}_t), Q)$ be a probability measure equivalent to $((\mathcal{F}_t), P)$. Let α_t be the Radon-Nikodym density of $(\mathcal{F}_t, Q)$ with respect to $(\mathcal{F}_t, P)$. Then the stochastic process $\{\alpha_t, t \in [0,T]\}$ is a positive martingale with respect to P. Therefore it can be represented as $\alpha_t = \alpha_t(f,g)$.

A localmartingale with respect to $((\mathcal{F}_t), Q)$ is called a Q-localmartingale. The following is an extension of Girsanov's theorem.

Theorem 2.3. *(c.f. [6], Theorem 6.2) With respect to $((\mathcal{F}_t), Q)$, we have*
1) $W^f(t) := W(t) - \int_0^t f(s)ds$ is a standard Brownian motion.
2) The compensator of N is $\hat{N}^g(dsdz) = e^{g(s,z)}ds\nu(dz)$, that is $\tilde{N}^g(dsdz) := N(dsdz) - \hat{N}^g(dsdz)$ is a martingale measure. Further if ψ belongs to $\Psi_{1,2}(\hat{N}^g)$, the stochastic integral

$$\tilde{N}_t^g(\psi) := \int_0^t \int_{\mathbf{R}^m} \psi(s,z)\tilde{N}^g(dsdz) \tag{2.8}$$

is well defined as a Q-localmartingale.

3) Let $X(t)$ be a Q-localmartingale. Then there exists a pair of predictable processes $(\phi(t), \psi(t,z))$ belonging to Φ and $\Psi_{1,2}(\hat{N}^g)$, respectively and $X(t)$ is represented by

$$X(t) = X(0) + \int_0^t (\phi(s), dW^f(s)) + \tilde{N}_t^g(\psi). \tag{2.9}$$

Remark. 1) $N(dtdz)$ is no longer a Poisson random measure with respect to Q unless g is a deterministic function.
2) Set $\mathcal{F}_t(f,g) = \sigma(W_s^f, \tilde{N}^g(dsdz); s \le t)$. Then it holds $\mathcal{F}_t(f,g) \subset \mathcal{F}_t$. The equality does not hold in general. The representation (2.9) is valid for localmartingale with respect to the filtration $(\mathcal{F}_t)$ but it is not clear if we have the similar representations for localmartingales with respect to the filtration $(\mathcal{F}_t(f,g))$.

Proof. The assertions (1) and (2) are shown in [6] in the case where α_t is a multiplicative functional of a Markov process. We give here an alternative proof. We first show that $W^f(t)$ is a Q-localmartingale. Set $X(t) = W^f(t)$. Then $X(t)$ is a Q-localmartingale if and only if the product $X(t)\alpha_t$ is a P-localmartingale. Note the equality

$$X(t)\alpha_t = \int_0^t X(s-)d\alpha_s + \int_0^t \alpha_{s-}dX(s) + [X,\alpha]_t. \tag{2.10}$$

The first term of the right hand side is a P-localmartingale. Since $[X,\alpha]_t = \langle X,\alpha\rangle_t = \int_0^t \alpha_{s-}f(s)ds$, we have $\int_0^t \alpha_{s-}dX(s) + [X,\alpha_t] = \int_0^t \alpha_{s-}dW(s)$, which is also a P-localmartingale. Therefore $X(t)\alpha_t$ is a

P-localmartingale or equivalently $X(t)$ is a continuous Q-localmartingale. It holds $[X]_t = [W]_t = t$, since the quadratic variation of $\int_0^t f(s)ds$ is 0. Hence $X(t) = W^f(t)$ is a Brownian motion with respect to Q.

We will next prove (2). Suppose first that $\psi(t,z)$ is bounded and $\int_0^T |\psi| d\hat{N} < \infty$ is satisfied. Then it holds valid $\int_0^T |\psi| e^g d\hat{N} < \infty$, since $g \in \Psi_{e,2}(\hat{N})$. Then $X(t) := \tilde{N}_t^g(\psi)$ is decomposed as $X(t) = \tilde{N}_t(\psi) - \int_0^t \int \psi(e^g - 1)d\hat{N}$. It holds (2.10) again. We have

$$\begin{aligned}\int_0^t \alpha_{s-} dX(s) + [X, \alpha]_t = &\int_0^t \alpha_{s-} \psi d\tilde{N} \\ &- \int_0^t \int_{\mathbf{R}^m} \alpha_{s-}\psi(e^g - 1)d\hat{N} + \int_0^t \int_{\mathbf{R}^m} \alpha_{s-}\psi(e^g - 1)dN,\end{aligned}$$

which is a P-localmartingale. Consequently $X(t)\alpha_t$ is again a P-localmartingale, proving that $X(t) = \tilde{N}_t^g(\psi)$ is a Q-localmartingale. It can be extended to any $\psi \in \Psi_{1,2}(\hat{N}^g)$.

We will prove (3). Suppose first that $X(t)$ is a Q-localmartingale such that its jumps are bounded. Then $M(t) := X(t)\alpha_t$ is a P-localmartingale. Since α_t^{-1} is a P-semimartingale, the product $X(t) = M(t)\alpha_t^{-1}$ is a P-semimartingale. Note that jumps of $X(t)$ are bounded. Then $X(t)$ is a P special semimartingale. Then it is decomposed uniquely as $X(t) - X(0) = N(t) + A(t)$, where $N(t)$ is a P locally square integrable martingale and $A(t)$ is a right continuous predictable process of bounded variation. Now, $N(t)$ is represented by $\int \phi dW + \int \psi d\tilde{N}$, where ψ is a bounded predictable process. Then we can rewrite $X(t)$ as

$$\begin{aligned}(2.11)\quad X(t) =& X(0) + \int_0^t (\phi(s), dW^f(s)) + N_t^g(\psi) \\ &+ \left\{ \int_0^t (\phi(s), f(s))ds + \int_0^t \int_{\mathbf{R}^m} \psi(e^g - 1)d\hat{N} + A(t) \right\}.\end{aligned}$$

The first and the second integrals of the right hand side are both Q-localmartingales. The last term $\{\cdots\}$ is a right continuous predictable process of bounded variation, which should be 0, since $X(t)$ is a Q-martingale. Therefore we get the representation of $X(t)$.

The representation can be extended to any Q locally square integrable martingale. Finally, the representation is valid for any Q-localmartingale. This can be verified through getting the exponential representation of positive Q-localmartingale similarly as in Theorem 2.1.

It is an interesting problem to find a condition for (f,g) which ensures that the localmartingale $\alpha_t(f,g)$ is a martingale. We give here a sufficient condition.

Theorem 2.4. *Suppose that $f \in \Phi$ and $g \in \Psi_{e.2}(\hat{N})$ satisfy*

$$(2.12)\quad E\Big[\exp\Big\{\int_0^T \Big((1+\epsilon)|f|^2 + \int_{g^+>\delta} e^{2(1+\epsilon)g^+}d\nu + 2(1+\epsilon)e^{2(1+\epsilon)\delta}\int_{|g|\le\delta} g^2 d\nu\Big)ds\Big\}\Big] < \infty,$$

for some $\epsilon > 0$ and $\delta > 0$, where $g^+ = \max(g, 0)$. Then $\alpha_t(f,g)$ is a martingale.

In particular, $\alpha_t(f,g)$ is a martingale if 1) $\int_0^T |f(s)|^2 ds$ is bounded a.s. and 2) $g^+, \hat{N}(g^+ > 1)$ and $\int_0^T \int_{|g|\le 1} |g|^2 \hat{N}(dsdz)$ are bounded a.s.

Proof. Let $\tau_n, n = 1,2,...$ be an increasing sequence of stopping times such that $P(\tau_n < T) \to 0$ as $n \to \infty$ and each stopped process $\alpha_{t\wedge\tau_n}$ is a martingale with mean 1. We want to prove that the above sequence of random variables (t is fixed) is uniformly integrable. If this property is verified, the limit process α_t is also a martingale. For this purpose it is sufficient to prove that $\sup_n E[\alpha_{t\wedge\tau_n}^p] < \infty$ holds for some $p > 1$. By a direct computation we can show that $\sup_n E[\alpha_{t\wedge\tau_n}^{1+\epsilon}] < \infty$, under the condition (2.12). Details are omitted.

§3. Processes with jumps and equivalent martingale measures

Let $\sigma(t) = (\sigma_j^i(t))$ be a $d \times m$ matrix valued predictable process, $b(t) = (b^i(t))$ be a d-vector predictable process and $v(t,z) = (v^i(t,z))$ be a d-vector predictable process continuous in $z \in \mathbf{R}^m$, which satisfy the integrability condition

$$(3.1)\quad \int_0^T |\sigma(t)|^2 + |b(t)|dt < \infty, \quad \int_0^T \int_{|z|\le 1} |v(s,z)|^2 ds\nu(dz) < \infty.$$

We shall consider a d-dimensional stochastic process with jumps defined by

$$(3.2)\quad \xi_t = \int_0^t \sigma(t)dW(t) + \int_0^t b(t)dt + \int_0^t \int_{|z|\le 1} v(t,z)\tilde{N}(dtdz) + \int_0^t \int_{|z|>1} v(t,z)N(dtdz),$$

where $W(t)$ is a m-dimensional standard Brownian motion and $N(dtdz)$ is a Poisson counting measure on $[0,T] \times \mathbf{R}^m$. To make the problem simple, we assume $d = m$ in this paper.

An equivalent probability measure Q such that ξ_t is a d-vector localmartingale with respect to $((\mathcal{F}_t), Q)$ is called an *equivalent martingale measure.* We denote by Γ the set of all equivalent martingale measures and by $\tilde{\Gamma}$ the set of all (f,g) such that $\alpha_t(f,g)dP \in \Gamma$. We shall characterize all equivalent martingale measures of a given process ξ_t by means of the pair (f,g).

Theorem 3.1. *Let $((\mathcal{F}_t), Q)$ be an equivalent probability measure and let $\alpha_t(f,g)$ be the density such that $dQ = \alpha_t(f,g)dP$, where $f \in \Phi$ and $g \in \Psi_{e,2}(\hat{N})$. Then the stochastic process ξ_t defined by (3.2) is a Q-localmartingale if and only if $v(s,z)(e^{g(s,z)} - 1_{\{|z|\le 1\}}) \in \Psi_1(\hat{N})$ and*

$$b(s) + \sigma(s)f(s) + \int_{\mathbf{R}^m} v(s,z)(e^{g(s,z)} - 1_{\{|z|\le 1\}})\nu(dz) = 0, \tag{3.3}$$

a.e. $\lambda \otimes P$, where λ is the Lebesgue measure.

Proof. In vector notation, we have by (2.2) and (3.2),

$$\begin{aligned}\xi_t\alpha_t &= \int_0^t \xi_{s-}d\alpha_s + \int_0^t \alpha_{s-}d\xi_s + [\xi,\alpha]_t \\ &= \text{a localmartingale} + \int_0^t \alpha_{s-}b(s)ds + \int_0^t \alpha_{s-}\sigma(s)f(s)ds \\ &\quad + \int_0^t \int_{\mathbf{R}^m} \alpha_{s-}v(s,z)(e^{g(s,z)} - 1_{\{|z|\le 1\}})N(dsdz).\end{aligned}$$

If it is a d-vector localmartingale, the integrand with respect to $N(dsdz)$ should be integrable with respect to $\hat{N}(dsdz)$ and the equality

(3.4)
$$\alpha_{s-}b(s) + \alpha_{s-}\sigma(s)f(s) + \alpha_{s-}\int_{\mathbf{R}^m} v(s,z)(e^{g(s,z)} - 1_{\{|z|\le 1\}})\nu(dz) = 0,$$

holds a.e. (Theorem 1.1). Then we have (3.3), since $\inf_s \alpha_{s-} > 0$ a.s. The converse will be shown similarly. The proof is complete.

An equivalent martingale measure $Q^0 = \alpha_t(f^0, g^0)dP$ is said to be standard if ξ_t is a locally square integrable martingale with respect to Q^0. We will show the existence of such an equivalent martingale measure.

Lemma 3.2. *Assume that $\sigma(t)$ is invertible and $\sigma(t)^{-1}$ and $b(t)$ are bounded a.e. $\lambda \otimes P$. Then there exists a standard equivalent martingale measure. Further, for any given pair of $\phi \in \Phi$ and $\psi \in \Psi_{1,2}(\hat{N})$, there* *exists a standard equivalent martingale measure* $Q^0 = \alpha_t(f^0, g^0)dP$ *such that*

$$M^{Q^0}(t) := \int_0^t (\phi(s), dW^{f^0}(s)) + \tilde{N}_t^{g^0}(\psi) \tag{3.5}$$

is well defined as a locally square integrable martingale with respect to Q^0.

Proof. We will show that there exists a predictable pair $(f^0(s), g^0(s,z))$ of $\tilde{\Gamma}$ satisfying

$$\int_0^T \int_{\mathbf{R}^m} \{|v(s,z)|^2 + |\psi(s,z)|^2\} e^{g^0(s,z)} ds\nu(dz) < \infty. \tag{3.6}$$

For each $s \in [0,T]$, set $E(s) = E(s,\omega) = \{z : |\psi(s,z)| > 1\} \cup \{|z| > 1\}$. Then $\int_0^T \nu(E(s))ds < \infty$. Take first a nonpositive predictable process $g'(s,z)$ supported by $E(s)$, $|g'(s,z)| > 1$ on $E(s)$ and

$$\int_{E(s)} \{1 + |v(s,z)| + |v(s,z)|^2 + |\psi(s,z)|^2\} e^{g'(s,z)} \nu(dz)$$

is bounded in (s,ω) a.e. Take next a bounded predictable process $g''(s,z)$ supported by $E(s)^c$, $|g''(s,z)| < 1$ and $\int_{E(s)^c} (1 + |v(s,z)|)|g''(s,z)|\nu(dz)$ is bounded in (s,ω) a.e. Define $g^0 = g' 1_E + g'' 1_{E^c}$. Then $g^0 \in \Psi_{e,2}(\hat{N})$ and

$$\int_0^T \int_{E(s)} (|v|^2 + |\psi|^2) e^{g^0} ds\nu(dz) < \infty, \quad a.s.$$

Since $\int_0^T \int_{|z|\le 1} |v|^2 ds\nu(dz) < \infty$ and $\int_0^T \int_{\mathbf{R}^m} |\psi_2|^2 ds\nu(dz) < \infty$ for $\psi_2 = \psi 1_{|\psi|\le 1}$, we have $\int_0^T \int_{E(s)^c} (|v|^2 + |\psi|^2) e^{g^0} ds\nu(dz) < \infty$, *a.s.* Therefore (3.6) is satisfied.

The process $a(s) = \int_{\mathbf{R}^m} v(s,z)(e^{g^0(s,z)} - 1_{\{|z|\le 1\}})\nu(dz)$ is well defined since

(3.7)

$$\begin{aligned}
|a(s)| \le \int_{E(s)} &|v(s,z)(e^{g'(s,z)} - 1_{|z|\le 1})|\nu(dz) \\
&+ \int_{E(s)^c} |v(s,z)(e^{g''} - 1_{|z|\le 1})|\nu(dz) \\
\le \int_{E(s)} &|v| e^{g'} \nu(dz) + \int_{E(s)^c} |v||g''|\nu(dz) \quad \text{bounded in } (s,\omega) \textit{ a.e.}
\end{aligned}$$

Then we can define $f^0(s)$ by $b(s) + \sigma(s)f^0(s) + a(s) = 0$. The pair (f^0, g^0) satisfies (3.3). Further, it satisfies conditions (1),(2) of Theorem 2.4. Indeed, we took g^0 so that it satisfies (2). By the estimation (3.7), $|a(s)|$ is bounded a.s. Since $|b(s)|$ is bounded and $\sigma(s)\sigma(s)^T$ is uniformly positive definite, $|f^0(s)|$ is also bounded a.s. Thus $f^0(s)$ satisfies (1) of the theorem. Then $\alpha_t(f^0, g^0)$ is a martingale.

Let $Q^0 = \alpha_T(f^0, g^0)dP$. We will show that ξ_t is a locally square integrable martingale with respect to Q^0. Observe (3.2) and (3.3). Then $\xi_t = (\xi_t^1, ..., \xi_t^d)$ is written as

$$\xi_t = \int_0^t \sigma(s)dW^{f^0}(s) + \int_0^t \int_{\mathbf{R}^m} v(s,z)\tilde{N}^{g^0}(dsdz).$$

The bracket process with respect to Q^0 is given by a $d \times d$ matrix

$$\left(\langle \xi^i, \xi^j \rangle_t^{Q^0}\right) = \int_0^t \sigma(s)\sigma(s)^T ds + \int_0^t \int_{\mathbf{R}^m} v(s,z)v(s,z)^T \hat{N}^{g^0}(dsdz).$$

It is finite a.s. This proves that ξ_t is a locally square integrable martingale.

Finally, $M^{Q^0}(t)$ of (3.5) is well defined as a locally square integrable martingale, because $\psi \in \Psi_2(\hat{N}^{g^0})$ by (3.6). The proof is complete.

We will fix the equivalent martingale measure Q^0 of Lemma 3.2. Set $\hat{f} = f - f^0$, $\hat{g} = g - g^0$. Then $\alpha_t(f,g)$ of Theorem 3.1 is decomposed to the product of two exponential semimartingales;

$$\alpha_t(f,g) = \alpha_t(f^0, g^0)\alpha_t^0(\hat{f}, \hat{g}), \tag{3.8}$$

(3.9)

$$\alpha_t^0(\hat{f},\hat{g}) = \exp\left\{\int_0^t (\hat{f}(s), dW^{f^0}(s)) - \frac{1}{2}\int_0^t |\hat{f}(s)|^2 ds \right.$$
$$\left. + N_t(\hat{g}_1) - \hat{N}_t^{g^0}(e^{\hat{g}_1} - 1) + \tilde{N}_t^{g^0}(\hat{g}_2) - \hat{N}_t^{g^0}(e^{\hat{g}_2} - 1 - \hat{g}_2) \right\}.$$

Since $dQ = \alpha_T(f,g)dP$ and $dQ^0 = \alpha_T(f^0, g^0)dP$, we have $dQ = \alpha_T^0(\hat{f}, \hat{g})dQ^0$. Hence Q is an equivalent martingale measure with respect to Q^0 and $\alpha_t^0(\hat{f}, \hat{g})$ is its density process. Then ξ_t, $\alpha_t^0(\hat{f}, \hat{g})$ and $\xi_t\alpha_t^0(\hat{f}, \hat{g})$ are all localmartingales with respect to Q^0.

Conversely if Q is an equivalent martingale measure with respect Q^0. The density process α_t^0 of Q with respect to Q^0 is represented by (3.9). We denote by $\hat{\Gamma}^0$ the set of all such density processes α_t^0 and by

$\hat{\Gamma}_2^0$ is the set of $\alpha_t^0 \in \hat{\Gamma}^0$ such that these are all locally square integrable martingales with respect to Q^0.

Let $\mathcal{M}_{loc}^2(Q^0)$ be the set of all locally square integrable martingales $M(t)$ with $M(0) = 0$ with respect to Q^0. Then the d-vector process ξ_t belongs to $\mathcal{M}_{loc}^2(Q^0)$. Further, ξ_t and $\alpha_t^0 - 1$ are orthogonal with respect to Q^0, if α_t^0 is locally square integrable with respect to Q^0. We claim;

Lemma 3.3. *Assume that $\sigma(t)$ is invertible and $\sigma(t)^{-1}$ and $b(t)$ are bounded a.e. $\lambda \otimes P$. Let Q^0 be a standard equivalent martingale measure. Then, with respect to Q^0, we have the orthogonal decomposition of $\mathcal{M}_{loc}^2(Q^0)$.*

$$\mathcal{M}_{loc}^2(Q^0) = \mathcal{L}(\xi_t^1, ..., \xi_t^d) \oplus \mathcal{L}\left(\alpha_t^0 - 1; \alpha_t^0 \in \hat{\Gamma}_2^0\right). \tag{3.10}$$

Proof. Let $\mathcal{K} = \mathcal{L}(\xi_t^1, ..., \xi_t^d)^\perp$ and let M be any element of $\mathcal{K}$ represented by $M = \int(\phi, dW^{f^0}) + \tilde{N}^{g^0}(\psi)$. Since it is orthogonal to $\xi_t^1, ..., \xi_t^d$ with respect to Q^0,

$$\langle \xi^i, M\rangle_t^{Q_0} = \int_0^t (\sigma^i(s), \phi(s))ds + \int_0^t \left(\int v^i(s,z)\psi(s,z)e^{g^0}\nu(dz)\right) ds = 0,$$
$$i = 1, ..., d.$$

Therefore, setting $\phi(t) = (\phi^1(t), ..., \phi^d(t))$ and $v(t,z) = (v^1(t,z), ..., v^d(t,z))$, we get

$$\sigma(t)\phi(t) + \int_{\mathbf{R}^d} \psi(t,z)v(t,z)e^{g^0(t,z)}\nu(dz) = 0, \quad \forall t.$$

We will show that

$$\mathcal{H} := \left\{\int(\phi, dW^{f^0}) + \tilde{N}^{g^0}(\psi) \in \mathcal{K};\ \psi \text{ are bounded}\right\}$$

is dense in $\mathcal{K}$. Let $M = \int(\phi, dW^{f^0}) + \tilde{N}^{g^0}(\psi)$ be any element of $\mathcal{K}$. We define trancated functions by $\psi_n = (\psi \wedge n) \vee (-n)$. Next define d-vector functions by $\phi_n(t) = -\sigma(t)^{-1} \int \psi^n(t,z)v(t,z)e^{g^0(t,z)}\nu(dz)$. Then it holds

$$\sigma(t)\phi^n(t) + \int \psi^n(t,z)v(t,z)e^{g^0(t,z)}\nu(dz) = 0, \quad \forall t.$$

Therefore $M^n = \int(\phi^n, dW^{f^0}) + \tilde{N}^{g^0}(\psi^n)$ belongs to $\mathcal{K}$. Further, since $\int_0^T \int |\psi_n - \psi|^2 e^{g^0}\nu(dz)ds \to 0$ holds valid as $n \to \infty$, $\int_0^T |\phi^n - \phi|^2 ds \to 0$ as $n \to \infty$. Therefore the sequence $\{M^n\}$ converges to M with respect to the topology of $\mathcal{M}_{loc}^2(Q^0)$. We have thus shown that $\mathcal{H}$ is dense in $\mathcal{K}$.

Let $\mathcal{J}$ be the set of all $M \in \mathcal{K}$ which is bounded from the below. Then we have $\mathcal{L}(\mathcal{J}) = \mathcal{L}(\alpha_t^0 - 1; \alpha_t^0 \in \hat{\Gamma}_2^0\}$. Further it holds $\mathcal{L}(\mathcal{J}) \supset \mathcal{L}(\mathcal{H})$. Indeed, we have $\{M^\tau(t); M \in \mathcal{H}\} \subset \mathcal{L}(\mathcal{J})$, where τ are stopping times such that $M^\tau(t) := M(t \wedge \tau)$ are bounded localmartingales. We have thus proved

$$\mathcal{L}(\xi_t^1, \ldots, \xi_t^d)^\perp = \mathcal{K} = \mathcal{L}(\mathcal{H}) \subset \mathcal{L}(\mathcal{J}) = \mathcal{L}(\alpha_t^0 - 1; \alpha_t^0 \in \hat{\Gamma}_2^0).$$

The proof is complete.

We are now in a position of stating a main result of the paper.

Theorem 3.4. *Assume that $\sigma(t)$ is invertible and $\sigma(t)^{-1}$ and $b(t)$ are bounded a.e. $\lambda \otimes P$. If $X(t)$ is a supermartingale for any equivalent martingale measure Q, then it is represented by*

$$X(t) = X(0) + \int_0^t (\varphi(s), d\xi_s) - A(t). \tag{3.11}$$

Here, $A(t)$ is a predictable increasing process and $\varphi(s)$ is a predictable process such that $\sigma(s)\varphi(s) \in \Phi$ and $(\varphi(s), v(s,z)) \in \Psi_{1,2}(\hat{N}^g)$ for any $(f,g) \in \tilde{\Gamma}$.

If $X(t)$ is a localmartingale for any equivalent martingale measure, then it is represented by

$$X(t) = X(0) + \int_0^t (\varphi(s), d\xi_s). \tag{3.12}$$

Proof. For each $Q \in \Gamma$, the supermartingale $X(t)$ is decomposed as $X(0) + M^Q(t) - A^Q(t)$, where $M^Q(t)$ is a Q-localmartingale with $M^Q(0) = 0$ and $A^Q(t)$ is a natural (=predictable) increasing process, by Doob-Meyer decomposition. The Q-localmartingale $M^Q(t)$ is represented by $M^Q(t) = \int_0^t (\phi(s), dW^f(s)) + \tilde{N}_t^g(\psi)$. We will show that the kernel (ϕ, ψ) does not depend on the choice of Q. Let Q^* be another equivalent martingale measure. Then M^{Q^*} is represented by $M^{Q^*} = \int (\phi^*(s), dW^{f^*}(s)) + \tilde{N}^{g^*}(\psi^*)$. Since $M^Q(t) - A^Q(t) = M^{Q^*}(t) - A^{Q^*}(t)$, we have

$$\begin{aligned}\tilde{N}_t^{g^*}(\psi^*) - \tilde{N}_t^g(\psi) = \left(\int_0^t (\phi(s), dW^f(s)) - \int_0^t (\phi^*(s), dW^{f^*}(s)) \right)& \\ - \left(A^Q(t) - A^{Q^*}(t) \right)&.\end{aligned}$$

The right hand side is a predictable process, so that it has no common jumps with the Poisson random measure $N(dtdz)$. So both sides of the above can not have jumps. This shows $\psi = \psi^*$ a.e. $\hat{N} \otimes P$. Hence the right hand side should be a predictable process of bounded variation. Therefore $\int_0^t (\phi(s), dW^f(s)) - \int_0^t (\phi^*(s), dW^{f^*}(s))$ is also a predictable process of bounded variation, which shows $\phi = \phi^*$ a.e. $\lambda \otimes P$. We have $\phi \in \Phi$ and $\psi \in \Psi_{1,2}(\hat{N}^g)$ for any $(f, g) \in \tilde{\Gamma}$.

We want to prove $A^Q = A^{Q^0}$ in the case where both Q and Q^0 are standard equivalent martingale measures such that $\psi \in \Psi_2(\hat{N}^g) \cap \Psi_2(\hat{N}^{g^0})$. Comparing two equations for $M^Q(t)$ and $M^{Q^0}(t)$, $M^{Q^0}(t)$ can be written as

$$\begin{aligned} M^{Q^0}(t) =& M^Q(t) + \int_0^t (\phi(s), \hat{f}(s))ds \\ &+ \int_0^t \int_{\mathbf{R}^m} \psi(s,z)(e^{\hat{g}(s,z)} - 1)e^{g^0(s,z)} ds\nu(dz), \end{aligned}$$

where $\hat{f} = f - f^0$ and $\hat{g} = g - g^0$, because $\psi, e^{\hat{g}} - 1 \in \Psi_2(\hat{N}^{g^0})$. Therefore we have

$$A^Q(t) = \int_0^t \left\{ (\phi, \hat{f}) + \int_{\mathbf{R}^m} \psi(e^{\hat{g}} - 1)e^{g^0} d\nu \right\} ds + A^{Q^0}(t).$$

We claim

$$(\phi, \hat{f}) + \int_{\mathbf{R}^m} \psi(e^{\hat{g}} - 1)e^{g^0} d\nu = 0, \tag{3.13}$$

a.e. $\lambda \otimes P$, in the case where $g \leq g^0$ or equivalently $\hat{g} \leq 0$. If it is not the case, then either the set

$$F = \left\{ (s, \omega); (\phi, \hat{f}) + \int_{\mathbf{R}^m} \psi(e^{\hat{g}} - 1)e^{g^0} d\nu > 0 \right\} \quad \text{or}$$

$$F' = \left\{ (s, \omega); (\phi, \hat{f}) + \int_{\mathbf{R}^m} \psi(e^{\hat{g}} - 1)e^{g^0} d\nu < 0 \right\}$$

is of positive measure with respect to $\lambda \otimes P$. Suppose that F is of positive measure. We define (f', g') by $f' = f^0 - n\hat{f}1_F$ and $g' = g^0 + \log\{1 - n(e^{\hat{g}} - 1)1_F\}$. Then it holds $e^{\hat{g}'} - 1 = -n(e^{\hat{g}} - 1)1_F$, where $\hat{g}' = g' - g^0$. Set $\hat{f}' = f' - f^0$. Then $\alpha'_t := \alpha^0_t(\hat{f}', \hat{g}')$ is a positive localmartingale with respect to Q^0. Further $\xi_t \alpha'_t$ is a localmartingale with respect to Q^0. Indeed, equalities

$$\langle \xi^i_t, \alpha'_t \rangle^{Q^0}_t = -n \int_0^t \alpha'_{s-} 1_F \left\{ (\sigma^i, \hat{f}) + \int_{\mathbf{R}^m} v^i (e^{\hat{g}} - 1)e^{g^0} \nu(dz) \right\} ds = 0,$$
$$i = 1, ..., d$$

hold valid since ξ_t^i and $\alpha_t^0(\hat{f},\hat{g})$ are orthogonal with respect to Q^0. Let $\{\tau_k, k=1,2,...\}$ be an increasing sequence of stopping times such that $P(\tau_k < T) \to 0$ as $k \to \infty$ and each stopped process $\alpha'_{t\wedge\tau_k}$ is a Q^0-martingale. Define a sequence of probability measures Q'_k by $dQ'_k = \alpha'_{\tau_k} dQ^0$. Then each Q'_k is an equivalent martingale measure for the stopped process $\xi_{t\wedge\tau_k}$. Then the stopped process $X(t\wedge\tau_k)$ is a supermartingale with respect to Q'_k for each k. Its Doob-Meyer decomposition is represented by

$$X(t\wedge\tau_k) = \int_0^{t\wedge\tau_k} (\phi(s), dW^{f'}) + \tilde{N}^{g'}_{t\wedge\tau_k}(\psi) - A^{\alpha'}(t\wedge\tau_k), \quad k=1,2,...$$

where $A^{\alpha'}(t)$ is a suitable predictable increasing process. It satisfies

$$A^{\alpha'}(t) = -n\int_0^t 1_F\left\{(\phi,\hat{f}) + \int_{\mathbf{R}^m} \psi(e^{\hat{g}}-1)e^{g^0}d\nu\right\}ds + A^{Q^0}(t).$$

This makes a contradiction since the right hand side is negative for sufficiently large n. Therefore we get $A^Q(t) = A^{Q^0}(t)$.

Now if F' is of positive measure instead of the set F, interchange the role of Q^0 and Q in the above discussion. Then we get the same conclusion. Further in the case where $g \geq g^0$, we get the same equality (3.13) by interchanging the role of Q^0 and Q.

We have thus seen that $M^Q(t) = M^{Q^0}(t)$ holds for any standard equivalent martingale measure Q such that its density process α_t^0 with respect to Q^0 is a locally square integrable martingale and $g \leq g^0$ or $g \geq g^0$ is satisfied. Let $\tilde{\Gamma}_2^0$ be the set of all $\alpha_t^0(\hat{f},\hat{g}) \in \Gamma_2^0$ such that Q with $dQ = \alpha^0(\hat{f},\hat{g})dP^0$ is a standard equivalent martingale measure and $\psi \in \Psi_2(\hat{N}^g)$. Then $M^{Q^0}(t)$ is orthogonal to any element of

$$\mathcal{N} = \left\{\alpha_t^0(\hat{f},\hat{g}) - 1;\ \alpha^0(\hat{f},\hat{g}) \in \tilde{\Gamma}_2^0,\ \hat{g} \leq 0 \text{ or } \hat{g} \geq 0\right\},$$

with respect to Q^0. Observe that $\mathcal{L}(\mathcal{N}) = \mathcal{L}(\alpha_t^0 - 1; \alpha_t^0 \in \Gamma_2^0)$. Then we see that $M^{Q^0}(t)$ belongs to $\mathcal{L}(\xi_t^1, ..., \xi_t^d)$ by the decomposition formula (3.10). Then it is represented by $\int_0^t (\varphi(s), d\xi_s)$ with respect to Q^0. Setting $A(t) = A^{Q^0}(t)$, we get the decomposition formula (3.11). Further, representation (3.11) should hold valid for any Q of Γ. Then, comparing this with the representation of Theorem 2.3, $\sigma(s)\varphi(s) \in \Phi$ and $(v(s,z),\varphi(s)) \in \Psi_{1,2}(\hat{N}^g)$.

The second assertion of the theorem follows immediately from the above discussion by setting $A^{Q^0}(t) = A^Q(t) \equiv 0$.

Applications to mathematical finance

We consider a simple market model, where the return process is given by a stochastic process ξ_t of (3.2) and the interest rate $r(t)$ is identically 0. Let $\pi(t)$ be a predictable process called a *strategy* or *portfolio* and C_t be a right continuous predictable increasing process called a *cumulative consumption process.* A stochastic process $X(t) = X^{x,\pi,C}(t)$ defined by

$$X(t) = x + \int_0^t (\pi_s, d\xi_s) - C(t)$$

is called a *wealth process.* We introduce admissible classes for pairs of portofolios and consumptions. Let $x > 0$. We denote by $\mathcal{A}^+(x)$ the set of the pair (π, C) such that $X^{x,\pi,C}(t) \geq 0$ holds a.s. for any $0 \leq t \leq T$. We denote by $\mathcal{A}^-(-x)$ the set of the pair (π, C) such that $X^{-x,\pi,C}(t) \leq 0$ for any $0 \leq t \leq T$.

A *Europian contingent claim* Y is a nonnegative $\mathcal{F}_T$-measurable random variable. The contingent claim is not always attainable, since the model is not complete due to jumps of the return process. We shall study the upper and lower hedging price. The *upper hedging price* and *lower hedging price* of the contingent claim Y are defined respectively by

$$\begin{aligned} h_{up} &= \inf\{x \geq 0; \\ &\qquad \text{there exists } (\pi, C) \in \mathcal{A}^+(x) \text{ such that } X^{x,\pi,C}(T) \geq Y \ a.s.\} \\ h_{low} &= \sup\{x \geq 0; \\ &\qquad \text{there exists } (\pi, C) \in \mathcal{A}^-(-x) \text{ such that } X^{-x,\pi,C}(T) \geq -Y \ a.s.\} \end{aligned}$$

Theorem 3.5. *Assume that $\sigma(t)\sigma(t)^T$ is uniformly positive definite, $v(t,z)$ is greater than -1 and $v(t,z) \neq 0$ a.e. $\lambda \times \nu \times P$. Let Y be an Europian contingent claim. We have*

$$h_{up} = \sup_{Q\in\Gamma} E_Q[Y] =: h, \tag{3.14}$$

$$h_{low} = \inf_{Q\in\Gamma} E[Y] =: f. \tag{3.15}$$

If h is finite (resp. f is positive), there exists a pair $(\pi, C) \in \mathcal{A}^+(h)$ (resp. $(\pi', C') \in \mathcal{A}^-(-f)$) such that

$$X^{h,\pi,C}(t) = \operatorname{ess\,sup}_{Q\in\Gamma} E_Q[Y|\mathcal{F}_t], \tag{3.16}$$

$$-X^{-f,\pi',C'}(t) = \operatorname{ess\,inf}_{Q\in\Gamma} E_Q[Y|\mathcal{F}_t], \tag{3.17}$$

holds for any t. *In particular,* $X^{h,\pi,C}(T) = Y$ *and* $X^{-f,\pi',C'}(T) = -Y$, *a.s.*

Proof. We consider the upper hedging price only. Set $h = \sup_{Q\in\Gamma} E_Q[Y]$. We want to prove $h = h_{up}$. We first show $h \le h_{up}$. The inequality is obvious if $h_{up} = \infty$. If $h_{up} < \infty$, let x be an arbitrary element in the set $\{\cdots\}$ appearing in the defintion of h_{up}. Then there exists a pair (π, C) of $\mathcal{A}^+(x)$ such that $X^{x,\pi,C}(T) \ge Y$. Then $X^{x,\pi,C}(t)$ is a supermartingale for any $Q \in \Gamma$. Therefore, $E_Q[Y] \le E_Q[X^{x,\pi,C}(T)] \le x$ holds for any $Q \in \Gamma$. Then we have $h \le x$, so that we have $h \le h_{up}$.

In order to prove the reverse inequality $h \ge h_{up}$, it is sufficient to construct $(\pi, C) \in \mathcal{A}^+(h)$ such that $X^{h,\pi,C}(t) = X(t)$, where $X(t) \equiv \operatorname{ess\,sup}_{Q\in\Gamma} E_Q[Y|\mathcal{F}_t]$. It is known that the process $X(t)$ is a supermartingale for any Q. We shall apply Theorem 3.4 to the return process ξ_t. Then $X(t)$ admits the decomposition (3.11) by Theorem 3.4. This implies $X(t) = X^{h,\pi,C}(t)$, by setting $\varphi = \pi$ and $A(t) = C(t)$. It is clearly nonnegative a.s. for any $t \in (0, T]$. Therefore (π, C) belongs to $\mathcal{A}^+(h)$.

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