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On the Characterization of the Set $\mathcal{D}_1$ of the Affine Weyl Group of Type $\tilde{A}_n$

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Dedicated to N. Iwahori

Abstract.

In this paper we show that a conjecture of Lusztig on distinguished involutions is true for the affine Weyl group of type $\tilde{A}_n$.

§1. Springer's formula and Lusztig's Conjecture

1.1. Let (W, S) be a Coxeter group with S the set of simple reflections. Let H be the Hecke algebra of (W, S) over $\mathcal{A} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ ($q^{\frac{1}{2}}$ an indeterminate). Then H is a free $\mathcal{A}$-module with a basis $\{T_w\}_{w\in W}$ and multiplication relations

$$(T_s - q)(T_s + 1) = 0 \qquad \text{if} \quad s \in S,$$

$$T_w T_u = T_{wu} \qquad \text{if} \quad l(wu) = l(w) + l(u),$$

where $l : W \longrightarrow \mathbf{N}$ is the length function.

In 1979, Kazhdan and Lusztig published a paper [KL] in which the famous Kazhdan-Lusztig polynomials are introduced, which are uniquely defined by the following properties. For each x in W, there exists a unique element

$$C_x = q^{-\frac{l(x)}{2}} \sum_{y \leq x} P_{y,x} T_y$$

(here $\leq$ is the Bruhat order on (W, S)) such that
(1) C_x is invariant under the ring involution $H \longrightarrow H$ defined by $q^{\frac{1}{2}} \longrightarrow q^{-\frac{1}{2}}$, $T_w \longrightarrow T_{w^{-1}}^{-1}$,
(2) $P_{y,x}$ are polynomials in q with degree less than or equal to $\frac{1}{2}(l(x) - l(y) - 1)$ if $y \leq x$ and $y \neq x$, and $P_{x,x} = 1$.

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$P_{y,x}$ are the famous Kazhdan-Lusztig polynomials, which play a great role in Lie Theory.

1.2. Suppose that $y \leq x$ and $y \neq x$. We then can write

$$P_{y,x} = \mu(y,x)q^{\frac{1}{2}(l(x)-l(y)-1)} + \text{lower degree terms.}$$

The coefficients $\mu(y,x)$ are important for understanding the Kazhdan-Lusztig polynomials and Kazhdan-Lusztig cells. We are interested in properties of the coefficients $\mu(y,x)$. We set $\mu(x,y) = \mu(y,x)$ if $\mu(y,x)$ is defined.

Assume that (W,S) is an affine Weyl group or a Weyl group. The following formula is due to Springer (see [Sp, S2]),

(a) $$\mu(y,x) = \sum_{d\in\mathcal{D}_0} \delta_{y^{-1},x,d} + \sum_{f\in\mathcal{D}_1} \gamma_{y^{-1},x,f}\pi(f).$$

We need explain the notations. Write

$$C_xC_y = \sum_{z\in W} h_{x,y,z}C_z, \qquad h_{x,y,z} \in \mathcal{A} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Define

$$a(z) = \min\{i \in \mathbf{N} \mid q^{-\frac{i}{2}}h_{x,y,z} \in \mathbf{Z}[q^{-\frac{1}{2}}] \text{ for all } x,y \in W\}.$$

If for any i, $q^{-\frac{i}{2}}h_{x,y,z} \notin \mathbf{Z}[q^{-\frac{1}{2}}]$ for some $x,y \in W$, we set $a(z) = \infty$. It is not clear that whether there exists a Coxeter group (W,S) such that $a(z) = \infty$ for some $z \in W$.

From now on, we assume that the function $a: W \to \mathbf{N}$ is bounded and (W,S) is crystallographic. Obviously, when W is finite, the function a is bounded. Lusztig showed that the function a is bounded for all affine Weyl groups (see [L1]). Following Lusztig and Springer, we define $\delta_{x,y,z}$ and $\gamma_{x,y,z}$ by the following formula,

$$h_{x,y,z} = \gamma_{x,y,z}q^{\frac{a(z)}{2}} + \delta_{x,y,z}q^{\frac{a(z)-1}{2}} + \text{ lower degree terms.}$$

Springer showed that $l(z) \geq a(z)$ (see [L2]). Let $\delta(z)$ be the degree of $P_{e,z}$, where e is the neutral element of W. Then actually one has $l(z) - a(z) - 2\delta(z) \geq 0$ (see [L2]). Set

$$\mathcal{D}_i = \{z \in W \mid l(z) - a(z) - 2\delta(z) = i\}.$$

The number $\pi(z)$ is defined by $P_{e,z} = \pi(z)q^{\delta(z)} +$ lower degree terms.

The elements of $\mathcal{D}_0$ are involutions, called distinguished involutions of (W,S) (see [L2]). Moreover, in a Weyl group or an affine Weyl group, each left cell (resp. right cell) contains exactly one element of $\mathcal{D}_0$, see [L2].

Example: Let S' be a subset of S such that the subgroup W' of W generated by S' is finite. Then the longest element of W' is in $\mathcal{D}_0$.

1.3. Comparing with the set $\mathcal{D}_0$, we know very little about the set $\mathcal{D}_1$. From the formula of Springer, we see that the set $\mathcal{D}_1$ is important for understanding the coefficients $\mu(y, x)$. Lusztig has an interesting conjecture for describing the set $\mathcal{D}_1$. For stating the conjecture we need the concept of cell.

We refer to [KL] for the definition of left cell, right cell and two-sided cell. For elements w, u in W we shall write $w \underset{L}{\sim} u$ (resp. $w \underset{R}{\sim} u$; $w \underset{LR}{\sim} u$) if w and u are in the same left (resp. right; two-sided) cell of W. Now we can state the conjecture of Lusztig (see[L3, S2]).

Conjecture: Let $z \in W$. Then z is in $\mathcal{D}_1$ if and only if there exists some d in $\mathcal{D}_0$ such that $z \underset{LR}{\sim} d$ and $\mu(z, d) \neq 0$.

To go further, we need some properties of $\gamma_{x,y,z}$ and $\delta_{x,y,z}$. The following are some properties of $\gamma_{w,u,v}$ (see [L2] for (a)-(d) and [L1] for (e)).

(a) If $\gamma_{w,u,v}$ is not equal to 0, then $w \underset{L}{\sim} u^{-1}$, $u \underset{L}{\sim} v$ and $w \underset{R}{\sim} v$. In particular we have $w \underset{LR}{\sim} u \underset{LR}{\sim} v$ if $\gamma_{w,u,v}$ is not equal to 0.

(b) $\gamma_{w,u,v} = \gamma_{u,v^{-1},w^{-1}}$ and $\gamma_{u^{-1},w^{-1},v^{-1}} = \gamma_{w,u,v}$.

(c) Let d be in $\mathcal{D}_0$. Then $\gamma_{w,d,u} \neq 0$ if and only if $w = u$ and $w \underset{L}{\sim} d$. Moreover $\gamma_{w,d,w} = \gamma_{d,w^{-1},w^{-1}} = \gamma_{w^{-1},w,d} = 1$.

(d) $w \underset{L}{\sim} u^{-1}$ if and only if $\gamma_{w,u,v}$ is not equal to 0 for some v.

(e) The positivity: as a Laurent polynomial in $q^{\frac{1}{2}}$, the coefficients of $h_{w,u,v}$ are non-negative. In particular, $\gamma_{w,u,v}$ and $\delta_{w,u,v}$ are non-negative for all w, u, v in W.

The following property is due to Springer, see [Sp] or [S2].
(f) If $w \underset{LR}{\sim} u$ and $\delta_{w,u,v} \neq 0$, then $w \underset{R}{\sim} v$ and $u \underset{L}{\sim} v$.

1.4. The observations of this subsection are due to Shi, Springer and Lusztig.

We can see easily that the "only if" part of the conjecture in subsection 1.3 is true. If z is in $\mathcal{D}_1$, then we can find some d in $\mathcal{D}_0$ such that $z \underset{R}{\sim} d$ (or equivalently, $z^{-1} \underset{L}{\sim} d$, note that $d = d^{-1}$). By 1.3 (c), we have

$\gamma_{z^{-1},d,z^{-1}} = 1$. By the positivity (see 1.3 (e)) and Springer's formula (see 1.2 (a)), we see that $\mu(z,d) \neq 0$. This observation and argument are due to Shi, see [S2]

Moreover, for $z \in W$ and $d \in \mathcal{D}_0$, if $\mu(z,d) \neq 0$, $z \underset{LR}{\sim} d$ and $z \underset{L}{\not\sim} z^{-1}$, it is easy to prove that z is in $\mathcal{D}_1$. Now we argue for this. We have

$$\mu(z,d) = \sum_{d' \in \mathcal{D}_0} \delta_{z^{-1},d,d'} + \sum_{f \in \mathcal{D}_1} \gamma_{z^{-1},d,f}\pi(f).$$

By 1.3 (c), $\gamma_{z^{-1},d,f} \neq 0$ implies that $f = z^{-1}$. Note that z is in $\mathcal{D}_1$ if and only if z^{-1} is in $\mathcal{D}_1$. If z is not in $\mathcal{D}_1$, then we have $\mu(z,d) = \sum_{d' \in \mathcal{D}_0} \delta_{z^{-1},d,d'}$. But if $\delta_{z^{-1},d,d'} \neq 0$, by 1.3 (f), we then have $z^{-1} \underset{R}{\sim} d'$ and $d \underset{L}{\sim} d'$. Thus we must have $d' = d$ since each left cell contains only one element of $\mathcal{D}_0$. So we must have $z^{-1} \underset{R}{\sim} d$. Since $\mu(z,d) = \mu(z^{-1},d^{-1}) = \mu(z^{-1},d)$, applying the same argument to z^{-1} we can see that $z \underset{R}{\sim} d$. In conclusion, we have $z \underset{R}{\sim} z^{-1}$, i.e., $z \underset{L}{\sim} z^{-1}$. This contradicts $z \underset{L}{\not\sim} z^{-1}$. Therefore we must have $z \in \mathcal{D}_1$. Essentially this observation and argument are due to Springer, see [S2] and [Sp].

The conjecture was also proved for Weyl groups by Lusztig with the following two exceptions (1) W is of type E_7 and $a(z) = 512$, (2) W is of type E_8 and $a(z) = 4096$ (see [L3]). In this paper we shall prove the following result.

Theorem 1.5. *Let (W,S) be an affine Weyl group of type $\tilde{A}_n$, then the conjecture of Lusztig in subsection 1.3 is true.*

We need some preparation to prove the theorem.

§2. Proof of Theorem 1.5

We shall need the star operations introduced by Kazhdan and Lusztig in [KL].

2.1. For w in W, set $L(w) = \{s \in S \mid sw \leq w\}$ and $R(w) = \{s \in S \mid ws \leq w\}$. Let s and t be in S such that st has order 3, i.e. $sts = tst$. Define

$$D_L(s,t) = \{w \in W \mid L(w) \cap \{s,t\} \text{ has exactly one element}\},$$

$$D_R(s,t) = \{w \in W \mid R(w) \cap \{s,t\} \text{ has exactly one element}\}.$$

If w is in $D_L(s,t)$, then $\{sw, tw\}$ contains exactly one element in $D_L(s,t)$, denoted by ${}^*w$, here $* = \{s,t\}$. The map: $D_L(s,t) \to D_L(s,t)$,

$w \to {}^*w$, is an involution and is called a **left star operation**. Similarly if $w \in D_R(s,t)$ we can define the **right star operation** $w \to w^* = \{ws, wt\} \cap D_R(s,t)$ on $D_R(s,t)$, where $* = \{s,t\}$. The following are some properties proved in [KL].

Let s and t be in S such that st has order 3 and set $* = \{s,t\}$. Assume that y, w are in $D_L(s,t)$. We have

(a) $\mu(y,w) = \mu({}^*y, {}^*w)$.

(b) $y \underset{R}{\sim} w$ if and only if ${}^*y \underset{R}{\sim} {}^*w$.

(c) $w \underset{L}{\sim} {}^*w$.

Let $* = \{s,t\}$. Assume that y, w are in $D_R(s,t)$. We have

(d) $\mu(y,w) = \mu(y^*, w^*)$.

(e) $y \underset{L}{\sim} w$ if and only if $w^* \underset{L}{\sim} y^*$.

(f) $w \underset{R}{\sim} w^*$.

2.2. Recall that we have assumed that (W, S) is crystallographic and the function $a : W \to \mathbf{N}$ is bounded. The following results (a-d) are proved in [X, section 1.4] and the assertion (e) is due to Shi (see [C, Theorem 1.10]).

Let s, t, s', t' be in S such that both st and $s't'$ have order 3. Assume that w is in $D_L(s,t) \cap D_R(s',t')$. Set $* = \{s,t\}$ and $\star = \{s',t'\}$. Then

(a) ${}^*w$ is in $D_R(s',t')$ and $w^\star$ is in $D_L(s,t)$. Moreover, we have ${}^*(w^\star) = ({}^*w)^\star$. We shall write ${}^*w^\star$ for ${}^*(w^\star) = ({}^*w)^\star$.

(b) $({}^*w)^{-1} = (w^{-1})^*$ and $({}^*w^\star)^{-1} = {}^\star(w^{-1})^*$

Let s, t, s', t' be as above and $* = \{s,t\}$ and $\star = \{s',t'\}$. Suppose that w is in $D_L(s,t)$ and u is in $D_R(s',t')$. Let v be in W such that $v \underset{L}{\sim} u$ and $v \underset{R}{\sim} w$. Then

(c) We have $v \in D_L(s,t) \cap D_R(s',t')$, so ${}^*v^\star$ is well defined and we have $h_{w,u,v} = h_{{}^*w, u^\star, {}^*v^\star}$, see subsection 1.2 for the definition of $h_{w,u,v}$.

Let s, t, s', t', s'', t'' be in S such that all st, $s't'$ and $s''t''$ have order 3. Suppose that w is in $D_L(s,t) \cap D_R(s'',t'')$ and u is in $D_L(s'',t'') \cap D_R(s',t')$. Set $* = \{s,t\}, \# = \{s'',t''\}$, and $\star = \{s',t'\}$. Let v be in W such that v is in $D_L(s,t) \cap D_R(s',t')$. Then we have

(d) $\gamma_{w,u,v} = \gamma_{{}^*w^\#, {}^\#u^\star, {}^*v^\star}$.

(e) Let w be in W such that $w = w^{-1}$. If w is in $D_L(s,t)$ or in $D_R(s,t)$, then ${}^*w^*$ is well defined for $* = \{s,t\}$. Moreover, if w is in $\mathcal{D}_0$, then ${}^*w^*$ is also in $\mathcal{D}_0$.

Lemma 2.3. *Let s, t be in S such that st has order 3, z in W and d in $\mathcal{D}_0$ such that $z \underset{LR}{\sim} d$. Assume that $\delta_{z^{-1},d,d} = 0$ and $\mu(z,d) \neq 0$. Then*
(a) z is in $\mathcal{D}_1$ and $\mu(z,d) = \pi(z)$.
(b) If $z \underset{L}{\sim} z^{-1} \underset{L}{\sim} d$ and ${}^*z^*$ is well defined for $* = \{s,t\}$, then ${}^*z^*$ is in $\mathcal{D}_1$ and $\mu({}^*z^*, {}^*d^*) = \pi({}^*z^*) = \pi(z) = \mu(z,d)$.

Proof. (a) Using Springer's formula 1.2 (a) and 1.3 (f), we see

$$\mu(z,d) = \delta_{z^{-1},d,d} + \sum_{f \in \mathcal{D}_1} \gamma_{z^{-1},d,f}\pi(f).$$

Now $\delta_{z^{-1},d,d} = 0$, so we get

$$\mu(z,d) = \sum_{f \in \mathcal{D}_1} \gamma_{z^{-1},d,f}\pi(f).$$

By 1.3 (c), $\gamma_{z^{-1},d,f} \neq 0$ implies that $f = z^{-1}$ and $\gamma_{z^{-1},d,f} = 1$. Hence z^{-1} is in $\mathcal{D}_1$, or equivalently z is in $\mathcal{D}_1$, and $\mu(z,d) = \pi(z^{-1}) = \pi(z)$.

(b) According to 2.1 (a) and 2.1 (d), we have $\mu({}^*z^*, {}^*d^*) = \mu(z,d) \neq 0$. By 2.2 (e), ${}^*d^*$ is in $\mathcal{D}_0$. By Springer's formula 1.2 (a), 1.3 (f) and 2.2 (b), we have

$$\mu({}^*z^*, {}^*d^*) = \delta_{{}^*(z^{-1})^*, {}^*d^*, {}^*d^*} + \sum_{f \in \mathcal{D}_1} \gamma_{{}^*(z^{-1})^*, {}^*d^*, f}\pi(f).$$

We need prove $\delta_{{}^*(z^{-1})^*, {}^*d^*, {}^*d^*} = 0$. By 2.1 (b-c), 2.1 (e-f) and 2.2 (c), $h_{{}^*(z^{-1})^*, {}^*d^*, {}^*d^*} = h_{(z^{-1})^*, {}^*d, d}$, so

$$\delta_{{}^*(z^{-1})^*, {}^*d^*, {}^*d^*} = \delta_{(z^{-1})^*, {}^*d, d}.$$

By Springer's formula 1.2 (a) and 2.2 (b), we have

$$\mu({}^*z, {}^*d) = \sum_{d' \in \mathcal{D}_0} \delta_{(z^{-1})^*, {}^*d, d'} + \sum_{f \in \mathcal{D}_1} \gamma_{(z^{-1})^*, {}^*d, f}\pi(f).$$

By 2.1 (a), $\mu({}^*z, {}^*d) = \mu(z,d)$, so

$$\mu({}^*z, {}^*d) = \mu(z,d) = \sum_{d' \in \mathcal{D}_0} \delta_{(z^{-1})^*, {}^*d, d'} + \sum_{f \in \mathcal{D}_1} \gamma_{(z^{-1})^*, {}^*d, f}\pi(f).$$

By 2.2 (d), we know that $\gamma_{(z^{-1})^*,{}^*d,f} = \gamma_{z^{-1},d,f}$. Thus, $\gamma_{(z^{-1})^*,{}^*d,f} \neq 0$ implies that $f = z^{-1}$ and $\gamma_{(z^{-1})^*,{}^*d,f} = 1$. By (a), we also have $\mu(z,d) = \pi(z) = \pi(z^{-1})$. As a consequence, we must have $\delta_{(z^{-1})^*,{}^*d,d'} = 0$ for any d' in $\mathcal{D}_0$. Therefore, $\delta_{{}^*(z^{-1})^*,{}^*d^*,{}^*d^*} = 0$. Since ${}^*d^*$ is in $\mathcal{D}_0$ (see 2.2 (e)), by 1.3 (c), we know $\gamma_{{}^*(z^{-1})^*,{}^*d^*,f} \neq 0$ implies that $f = {}^*(z^{-1})^*$ and $\gamma_{{}^*(z^{-1})^*,{}^*d^*,f} = 1$. Therefore ${}^*(z^{-1})^* = ({}^*z^*)^{-1}$ and ${}^*z^*$ are in $\mathcal{D}_1$. Moreover we have $\mu(z,d) = \mu({}^*z^*, {}^*d^*) = \pi({}^*(z^{-1})^*) = \pi({}^*z^*)$. The lemma is proved. □

2.4. Now we can prove the theorem. When $n = 1$, Theorem 1.5 is clearly true. Now assume that $n \geq 2$. By the discussion in 1.4, we only need to prove that for some $d \in \mathcal{D}_0$, if $z \underset{L}{\sim} z^{-1} \underset{L}{\sim} d$ and $\mu(z,d) \neq 0$, then $z \in \mathcal{D}_1$. According to [S1, 18.3.2], there exists a sequence of right star operations such that its composition sends z (resp. d) to some $z'w$ (resp. yw) for the longest element w of a parabolic subgroup of W and such that $z'w \underset{L}{\sim} w$ (resp. $yw \underset{L}{\sim} w$). Using 2.2 (a), we can apply the corresponding (in the same order) left star operations to $z'w$ (resp. yw). Then we obtain some element xw (resp. w, here 2.2 (e) is needed) such that $xw \underset{L}{\sim} (xw)^{-1} \underset{L}{\sim} w$. By 2.1 (a) and 2.1 (d), clearly we have $\mu(xw,w) = \mu(z,d) \neq 0$. Note that for x_1 and x_2 in a Coxeter group, we have $R(x_1) = R(x_2)$ (resp. $L(x_1) = L(x_2)$) if x_1 and x_2 are in the same left (resp. right) cell of the Coxeter group, see [KL]. By 2.1 (b) and 2.1 (e), we have $R(z'w) = R(yw) = R(xw) = L(xw) = R(w) = L(w)$. Thus it is obvious that $\delta_{xw,w,w} = 0$. By Lemma 2.3 (a), we see that xw is in $\mathcal{D}_1$. By Lemma 2.3 (b) we know that z is in $\mathcal{D}_1$. The theorem is proved.

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