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Gauge Theorems for Stieltjes Exponentials

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Abstract.

A gauge theorem for the Stieltjes exponential of a right continuous additive functional satisfying a general Kato type condition is established. Results for the ordinary exponential are then obtained as corollaries.

§1. Introduction

In a series of recent papers Chen and Song [CS02], [CS03] and Chen [C02] have made remarkable progress in establishing the gauge and conditional gauge theorem under quite general hypotheses. The gauge theorem has the following structure: Given a multiplicative function $M = (M_t)$ and a terminal time τ of a strong Markov process X the gauge function $g(x) := E^x(M_\tau)$ is bounded on $\{g < \infty\}$ under suitable hypotheses on X and M. Usually X is assumed to satisfy some irreducibility hypothesis which then implies that g is either bounded or identically infinite. Also M usually is of the form $M_t = \exp(A_t)$ where A is an additive functional. See the above cited papers and also the book of Chung and Zhao [CZ95] for some history of the subject. Before the above cited papers A was usually assumed to be continuous and often of the form $A_t = \int_0^t q(X_s)ds$ where q is a function on the state space of X. See however [CR88], [So93] and [St91] for notable exceptions.

In their papers Chen and Song and Chen consider both continuous and a class of discontinuous additive functionals. The arguments in the two cases are similar in structure but somewhat different in detail. It turns out that by modifying slightly their approach one can prove a gauge theorem for arbitrary right continuous additive functionals in a unified way, assuming only that the underlying process X is a Borel right

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process. The present paper is devoted to spelling this out in some detail. We obtain a slightly sharper result even in the case considered by Chen and Song, but our emphasis is the generality and the simplicity of the result obtained. We work directly with Stieltjes exponentials of additive functionals obtaining a gauge theorem for such exponentials. Results for ordinary exponentials then appear as simple corollaries. Our results are general enough to apply to infinite dimensional processes since we do not assume any absolute continuity condition.

One of the main purposes of the Chen, Song papers was to prove a gauge theorem in enough generality that it could be applied to prove a conditional gauge theorem. A critical hypothesis for their result for a conditional gauge theorem is that X be in strong duality with another Borel right process X. Under this duality hypothesis, the hypotheses and argument in section 3 of [CS00] can be adapted to prove a conditional gauge result for Stieltjes exponentials along the lines of this paper. We leave the precise formulation to the interested reader.

We close this introduction with some words on notation. If $(F, \mathcal{F}, \mu)$ is a measure space, then we use $\mathcal{F}$ also to denote the class of all $\overline{\mathbb{R}} = [-\infty, \infty]$ valued $\mathcal{F}$ measurable functions. If $\mathcal{M} \subset \mathcal{F}$, then $b\mathcal{M}$(resp.$p\mathcal{M}$) denotes the class of bounded (resp. $[0, \infty]$ valued) functions in $\mathcal{M}$. For $f \in \mathcal{F}, \mu(f)$ denotes the integral $\int f d\mu$. If $(E, \mathcal{E})$ is a second measurable space and $K = K(x, dy)$ is a kernel from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ (i.e. $x \to K(x, A)$ is in $\mathcal{F}$ for each $A \in \mathcal{E}$ and $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$ for each $x \in F$), then we write μK for the measure $A \to \int \mu(dx) K(x, A)$ and Kf for the function $x \to \int K(x, dy) f(y)$. The symbol ":=" stands for "is defined to be".

§2. Preliminaries

Throughout this paper $(P_t, t \geq 0)$ will denote a Borel right semigroup on a Lusin state space $(E, \mathcal{E})$, and $X = (X_t, P^x)$ will denote the canonical realization of (P_t) as a right continuous strong Markov process. A (positive) σ-finite measure m on $(E, \mathcal{E})$ is *excessive* provided $mP_t \leq m$ for all $t \geq 0$. Since (P_t) is a right semigroup, it follows that $mP_t \uparrow m$ setwise as $t \downarrow 0$. See [DM87; XII, 36-37]. We fix an excessive measure m to serve as a background measure. In general we shall use the standard notation for Markov processes without special mention. See, for example, [BG68], [DM87], [S88] and [G90]. In particular, $U^q := \int_0^\infty e^{-qt} P_t dt, q \geq 0$, denotes the resolvent and $U := U^0$ the potential kernel of (P_t) or X. We assume only that (P_t) is sub-Markovian and so a point Δ is adjoined to E as an isolated point to serve as a cemetary and $\zeta := \inf\{t : X_t = \Delta\}$ is the lifetime of X and $P^x(\zeta > t) = P_t 1(x)$.

As usual a function f on E is extended to Δ by $f(\Delta) = 0$ unless explicitly stated otherwise. Thus for example, if $f \geq 0$

$$U^q f(x) = E^x \int_0^\zeta e^{-qt} f(X_t)dt = E^x \int_0^\infty e^{-qt} f(X_t)dt.$$

We shall assume throughout that X is transient or equivalently that U is proper. More precisely we shall assume:

(2.1) Transience Assumption. *There exists a function $b \in \mathcal{E}, 0 < b \leq 1$ with $Ub \leq 1$; reducing b if necessary we may also suppose that $m(b) < \infty$.*

Recall that a set $B \in \mathcal{E}^n$ is m-polar (resp. m-semipolar) provided $\{t : X_t \in B\}$ is empty (resp. at most countable) P^m a.s. Here $\mathcal{E}^n$ denotes the σ-algebra of nearly Borel sets. A nearly Borel set N is m-**inessential** provided it is m-polar and $E \backslash N$ is absorbing for X. By [GS 84; (6.12)] any m-polar set is contained in a Borel m-inessential set. A property or statement $P(x)$ is said to hold quasi-everywhere (q.e.) or for quasi-every x provided it holds for all x outside some m-polar set. The exceptional set may then be supposed to be m-inessential. We also write a.e. for m-a.e.

(2.2) Definition. *A positive additive functional (PAF) is an $(\mathcal{F}_t)$-adapted increasing process $A = (A_t; t \geq 0)$ with values in $[0, \infty]$, for which there exist a defining set $\Omega_A \in \mathcal{F}$ and a Borel m-inessential set N_A (called an exceptional set for A) such that*

- *(i)* $P^x(\Omega_A) = 1$ *for all* $x \notin N_A$;
- *(ii)* $\theta_t \Omega_A \subset \Omega_A$ *for all* $t \geq 0$;
- *(iii) For all* $\omega \in \Omega_A$ *the mapping* $t \to A_t(\omega)$ *is right continuous on* $[0, \infty[$, *finite valued on* $[0, \zeta(\omega)[$ *with* $A_0(\omega) = 0$;
- *(iv) For all* $\omega \in \Omega_A$ *and* $s, t \geq 0, A_{s+t}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$;
- *(v) For all* $t \geq 0, A_t([\Delta]) = 0$ *where* $[\Delta]$ *is the dead path identically equal to* Δ.
- *(vi) For* $\omega \in \Omega_A, A_t(\omega) = A_{\zeta(\omega)-}(\omega)$ *for* $t \geq \zeta(\omega)$.

We let $\widetilde{\mathcal{A}}^+$ denote the class of all PAFs. If in (2.2-iii) right continuous is replaced by continuous, then A is a positive continuous additive functional (PCAF) and we write $\widetilde{\mathcal{A}}_c^+$ for the class of such functionals. Two PAFs A and B are m-equivalent provided $P^m(A_t \neq B_t) = 0$ for all $t \geq 0$. One can check that A and B are m-equivalent if and only if they have a common defining set Λ and a common exceptional set N such that $A_t(\omega) = B_t(\omega)$ for all $t \geq 0$ and $\omega \in \Lambda$. See the argument just below Definition 3.1 in [FG 96]. Equality between elements of $\widetilde{\mathcal{A}}^+$ will mean m-equivalence unless explicitly mentioned otherwise. An $A \in \widetilde{\mathcal{A}}^+$ may be decomposed as $A = A^c + A^d$ where A^c is a PCAF and $A_t^d = \sum_{0<s\leq t} \Delta A_s$ is the sum of the jumps of $A, \Delta A_s = A_s - A_{s-}$. Because of *(vi)*, $\Delta A_\zeta = 0$ on Ω_A. Of course $\Omega_A = \Omega$ and $N_A = \phi$ are allowed in Definition 2.2. By restricting X to the Borel absorbing set $E \backslash N_A$ one may usually reduce to the case in which N_A is empty. In order to keep this exposition as simple possible I will restrict attention to this case. To be precise we define

$$\mathcal{A}^+ = \{A \in \widetilde{\mathcal{A}}^+ : N_A = \phi\}. \tag{2.3}$$

Our results will be stated for $A \in \mathcal{A}^+$, but the interested reader should have no difficulty in formulating them for $A \in \widetilde{\mathcal{A}}^+$. If $A \in \mathcal{A}^+$, then $P^x(\Omega_A) = 1$ for all x. Hence A is almost perfect as defined in [S88, p.173].

If $A \in \mathcal{A}^+$, then its characteristic (Revuz) measure μ_A is defined by

$$\mu_A(f) :=\uparrow \lim_{t\to 0} \frac{1}{t}\, E^m \int_0^t f(X_s)dA_s \tag{2.4}$$

for $f \in p\mathcal{E}^n$. Moreover $\mu_A(f) =\uparrow \lim_{q\to\infty} q \cdot mU_A^q f$ where

$$U_A^q f(x) := E^x \int_0^\zeta e^{-qt} f(X_t)dA_t \tag{2.5}$$

is the q-potential operator of A. As usual $U_A := U_A^0$. Clearly μ_A does not charge m-polars.

We shall also need the Stieltjes exponential $\mathrm{Exp}(A)$ of $A \in A^+$:

$$\mathrm{Exp}(A_t) := e^{A_t^c} \prod_{0<s\leq t} (1 + \Delta A_s). \tag{2.6}$$

Clearly $\mathrm{Exp}(A)$ is increasing, right continuous and finite on $\{(t,\omega) : 0 \leq t < \zeta(\omega), \omega \in \Omega_A\}$ and $\mathrm{Exp}(A_0) = 1$. On Ω_A the compensated powers $A^{(n)}$ of A are defined by $A_t^{(0)} = 1$ and for $t < \zeta$

$$A_t^{(n)} = n \int_{]0,t]} A_{s-}^{(n-1)} dA_s; \quad n \geq 1.$$

It is well-known that $\mathrm{Exp}(A_t) = \sum_{n\geq 0} \frac{1}{n!} A_t^{(n)}$, see [DD70, p189]. Recall that a terminal time τ for X is a stopping time which satisfies $t+\tau\circ\theta_t = \tau$ on $\{t < \tau\}$. A straightforward induction argument yields the following result. See [SS00] for much more general results.

(2.7) Lemma. *Let $A \in \mathcal{A}^+$ and let σ be either a terminal time or a constant time. Let $\tau = \sigma \wedge \zeta$. If $E^x(A_{\tau-}) \leq C < 1$ for all x, then $E^x[\mathrm{Exp}(A_{\tau-})] \leq (1-C)^{-1}$ for all x.*

If $B \in \mathcal{E}^n, \tau(B) := \inf\{t > 0 : X_t \notin B]$ denotes the exit time from B. Evidently $\tau(B) \leq \zeta$.

(2.8) Proposition. *Let $A \in \mathcal{A}^+$ and suppose that A has bounded jumps; that is, there exists $0 \leq c < \infty$ such that $\sup_t \Delta A_t \leq c$ a.s. Then there exists an increasing sequence (G_n) of finely open nearly Borel sets with $E = \cup G_n, \tau(G_n) \uparrow \zeta$ a.s. and for each $n, \mu_A(G_n) < \infty$ and $U_A 1_{G_n} \leq (2c+1)n$.*

Proof. Suppose first that $c < 1$. Define

$$M_t := \mathrm{Exp}(-A_t) = e^{-A_t^c} \prod_{0<s\leq t} (1 - \Delta A_s).$$

Then a.s., $t \to M_t$ is right continuous and decreasing, $M_t > 0$ if $t < \zeta$ and $M_0 = M_{0+} = 1$. It is well-known and easily verified that $M_{t+s} = M_t M_s \circ \theta_t$ and $d(M_t^{-1}) = M_t^{-1} dA_t$. Define

$$g(x) := E^x \int_0^\zeta M_t b(X_t) dt \tag{2.9}$$

where b is the function in (2.1). Clearly $g > 0$, and

$$\begin{aligned} U_A g &= E^{\cdot} \int_0^\zeta g(X_t) dA_t = E^{\cdot} \int_0^\zeta M_t^{-1} \int_t^\zeta M_s b(X_s) ds\, dA_t \\ &= E^{\cdot} \int_0^\zeta M_s b(X_s) \int_0^s M_t^{-1} dA_t ds = Ub - g. \end{aligned}$$

Hence $0 < g \leq Ub$ and $U_A g \leq Ub$. It is easily checked that g is excessive relative to (X, M) – the M subprocess of X– and consequently g is nearly Borel and finely continuous. Thus the sets $G_n := \{g > \frac{1}{n}\}$ form an increasing sequence of finely open nearly Borel subsets of E with $\cup G_n = E$. Let τ_n be the exit time from G_n. Since G_n^c is finely closed, $g(X_{\tau_n}) \leq \frac{1}{n}$, a.s. on $\{\tau_n < \zeta\}$. Hence

$$\frac{1}{n} \geq E^{\cdot}[g(X_{\tau_n})] = E^{\cdot}[M_{\tau_n}^{-1} \int_{\tau_n}^\zeta M_t b(X_t) dt] \geq E^{\cdot} \left[\int_{\tau_n}^\zeta M_t b(X_t) dt \right].$$

But $b > 0$ and $M_t > 0$ on $[0, \zeta[$ a.s., so we must have $\lim_n \tau_n = \zeta$ a.s. Since $U_A g \leq Ub$ one has $U_A 1_{G_n} \leq nUb \leq n$. Therefore $\mu_A(G_n) \leq n\mu_A Ub \leq n \cdot m(b) < \infty$ where the second inequality comes from (the proof of) the lemma at the bottom of page 508 in [Re 70]. This establishes (2.8) when $c < 1$. If $c \geq 1$, let $A_t^* = (2c)^{-1}A_t$ so that A^* has jumps bounded by $\frac{1}{2}$. Let g be defined as in (2.9) but with A replaced by A^*. Since $U_A = 2cU_{A^*}$ and $\mu_A = 2c\mu_{A^*}, G_n := \{g > \frac{1}{n}\}$ has the desired properties. ∎

The proof of (2.8) is easily modified to prove the following:

(2.10) Proposition. *Let $A \in \widetilde{\mathcal{A}}^+$ and suppose that $\sup_t \Delta A_t \leq c < \infty$ a.s. P^x for $x \in E\backslash N_A$. Then there exists an increasing sequence (G_n) of finely open nearly Borel sets such that $E\backslash \cup G_n$ is m-polar, $\tau(G_n) \uparrow \zeta$ a.s. P^x for $x \in E\backslash N_A$, and for each n, $\mu_A(G_n) < \infty$ and $U_A 1_{G_n} \leq (2c+1)n$ on $E\backslash N_A$.*

From time to time we will spell out the situation when $\widetilde{\mathcal{A}}^+$ replaces $\mathcal{A}^+$. Usually it is just a matter of keeping track of the exceptional set N_A as illustrated in (2.10).

§3. Kato Classes

In this section we introduce some Kato classes of additive functionals. The definitions are modifications of those in [C02]. Let $\|\cdot\|_\infty$ denote the norm in $L^\infty(m)$; that is for $f \in \mathcal{E}^n$, $\|f\|_\infty = m\text{-}\operatorname{ess\,sup}_x |f(x)|$. We shall also use the q.e. supremum norm for $f \in \mathcal{E}^n$; that is $\|f\|_{qe} = \inf\{\beta : |f| \leq \beta\, q.e.\}$. Clearly $\|f\|_{qe} \geq \|f\|_\infty$. Recall that $f \in \mathcal{E}^n$ is quasi-finely continuous (qfc) provided it is finely continuous off an m-polar set which may be assumed to be m-inessential. Since an m-null finely open set is m-polar, it follows that if f is qfc, then $\|f\|_\infty = \|f\|_{qe}$.

(3.1) Definition. *Let $0 < \beta < \infty$. Then $\widetilde{K}_\beta$ (resp. $\widetilde{K}_\beta^*$) consists of those $A \in \widetilde{\mathcal{A}}^+$ that have bounded jumps as defined in (2.10) and such that there exist a positive measure ν on E, a set $K \in \mathcal{E}$ with $\nu(K) < \infty$ and $\delta = \delta(\nu, K) > 0$ with the following property:*

(3.2) *If $B \subset \mathcal{E}^n$ with $B \subset K$ and $\nu(B) < \delta$, then*
$$\|U_A 1_{B\cup K^c}\|_\infty \leq \beta \text{ (resp. } \|E^{\cdot}(A_{\tau(B\cup K^c)-}\|_{qe} \leq \beta).$$

Remarks. For and $D \in \mathcal{E}^n, U_A 1_D$ is excessive for X restricted to $E\backslash N_A$ and hence qfc. But $E^{\cdot}(A_{\tau(D)-}) \leq E^{\cdot} \int 1_D(X_t)dA_t = U_A 1_D$ and therefore $\widetilde{K}_\beta \subset \widetilde{K}_\beta^*$. Replacing ν by $\nu|_K$ one may suppose that ν is finite when convenient.

Once again to keep the exposition simple we are going to eliminate the exceptional sets.

(3.3) Definition. *If $0 < \beta < \infty$, then $A \in K_\beta$ (resp. K^*_β) provided $A \in \mathcal{A}^+$ and has bounded jumps as in (2.8) and there exist ν, K and δ as in (3.1) such that $B \subset K, \nu(B) < \delta$ imply*

$$\sup_{x\in E} U_A 1_{B\cup K^c}(x) \leq \beta \ (\textit{resp.} \sup_{x\in E} E^x(A_{T(B\cup K^c)-}) \leq \beta).$$

We are going to work under (3.3) and leave the straightforward extension to the more general situation to the interested reader. Obviously if $0 < \beta < \gamma$, then $K_\beta \subset K_\gamma$ and $K^*_\beta \subset K^*_\gamma$. It is convenient to define

$$K_0 := \cap_{\beta>0} K_\beta \quad \text{and} \quad K^*_0 := \cap_{\beta>0} K^*_\beta. \tag{3.4}$$

It will turn out that the K^*_β are the appropriate classes for the gauge theorem. Moreover in an important special case a sufficient condition that $A \in K^*_\beta$ is that $A^p \in K^*_\beta$ where A^p is the dual predictable projection of A. We now describe this result. Let $A \in \mathcal{A}^+$ have bounded jumps. Then there exists a unique predictable element $A^p \in \mathcal{A}^+$– the dual predictable projection of A– such that for any positive predictable process (Z_t)

$$E^x \int_0^\infty Z_t \, dA_t = E^x \int_0^\infty Z_t \, dA^p_t. \tag{3.5}$$

See [S88, §31].

(3.6) Proposition. *Let $A \in \mathcal{A}^+$ have bounded jumps with bound c as in (2.8). If the dual predictable projection A^p of A is continuous, $A^p \in K^*_\beta$ if and only if $A \in K^*_\beta$.*

Proof. If T is a stopping time, $1_{]0,T]}(t)$ is predictable. Therefore since A^p is continuous

$$E^x(A_T) - c \leq E^x(A_{T-}) \leq E^x(A_T) = E^x(A^p_T) = E^x(A^p_{T-}),$$

and this establishes (3.6). ∎

The next two propositions are taken from [C02]. We give proofs for the convenience of the reader. For $A \in \mathcal{A}^+, u_A := U_A 1 = E^\cdot(A_{\zeta-})$ is the potential (function) of A.

(3.7) Proposition. *Suppose $A \in K_\beta, \beta > 0$. Then $\sup_{x\in E} u_A(x) < \infty$.*

Proof. Let $\nu, K \in \mathcal{E}$ and δ be as in Definition 3.3. Then K contains at most a finite number of points $\{x_1, \ldots, x_n\}$ with $\nu(\{x_j\}) > \delta$. It follows from a result of Saks (see [DS58, p308]) that $K \backslash \{x_1, \ldots x_n\}$ can be written as the disjoint union of a finite number $B_1, \ldots, B_k$ of sets in $\mathcal{E}$ with $\nu(B_j) \leq \delta$. From (2.8) there exists (G_j) with $G_j \uparrow E$ and $U_A 1_{G_j} \leq (2j+1)c$ for each j. Let $F = \{x_1, \ldots, x_n\}$. Since $E = \cup G_j$, there exists an ℓ with $1_F \leq 1_{G_\ell}$. Hence

$$u_A = U_A 1_{K^c} + U_A 1_F + \sum_{j=1}^{k} U_A 1_{B_j} \leq (k+1)\beta + (2\ell+1)c. \qquad \blacksquare$$

The definition of K_β depends on what appears to be an arbitrary choice of the measure ν. The next proposition gives an intrinsic criterion for A to be in K_β, at least when m is a reference measure. If $f : E \to \overline{\mathbb{R}}$, let $\|f\| = \sup_{x \in E} |f(x)|$.

(3.8) Proposition. *Let $A \in \mathcal{A}^+$ and $\beta > 0$. (i) If $A \in K_\beta$, then for every decreasing sequence $(D_n) \subset \mathcal{E}^n$ with $\cap D_n = \phi, \lim_n \|U_A 1_{D_n}\| \leq \beta$. (ii) If m is a reference measure and if for every decreasing sequence $(D_n) \subset \mathcal{E}^n$ with $\cap D_n = \phi, \lim_n \|U_A 1_{D_n}\| < \frac{\beta}{2}$ and A has bounded jumps, then $A \in K_\beta$.*

Proof. (i) Suppose $\beta > 0$ and $A \in K_\beta$. Let $D_n \downarrow \phi$. Then there exists an N such that $\nu(K \cap D_n) \leq \delta$ for $n \geq N$. Thus for $n \geq N, U_A 1_{D_n} \leq U_A 1_{(D_n \cap K) \cup K^c}$ and so $\|U_A 1_{D_n}\| \leq \beta$ for $n \geq N$. (ii) Let $A \in \mathcal{A}^+$ have bounded jumps. By (2.8) there exists an increasing sequence (G_n) of finely open sets in $\mathcal{E}^n$ with $\mu_A(G_n) < \infty$ and $\cup G_n = E$. Let $D_n = E \backslash G_n$. Then $\cap D_n = \phi$ and so $\lim_n \|U_A 1_{D_n}\| < \frac{\beta}{2}$. Fix an n with $\|U_A 1_{D_n}\| < \frac{\beta}{2}$ and put $K = G_n$. We claim that there exists a $\delta > 0$ such that if $B \subset K$ and $\mu_A(B) < \delta$, then $\|U_A 1_{B \cup K^c}\| \leq \beta$. Suppose no such $\delta > 0$ exists. Then for each n there exists $B_n \subset K$ with $\mu_A(B_n) \leq 2^{-n-1}$ and $\|U_A 1_{B_n \cup K^c}\| > \beta$. Let $F_n = \cup_{k \geq n} B_k$. Then (F_n) is a decreasing sequence with $\mu_A(F_n) \leq 2^{-n}$. If $F := \cap F_n$, then $\mu_A(F) = 0$. Hence $0 = \mu_A(F) = \uparrow \lim_{q \to \infty} qmU_A^q 1_F$ and so $mU_A^q 1_F = 0$ for $q > 0$. Letting $q \downarrow 0$ we see that $U_A 1_F = 0$ a.e. m and thus everywhere since $U_A 1_F$ is excessive and m is a reference measure. Consequently $U_A 1_{F_n} = U_A 1_{F_n \backslash F}$ and since $F_n \backslash F \downarrow \phi, \lim_n \|U_A 1_{F_n}\| < \frac{\beta}{2}$. Choose n with $\|U_A 1_{F_n}\| < \frac{\beta}{2}$. Now $B_n \subset F_n$ so

$$\beta < \|U_A 1_{F_n \cup K^c}\| \leq \|U_A 1_{F_n}\| + \|U_A 1_{K^c}\| < \beta$$

and this contradiction establishes (3.8). $\blacksquare$

Remarks. We emphasize that the measure constructed in the proof of (3.8) is $\nu = \mu_A$. The only place in the proof that the fact that m is a reference measure is used is concluding that $U_A 1_F = 0$ from $U_A 1_F = 0$ a.e. and of course then q.e. Consequently the proof is easily adapted using (2.10) in place of (2.8) to show:

(3.9) Proposition. *Let $A \in \widetilde{\mathcal{A}}^+$ have bounded jumps as in (2.10). If for every decreasing sequence $(D_n) \subset \mathcal{E}^n$ with $\cap D_n = \phi, \lim_n \|U_A 1_{D_n}\|_\infty < \frac{\beta}{2}$, then $A \in \widetilde{K}_\beta$.*

(3.10) Proposition. *Suppose $A^j \in K_{\beta_j}$ (resp. $K^*_{\beta_j}$) for $j = 1, 2$. Then $A^1 + A^2 \in K_{\beta_1+\beta_2}$ (resp. $K^*_{\beta_1+\beta_2}$).*

Proof. Let ν_j, K_j and δ_j serve for $A^j \in K_{\beta_j}, j = 1, 2$. We may suppose that ν_j is carried by $K_j, j = 1, 2$. Define $\nu = \nu_1 + \nu_2, K = K_1 \cup K_2$ and $\delta = \delta_1 \wedge \delta_2$. Then

$$\begin{aligned} \nu(K) &= \nu(K_1 \cap K_2^c) + \nu(K_1^c \cap K_2) + \nu(K_1 \cap K_2) \\ &\leq 2[\nu_1(K_1) + \nu_2(K_2)] < \infty. \end{aligned}$$

Suppose $B \subset K$ with $\nu(B) < \delta$. Then $\nu_j(B) \leq \delta_j$ for $j = 1, 2$. Note that $B \cup K^c \subset (B \cap K_j) \cup K_j^c$ for $j = 1, 2$ and so

$$U_{A_1+A_2} 1_{B \cup K^c} \leq \sum_{j=1}^{2} U_{A_j} 1_{(B \cap K_j) \cup K_j^c} \leq \beta_1 + \beta_2.$$

Of course $A_1 + A_2$ has bounded jumps. The same argument works when the K_β are replaced by K^*_β. ■

§4. Gauge Theorems

Gauge theorems are usually stated for fluctuating additive functionals. Formally let $\mathcal{A} := \mathcal{A}^+ - \mathcal{A}^-$ and introduce the obvious notion of equality: If $A_j = A_j^+ - A_j^-, A_j^\pm \in \mathcal{A}^+$ for $j = 1, 2$, then $A_1 = A_2$ provided $A_1^+ + A_2^- = A_1^- + A_2^+$ in $\mathcal{A}^+$. Then it is known that $A \in \mathcal{A}$ can be written uniquely as $A = A^+ - A^-$ with $A^+, A^- \in \mathcal{A}^+$ having a common defining set Ω_A and such that the measures $dA_t^+(\omega)$ and $dA_t^-(\omega)$ on $[0, \zeta(\omega)[$ are orthogonal for $\omega \in \Omega_A$. Actually the only thing that we shall use is that a.s., A^+ and A^- have no common discontinuities. Of course $A \in \mathcal{A}$ can be decomposed as $A = A^c + A^d$ where $A^c \in \mathcal{A}$ is continuous and $A^d \in \mathcal{A}$ is purely discontinuous,

$$A_t^d = \sum_{0<s\leq t} \Delta A_s = \sum_{0<s\leq t} \Delta A_s^+ - \sum_{0<s\leq t} \Delta A_s^-$$

with $\sum_{0<s\leq t} |\Delta A_s| < \infty$ if $t < \zeta$. In particular $t \to A_t$ is of bounded variation on compact intervals in $[0, \zeta[$ a.s. Fix $A = A^+ - A^- \in \mathcal{A}$. Define for $t < \zeta$,

$$L_t^+ := \mathrm{Exp}(A_t^+),\ L_t^- := \mathrm{Exp}(A_t^-).$$

It turns out that the appropriate multiplicative functional to consider is

$$L_t := L_t^+/L_t^- = e^{A_t^c} \prod_{0<s\leq t} \frac{(1+\Delta A_s^+)}{(1+\Delta A_s^-)}. \tag{4.1}$$

Clearly a.s., $t \to L_t^{\pm}$ is increasing and finite on $[0, \zeta[$ and $\Delta A_\zeta^{\pm} = 0$. Hence $L_{\zeta-}^{\pm} = L_\zeta^{\pm}$ and so $L_{\zeta-} = L_\zeta$ where $\infty/\infty = 0$ by convention. Moreover $t \to L_t$ is right continuous on $[0, \zeta[$ and is of bounded variation on compact subintervals of $[0, \zeta[$, a.s. Henceforth we shall omit the qualifier "a.s" in places where it is obviously required such as in the preceding two sentences. Note that $L_0 = L_{0+} = 1$. The function

$$g(x) := E^x[L_\zeta] = E^x[L_{\zeta-}] \tag{4.2}$$

is called the **gauge** of A (or L).

(4.3) Proposition. *The gauge g is nearly Borel measurable and finely continuous. If $F := \{g < \infty\}$, then F is absorbing.*

Remark. Since g may take the value $+\infty, g$ is finely continuous as a map from E to $[0, \infty]$.

Proof. Since $L_\zeta \in \mathcal{F}, g$ is universally measurable and because L is a multiplicative functional, $L_\zeta \circ \theta_t = L_{\zeta-} \circ \theta_t = L_{\zeta-}/L_t$ if $t < \zeta$. Let $M_t = (L_t^-)^{-1}$. Then M is a decreasing, right continuous multiplicative functional that is strictly positive for $t < \zeta$. Now

$$\begin{aligned} E^x[g(X_t)M_t] &= E^x(M_t L_{\zeta-}/L_t\,; t < \zeta) \\ &= E^x[L_{\zeta-}/L_t^+; t < \zeta] \uparrow g(x) \end{aligned} \tag{4.4}$$

as $t \downarrow 0$. Therefore g is excessive relative to (X, M)– the M-subprocess of X– and hence g is nearly Borel and finely continuous. In particular $F = \{g < \infty\}$ is finely open and nearly Borel. The computation in (4.4), aside from taking the limit as $t \downarrow 0$, holds with t replaced by a stopping time T. Hence $E^x(M_T g(X_T)) \leq g(x)$ for any stopping time T. Let $D = \{g = \infty\} = E \backslash F$. Then D is finely closed and so $g \circ X_{T(D)} = \infty$ a.s. on $\{T(D) < \zeta\}$ where $T(D) := \inf\{t > 0 : X_t \in D\}$ is the hitting time of D. Hence if $x \in F$

$$\infty > g(x) \geq E^x(M_{T(D)} g \circ X_{T(D)}; T(D) < \zeta)$$

and since $M_{T(D)} > 0$ on $\{T(D) < \zeta\}$, this forces $P^x(T(D) < \zeta) = 0$. Hence F is absorbing. ∎

Of course $g(x) \leq E^x(L_\zeta^+)$ and so we may obtain bounds on g by estimating $E^x(L_\zeta^+)$. Thus in what follows we are going to assume that $A \in \mathcal{A}^+$. For $A \in \mathcal{A}^+$ let $c(A)$ denote the infimum of the c such that $\sup_{t<\zeta} \Delta A_t \leq c$ a.s. Then $A \in \mathcal{A}^+$ has bounded jumps provided $c(A) < \infty$. We come now to the main result of this section. The proof is borrowed from Chen and Song [CS02].

(4.5) Theorem. *Suppose $A \in K_\beta^*$ with $\beta < 1$. Then the gauge $g(x) = E^x(L_\zeta)$ is bounded on $\{g < \infty\}$.*

Proof. Let ν, K and δ be as in (3.3) for $A \in K_\beta^*$. Since $\nu(K) < \infty$ we may choose M large enough that $\nu(K \cap \{M < g < \infty\}) < \delta$. Let $B := K^c \cup \{M < g < \infty\} = K^c \cup (K \cap \{M < g < \infty\})$. Then $E^{\cdot}(A_{\tau(B)-}) \leq \beta$. Consequently by (2.7)

$$E^{\cdot}(L_{\tau(B)-}) \leq \gamma := (1-\beta)^{-1} < \infty. \tag{4.6}$$

Fix an x. Then

$$\begin{aligned} g(x) &= E^x[L_{\tau(B)-}; \tau(B) = \zeta] + E^x[L_{\zeta-}; \tau(B) < \zeta] \\ &\leq \gamma + E^x[L_{\tau(B)} g(X_{\tau(B)}); \tau(B) < \zeta]. \end{aligned}$$

Let $F = \{g < \infty\}$ and suppose that $x \in F$. But F is absorbing and so $g(X_t) < \infty$ a.s. P^x on $[0, \zeta[$. Hence $g \circ X_{\tau(B)} \leq M$ on $\{\tau(B) < \zeta\}$ a.s. P^x since g is finely continuous. Therefore $g(x) \leq \gamma + M E^x[L_{\tau(B)}; \tau(B) < \zeta]$ on F. Since a.s., $L_{\tau(B)} \leq (1+c) L_{\tau(B)-}$ where $c = c(A)$, we see that $g \leq \gamma + M\gamma(1+c)$ on $F = \{g < \infty\}$. ∎

(4.7) Remark. If one only assumes that $A \in \widetilde{K}_\beta^*$ with $\beta < 1$, then g is only defined on $E \backslash N_A$. It follows that g is finely continuous on $E \backslash N_A$ and g is bounded on $\{g < \infty\} \cap (E \backslash N_A)$. Under the hypotheses of (4.5), $\{g = \infty\} = \{g > M\}$ where $M = \sup\{g(x) : g(x) < \infty\}$. Thus both $\{g < \infty\}$ and $\{g = \infty\}$ are finely open. Therefore if E cannot be written as the disjoint union of two finely open nearly Borel sets one of which is absorbing, in particular if E is finely connected, then g is either bounded or identically infinite. This is the classical form of a gauge theorem.

(4.8) Corollary. *Let $A \in \mathcal{A}^+$ and define $B_t := A_t^c + \sum_{0<s\leq t} (e^{\Delta A_s} - 1)$. Then $B \in \mathcal{A}^+$. If $B \in K_\beta^*$ for some $\beta < 1$, then $g_A(x) := E^x(e^{A_\zeta})$ is finely continuous and bounded on $\{g_A < \infty\}$.*

Proof. Since $\sum_{0<s\leq t} \Delta A_s < \infty$ on $[0,\zeta[$ on Ω_A it is clear that $B \in \mathcal{A}^+$ and has the same defining set as A. Now (4.8) is evident because $e^{A_t} = \mathrm{Exp}(B_t)$. ∎

Remark. If $A = A^+ - A^- \in \mathcal{A}$ and one defines $B_t^{\pm} := A_t^{\pm c} + \sum_{0<s\leq t} (e^{\Delta A_s^{\pm}} - 1)$, then $B_t := B_t^+ - B_t^- \in \mathcal{A}$ and $e^{A_t} = \mathrm{Exp}(B_t^+)/\mathrm{Exp}(B_t^-)$. Therefore $g_A(x) := E^x(e^{A_\zeta})$ is nearly Borel, finely continuous and $\{g_A < \infty\}$ is absorbing. Here $e^{A_\zeta} = e^{A_\zeta^+} e^{-A_\zeta^-}$ with $\infty \cdot 0 = 0$ as customary.

Suppose $A \in \mathcal{A}^+$ is purely discontinuous ($A = A^d$) and all of its jumps are totally inaccessible. In this case the dual predictable projection A^p of A has an especially nice form which we now describe. Let

$$J := \{(t,\omega) : X_{t-}(\omega) \neq X_t(\omega), X_{t-}(\omega) \in E\} \tag{4.9}$$

be the set of totally inaccessible discontinuities of X. Here X_{t-} denotes the left limit in the Ray topology. A Lévy system (N, H) for X consists of a kernel $N = N(x, dy)$ on E with $N(x, \{x\}) = 0$ and a PCAF, H, with empty exceptional set and bounded one potential such that if $F \in (\mathcal{E}^n \otimes \mathcal{E}^n)$ with $F \geq 0$ and $Z = (Z_t)$ with $Z_t \geq 0$ is predictable, then

$$E^x \sum_{s\in J} Z_s F(X_{s-}, X_s) = E^x \int_0^\infty Z_t NF(X_t) dH_t \tag{4.10}$$

where $NF(x) = \int F(x,y)N(x,dy)$. If $A \in \mathcal{A}^+$ is purely discontinuous with totally inaccessible jumps, then there exists such an F vanishing on the diagonal such that

$$A_t = A_t^F = \sum_{s\in J, s\leq t} F(X_{s-}, X_s). \tag{4.11}$$

See §73 of [S88]. If X is a special standard process and X_{t-}^0 is the left limit in the original topology of E, then X_{t-} and X_{t-}^0 are indistinguishable on $[0,\zeta[$ and so X_{s-} may be replaced by X_{s-}^0 in (4.11) and s is automatically in J when $X_{s-}(= X_{s-}^0) \neq X_s, s < \zeta$. See [S88, (47.10)]. Moreover if $A = A^F$ then

$$(NF * H)_t := \int_0^t NF(X_s) dH_s$$

is the dual optional projection of A^F. Suppose F is bounded. Since $NF * H$ is continuous, (3.6) implies that $A^F \in K_\beta^*$ whenever $NF * H \in K_\beta^*$. In particular if $NF * H \in K_\beta$. The next proposition treats an important special case. It is the case considered in [C02]. It is an immediate consequence of (3.6), (3.10), (4.5) and (4.8).

(4.12) Proposition. *Let $A \in \mathcal{A}^+$ have the form $A = A^c + A^F$ where A^c is continuous and A^F is as in (4.11) with F bounded. If $A^c \in K^*_\beta$ and $NF * H \in K^*_\gamma$ with $\beta + \gamma < 1$, then g is bounded on $\{g < \infty\}$ where g is defined in (4.2). If $G = e^F - 1$ and $NG * H \in K^*_\gamma$ with $\beta + \gamma < 1$, then g_A is bounded on $\{g_A < \infty\}$ where g_A is defined (4.8).*

§5. Additional Conditions for the Gauge to be Bounded

In this section we develop some conditions that are equivalent to the boundedness of the gauge function. We follow a well-trodden path that was originally broken by Chung and Rao [CR88] and explored further by Chen and Song and Chen in their papers. The direct results using the Stieltjes exponential appear to be new. In what follows $A = A^+ - A^- \in \mathcal{A}$ as in the beginning of section 4 and L_t is defined in (4.1). The gauge g is defined in (4.2). We begin with the following proposition which is the part of Lemma 9 in [CR88] and Lemma 7 in [C02] that carries over to the present situation.

(5.1) Proposition. *Suppose that $A^+ \in K^*_\beta$ for $\beta < 1$ with $A^+_\zeta < \infty$ a.s. and that $E^\cdot(A^-_\zeta)$ is bounded. Let $\epsilon > 0$. Define*

$$\tau_n := \inf\{t : A^+_t > n\epsilon\}, \ n \geq 1$$

where as usual the infimum of the empty set is $+\infty$. If the gauge g is bounded, then

$$\lim_n \sup_x E^x[L_{\tau_n}; \tau_n < \zeta] = 0$$

Remark. If $A^+ \in K_\beta$, then $E^\cdot(A^+_\zeta)$ is bounded according to (3.7) and so $A^+_\zeta < \infty$ a.s. in this case.

Proof. Since the proof is the same for all $\epsilon > 0$, we shall give it for $\epsilon = 1$, which is the only case used later. Let K, ν, δ be as in the requirement that $A^+ \in K^*_\beta$. Thus $E^\cdot[A^+_{\tau(B \cup K^c)-}] \leq \beta$ when $B \subset K$ with $\nu(B) < \delta$. Since $A^+_\zeta < \infty$ it follows that $\{\tau_n < \zeta\} \downarrow \phi$. Here and in the remainder of this section we omit the qualifier "a.s." where it is obviously required. Therefore by Egorov's theorem, since $E^x(L_\zeta) < \infty, E^x[L_\zeta; \tau_n < \zeta] \downarrow 0$ almost uniformly on K with respect to ν. Recall $\nu(K) < \infty$. Hence given $\epsilon > 0$ there exists a closed set $D \subset K$ with $\nu(K \backslash D) < \delta$ and an N such that if $n \geq N$, then

$$\sup_{x \in D} E^x[L_\zeta; \tau_n < \zeta] < \epsilon. \tag{5.2}$$

Now $D^c = K^c \cup K\backslash D$ and $\tau(D^c) = T(D) \wedge \zeta$ where $T(D)$ is the hitting time of D. But $A_t^+ = A_{\zeta-}^+$ if $t \geq \zeta$ and so $E^{\cdot}(A^+_{T(D)-}) = E^{\cdot}(A^+_{\tau(D^c)-}) \leq \beta < 1$. Thus by (2.7)

$$(5.3) \qquad E^{\cdot}[L^+_{T(D)-}] \leq (1-\beta)^{-1} < \infty.$$

Define $B_t = A_t^{+c} + \sum_{0<s\leq t} \log(1+\Delta A_s^+)$. Note that $e^{B_t} = L_t^+$ and so $E^x[\exp(B_{T(D)-})] \leq (1-\beta)^{-1}$. Since A^+, and hence, B has bounded jumps, $\sup_x E^x[\exp(B_{T(D)})] < \infty$. Consequently it follows from Corollary 4.2 in [SS00] that there exists $p > 1$ with $\sup_x E^x[\exp(pB_{T(D)})] < \infty$. Finally $e^{pB_t} = [\mathrm{Exp}(A_t^+)]^p = (L_t^+)^p$ and so

$$(5.4) \qquad \sup_x E^x[(L^+_{T(D)})^p] = M' < \infty.$$

Let $n > \max(N, c(A^+))$ where $c(A^+)$ is defined in the paragraph above (4.5). Then

$$\begin{aligned} E^x[L_\zeta; \tau_{3n} < \zeta] = E^x[L_\zeta; \tau_n \leq T(D), \tau_{3n} < \zeta] \\ +E^x[L_\zeta; T(D) < \tau_n, \tau_{3n} < \zeta] = I + II. \end{aligned}$$

If $T(D) < \tau_n$, then $T(D) + \tau_n \circ \theta_{T(D)} \leq \tau_{3n}$ since $\sup_s \Delta A_s^+ < n$ and $\tau_n < \tau_{3n}$ on $\{\tau_{3n} < \zeta\}$. Therefore writing $T_D = T(D)$ when convenient,

$$\begin{aligned} (5.5) \quad II &= E^x[L_\zeta; T(D) < \tau_n < \tau_{3n} < \zeta] \\ &\leq E^x[L_{T(D)} L_\zeta \circ \theta_{T(D)}; T(D) + \tau_n \circ \theta_{T(D)} < \zeta, T(D) < \zeta] \\ &= E^x[L_{T(D)} E^{X(T_D)}[L_\zeta; \tau_n < \zeta]; T(D) < \zeta]. \end{aligned}$$

Noting that $L_\zeta = L_{T(D)} L_\zeta \circ \theta_{T(D)}$ even if $T(D) \geq \zeta$ because $\zeta \circ \theta_{T(D)} = 0$ in that case, and that $L_{T(D)} \leq L^+_{T(D)}$ one obtains from (5.4),

$$I \leq \|g\| E^x[L^+_{T(D)}; \tau_n \leq T(D),] \leq \|g\| E^x[(L^+_{T(D)})^p]^{1/p} P^x[\tau_n \leq T(D)]^{1/q}$$

where $1/p + 1/q = 1$. Moreover $\{\tau_n \leq T(D)\} \subset \{A^+_{T(D)} \geq n\}$ and so

$$P^x[\tau_n \leq T(D)] \leq \frac{1}{n}[E^x[A^+_{T(D)-}] + c(A^+)] \leq \frac{1}{n}[\beta + c(A^+)].$$

Consequently I approaches zero uniformly in x as $n \to \infty$. On the otherhand $X(T_D) \in D$ on $\{T(D) < \zeta\}$ since D is closed. Thus from (5.2), (5.3) and (5.5)

$$II \leq \epsilon E^x[L^+_{T(D)}] \leq \epsilon(1-\beta)^{-1}[1 + c(A^+)].$$

Combining these estimates we find that

$$\lim_n \sup_x E^x[L_\zeta; \tau_n < \zeta] = 0. \tag{5.6}$$

Next observe that $L_\zeta \geq (L_\zeta^-)^{-1} \geq e^{-A_\zeta^-}$. Therefore if $c = \sup_x E^x(A_\zeta^-)$, Jensen's inequality implies that $E^x(L_\zeta) \geq e^{-c}$. Hence

$$E^x[L_{\tau_n}; \tau_n < \zeta] \leq e^c E^x[L_{\tau_n} E^{X(\tau_n)}(L_\zeta); \tau_n < \zeta] = e^c E^x[L_\zeta; \tau_n < \zeta]$$

and combining this with (5.6) completes the proof of Proposition 5.1 ■

We come now to the main result of this section. This should be compared with Corollary 2.16 in [C02].

(5.7) Theorem. *Let A^+ and A^- satisfy the hypotheses of (5.1) and suppose in addition that A^- has bounded jumps. Then the following are equivalent:*

(i) $\sup_x E^x[L_\zeta] < \infty$;
(ii) $\sup_x E^x \int_0^\zeta L_{t-} dA_t^+ < \infty$;
(iii) $\sup_x E^x \int_0^\zeta L_t dA_t^+ < \infty$;
(iv) $\sup_x E^x[\sup_{t<\zeta} L_t] < \infty$;

Proof. Since $L_t = L_{t-}(1 + \Delta A_t^+)(1 + \Delta A_t^-)^{-1} \leq [1 + c(A^+)]L_{t-}$ and $L_{t-} \leq [1 + c(A^-)]L_t$, the equivalence of *(ii)* and *(iii)* is clear. Also $dL_t^+ = L_{t-}^+ dA_t^+$ and so

$$\int_{[0,t[} L_{s-} dA_s^+ = \int_{[0,t[} (L_{s-}^-)^{-1} dL_s^+ \geq (L_{t-}^-)^{-1}[L_{t-}^+ - 1] \geq [L_{t-} - 1]$$

for $t \leq \zeta$. Taking $t = \zeta$, *(ii)* implies *(i)*. Also taking the supremum over $t \in [0, \zeta[$,

$$\sup_{t<\zeta} L_t = \sup_{t<\zeta} L_{t-} \leq 1 + \int_{[0,\zeta[} L_{t-} dA_t^+, \tag{5.8}$$

and so *(ii)* implies *(iv)*. Clearly *(iv)* implies *(i)*. Thus it suffices to show that *(i)* implies *(ii)* to complete the proof of (5.7). Therefore suppose that *(i)* holds. Using (5.1) with $\epsilon = 1$ choose $N > c(A^+)$ such that

$$\lambda := \sup_x E^x[L_{\tau_N}; \tau_N < \zeta] < 1.$$

Define $\tau_N^0 = 0$ and $\tau_N^{k+1} = \tau_N^k + \tau_N \circ \theta_{\tau_N^k}$ for $k \geq 0$. Then using the strong Markov property, $\sup_x E^x[L_{\tau_N^k}; \tau_N^k < \zeta] \leq \lambda^k$. We claim that $\tau_{kN} \leq \tau_N^k$ and hence $\lim_k \tau_N^k \geq \zeta$. This is obvious when $k = 1$. Assume that it holds for a fixed $k \geq 0$. Writing $A^+(t) = A_t^+$ for typographical simplicity we have

$$\tau_N^{k+1} = \tau_N^k + \sup\{t : A^+(\tau_N^k + t) - A^+(\tau_N^k) > N\}.$$

If $\tau_{kN} \leq \tau_N^k$, $A^+(\tau_N^k) \geq kN$ and so $\tau_N^{k+1} \geq \tau_{(k+1)N}$. This establishes the claim. Now

$$\begin{aligned} E^x \int_0^\zeta L_{t-} dA_t^+ &= \sum_{k\geq 0} E^x \int_{[\tau_N^k, \tau_N^{k+1}[} L_{t-} dA_t^+ ; \tau_N^k < \zeta] \\ &= \sum_{k\geq 0} E^x [L_{\tau_N^k} E^{X(\tau_N^k)} \int_{[0,\tau_N[} L_{t-} dA_t^+ ; \tau_N^k < \zeta]. \end{aligned}$$

But

$$\begin{aligned} E^x \int_{[0,\tau_N[} L_{t-} dA_t^+ &\leq E^x \int_{[0,\tau_N[} L_{t-}^+ dA_t^+ \\ &= E^x [L_{\tau_{N-}}^+ - 1] \leq E^x e^{A^+(\tau_N -)} \leq e^N, \end{aligned}$$

and combining these estimates we obtain

$$E^x \int_0^\zeta L_{t-} dA_t^+ \leq e^N \sum_{k\geq 0} E^x [L_{\tau_N^k} ; \tau_N^k < \zeta] \leq e^N (1-\lambda)^{-1}.$$

Hence *(i)* implies *(ii)* establishing (5.7). ■

(5.9) Remark. Suppose in addition to the hypotheses in (5.7), that E can not be written as the disjoint union of two finely open sets one of which is absorbing as in (4.7). Then the condition (5.7 - i) is equivalent to $E^x(L_\zeta) < \infty$ for at least one $x \in E$. But in view of (5.8), the remaining conditions in (5.7) are equivalent to the corresponding condition with $\sup_x$ replaced by for at least one $x \in E$.

We next give a sufficient condition that the gauge g is bounded. The integral condition in the following result should be compared to the conditions in Theorem 5.7.

(5.10) Theorem. *Let $A^+ \in K_\beta$ with $\beta < 1$ and suppose that $\zeta < \infty$ a.s. If $\sup_x E^x \int_0^\zeta L_t dt < \infty$, then g is bounded.*

In the course of the proof we shall need the following lemma.

(5.11) Lemma. *Let $A \in K_\beta$ with $\beta < \infty$. Then $\lim_{t\downarrow 0} \sup_x E^x(A_t) \leq \beta$. If $\beta < 1$, then there exist $C < \infty$ and $\lambda > 0$ such that $\sup_x E^x[L_t] \leq Ce^{t\lambda}$.*

Proof. Proposition 2.3 in [C02] asserts that $\lim_{\alpha\to\infty} \sup_x E^x \int_0^\infty e^{-\alpha t} dA_t \leq \beta$. The proof in [C02] works perfectly well for right continuous A. If $\eta > 0$,

$$\begin{aligned} E^x \int_0^\infty e^{-\alpha t} dA_t &= \alpha E^x \int_0^\infty e^{-\alpha t} A_t dt \\ &= \int_0^\infty e^{-t} E^x(A_{t/\alpha}) dt \geq E^x[A_{\eta/\alpha}] e^{-\eta}. \end{aligned}$$

This implies that $\lim_{t\to 0} \sup_x E^x(A_t) \leq \beta e^\eta$ and letting η fall to zero yields the first assertion in (5.11). If $\beta < 1$, then it follows from (2.7) that there exists $t > 0$ such that $\sup_x E^x(L_t) < \infty$. Since $Q_t f(x) := E^x[f(X_t)L_t]$ defines a semigroup, it is well-known and easily checked that this implies the final assertion in (5.11). ■

We now turn to the proof of (5.10). By (5.11), there exist $C < \infty$ and $\lambda > 0$ such that $E^x[L_t^+] \leq Ce^{\lambda t}$. Since $\zeta < \infty$,

$$\begin{aligned} g(x) &= \sum_{n\geq 0} E^x[L_\zeta; n < \zeta \leq n+1] \\ &= \sum_{n\geq 0} E^x[L_n E^{X(n)}[L_\zeta; \zeta \leq 1];; n < \zeta] \\ &\leq \sum_{n\geq 0} E^x[L_n E^{X(n)}[L_1^+];; n < \zeta] \leq Ce^\lambda \sum_{n\geq 0} E^x[L_n; n < \zeta]. \end{aligned}$$

If $n \leq t < n+1$, then writing $c = Ce^\lambda$

$$\begin{aligned} &E^x[L_{n+1}; n+1 < \zeta] \leq E^x[L_{n+1}; t < \zeta] \\ &\quad = E^x[L_t E^{X(t)}[L_{n+1-t}]; t < \zeta] \leq cE^x[L_t; t < \zeta]. \end{aligned}$$

Consequently

$$\sum_{n\geq 0} E^x[L_n; n < \zeta] \leq 1 + cE^x \int_0^\zeta L_t dt,$$

and hence g is bounded. ■

Remarks. The proof of (5.10) is just the argument on page 831 of [CR88]. Under the hypotheses in the first sentence of (5.10), the proof shows that for x fixed, $E^x \int_0^\zeta L_t dt < \infty$ implies that $g(x) < \infty$. Note that in (5.10) it is not assumed that $E^\cdot(A_\zeta^-)$ is bounded. If one assumes in addition that $E^\cdot(\zeta)$ and $E^\cdot(A_\zeta^-)$ are bounded, then the proof of Theorem 6 in [CR88] may be modified to show that $\sup_x E^x \int_0^\zeta L_{t-} dt < \infty$

is necessary for g to be bounded. This requires showing first that $\lim_{n\to\infty} E^x[L_n; n < \zeta] = 0$ uniformly in x, which may be proved by an argument that is similar to, but simpler than, the proof of (5.1). We leave the details to the interested reader.

References

[BG68] R.M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York, 1968.

[C02] Z-Q. Chen, *Gaugeability and conditional gaugeability*, Trans. Amer. Math. Soc., **354**(2002), 4639-4679.

[CS02] Z-Q. Chen and R. Song, *General gauge and conditional gauge theorems*, Ann. Probab., **30**(2002), 1313-1339.

[CS03] Z-Q. Chen and R. Song, *Conditional gauge theorem for non-local Feynman-Kac transforms*, Probab. Theory Related Fields, **125**(2003), 45-72.

[CR88] K. L. Chung and K. M. Rao, *General gauge theorem for multiplicative functionals*, Trans. Amer. Math. Soc., **306**(1988), 819-836.

[CZ95] K. L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's equation*, Springer, Berlin, 1995.

[DD70] C. Doléans-Dade, *Quelques applications de la formule de changement de variables pour les semimartingales*, Z. Wahrsch, **16**(1970), 181-194.

[DM87] C. Dellacherie and P. -A. Meyer, *Probabilités et Potentiel, Chapitres XII - XVI*, Hermann, Paris, 1987.

[DS58] N. Dunford and J. T. Schwartz, *Linear Operators: Part I*, Interscience, New York, 1958.

[FG96] P.J. Fitzsimmons and R.K. Getoor, *Smooth measures and continuous additive functionals of right Markov processes.* In Itô's Stochastic Calculus and probability Theory, Springer, Tokyo, 1996, pp.31-49.

[G90] R. K. Getoor, *Excessive Measures*, Birkhäuser, Boston, 1990.

[GS84] R. K. Getoor and M. J. Sharpe, *Naturality, standardness and weak duality*, Z. Wahrsch., **67**(1984), 1-62.

[Re70] D. Revuz, *Measures aassociées aux fonctionnelles additives de Markov I*, Trans. Amer. Math. Soc., **148**(1970), 501-531.

[S88] M. Sharpe, *General Theory of Markov Processes*, Academic Press, Boston, 1988.

[So93] R. Song, *Probabilistic approach to the Dirichlet probem of perturbed stable processes*, Prob. Th. Rel. Fields **95**(1993), 371-389.

[SS00] W. Stummer and K. -Th Sturm, *On exponentials of additive functionals*, Stochastic Process. Appl., **85** (2000), 45-60.

[St91] K.-Th. Sturm, *Gauge theorems for resolvents with application to Markov processes*, Probab. Theory Related Fields, **89**(1991), 387-406.

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