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Crystal Bases and Diagram Automorphisms

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Abstract.

We prove that the action of an ω-root operator on the set of all paths fixed by a diagram automorphism ω of a Kac–Moody algebra $\mathfrak{g}$ can be identified with the action of a root operator for the orbit Lie algebra $\breve{\mathfrak{g}}$. Moreover, we prove that there exists a canonical bijection between the elements of the crystal base $\mathcal{B}(\infty)$ for $\mathfrak{g}$ fixed by ω and the elements of the crystal base $\breve{\mathcal{B}}(\infty)$ for $\breve{\mathfrak{g}}$. Using this result, we give twining character formulas for the "negative part" of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ and for certain modules of Demazure type.

§0. Introduction.

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{Q}$ associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ with Cartan subalgebra $\mathfrak{h}$ and Weyl group $W = \langle r_i \mid i \in I \rangle$. A path is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathfrak{h}^*$ such that $\pi(0) = 0$, where $[0,1] := \{t \in \mathbb{Q} \mid 0 \le t \le 1\}$. We denote by $\mathbb{P}$ the set of all paths (modulo reparametrization). In [13], Littelmann defined root operators $e_i, f_i : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$, where θ is an extra element, and introduced the notion of Lakshmibai–Seshadri paths of shape λ, where $\lambda \in \mathfrak{h}^*$ is a dominant integral weight. By using root operators, we can make the set $\mathbb{B}(\lambda)$ of Lakshmibai–Seshadri paths of shape λ into a crystal which is isomorphic to the crystal base $\mathcal{B}(\lambda)$ of an integrable highest weight $U_q(\mathfrak{g})$-module of highest weight λ (see [3] and [10]), where $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{Q}(q)$.

Let $\omega \in \mathrm{Aut}(\mathfrak{g})$ be a diagram automorphism of $\mathfrak{g}$, and $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ the contragredient map of the restriction $\omega|_{\mathfrak{h}}$ of ω to $\mathfrak{h}$. For a path $\pi \in \mathbb{P}$, we define a path $\omega(\pi) \in \mathbb{P}$ by $(\omega(\pi))(t) := \omega^*(\pi(t))$ for $t \in [0,1]$. In [20] and [21], we introduced ω-root operators $\widetilde{e}_i$ and $\widetilde{f}_i$ (see (2.2.2)), and then proved that the Lakshmibai–Seshadri paths fixed by ω can be

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identified with the Lakshmibai–Seshadri paths for the orbit Lie algebra $\breve{\mathfrak{g}}$, which is a certain Kac–Moody algebra corresponding to ω.

In this paper, we first prove that the action of an ω-root operator on the set of all paths fixed by ω can be identified with the action of a root operator for the orbit Lie algebra $\breve{\mathfrak{g}}$, generalizing results in [20] and [22]. Then, using results in [20] and [21], we show that there exists a canonical bijection between the elements of the crystal base $\mathcal{B}(\infty)$ of the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ fixed by ω and the elements of the crystal base $\breve{\mathcal{B}}(\infty)$ of the negative part $U_q^-(\breve{\mathfrak{g}})$ of $U_q(\breve{\mathfrak{g}})$. In addition, we give twining character formulas for $U_q^-(\mathfrak{g})$ and for certain modules $(U_w^-)_q(\mathfrak{g})$ of Demazure type.

Let us explain our results more precisely. We set $(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}$, $\widetilde{W} := \{w \in W \mid \omega^* w = w\omega^*\}$. Note that there exist a natural $\mathbb{Q}$-linear isomorphism $P_\omega^* : \widehat{\mathfrak{h}} \to (\mathfrak{h}^*)^0$ and a group isomorphism $\Theta : \widehat{W} \to \widetilde{W}$, where $\widehat{\mathfrak{h}}$ is the Cartan subalgebra of the orbit Lie algebra $\breve{\mathfrak{g}}$ and $\widehat{W}$ is the Weyl group of $\breve{\mathfrak{g}}$. Denote by $\widehat{\mathbb{P}}$ the set of all paths (modulo reparametrization) for the orbit Lie algebra $\breve{\mathfrak{g}}$, and by $\widehat{e}_i$, $\widehat{f}_i : \widehat{\mathbb{P}} \cup \{\theta\} \to \widehat{\mathbb{P}} \cup \{\theta\}$ root operators for $\breve{\mathfrak{g}}$. For a path $\widehat{\pi} \in \widehat{\mathbb{P}}$, we define a path $P_\omega^*(\widehat{\pi}) \in \mathbb{P}$ by $(P_\omega^*(\widehat{\pi}))(t) := P_\omega^*(\widehat{\pi}(t))$ for $t \in [0,1]$, and set $P_\omega^*(\theta) = \theta$.

In [20] and [22], we proved that the equalities $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ hold on a certain subset of $\widehat{\mathbb{P}}$. In this paper, we extend this result to the whole of $\widehat{\mathbb{P}}$.

Theorem 1. *The set $\mathbb{P}^0 \cup \{\theta\}$ is stable under the ω-root operators, where $\mathbb{P}^0 := \{\pi \in \mathbb{P} \mid \omega(\pi) = \pi\}$. Furthermore, we have $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ on $\widehat{\mathbb{P}}$.*

Denote by $e_i, f_i : \mathcal{B}(\infty) \cup \{0\} \to \mathcal{B}(\infty) \cup \{0\}$ the Kashiwara operators for the crystal base $\mathcal{B}(\infty)$. Let $w \in W$, and $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. We define a subset $\mathcal{B}_w(\infty)$ of $\mathcal{B}(\infty)$ by

$$\mathcal{B}_w(\infty) := \{f_{i_1}^{m_1} f_{i_2}^{m_2} \ldots f_{i_k}^{m_k} \overline{v}_\infty \mid m_j \in \mathbb{Z}_{\geq 0}\},$$

where $\overline{v}_\infty$ is the (unique) highest weight element of $\mathcal{B}(\infty)$. We know from [8] that $\mathcal{B}_w(\infty)$ is the crystal base of the following module $(U_w^-)_q(\mathfrak{g})$ of Demazure type:

$$(U_w^-)_q(\mathfrak{g}) = \sum_{m_j \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) y_{i_1}^{m_1} y_{i_2}^{m_2} \cdots y_{i_k}^{m_k} \subset U_q^-(\mathfrak{g}),$$

where y_i, $i \in I$, are the Chevalley generators corresponding to negative roots. We also know that $\mathcal{B}_w(\infty)$ (and hence $(U_w^-)_q(\mathfrak{g})$) does not depend on the choice of the reduced expression of w.

There exists a canonical $\mathbb{Q}(q)$-algebra automorphism $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ of $U_q(\mathfrak{g})$ induced from the diagram automorphism ω. Since the crystal lattice $\mathcal{L}(\infty)$ of $U_q^-(\mathfrak{g})$ is stable under ω, we obtain a $\mathbb{Q}$-linear automorphism $\omega : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ induced from $\omega : \mathcal{L}(\infty) \to \mathcal{L}(\infty)$. Note that the crystal base $\mathcal{B}(\infty)$ and its subset $\mathcal{B}_w(\infty)$ for $w \in \widetilde{W}$ are stable under ω. We set

$$\mathcal{B}^0(\infty) := \{b \in \mathcal{B}(\infty) \mid \omega(b) = b\}, \quad \mathcal{B}_w^0(\infty) := \{b \in \mathcal{B}_w(\infty) \mid \omega(b) = b\}.$$

We denote by $\widehat{e}_i, \widehat{f}_i : \breve{\mathcal{B}}(\infty) \cup \{0\} \to \breve{\mathcal{B}}(\infty) \cup \{0\}$ the Kashiwara operators for the crystal base $\breve{\mathcal{B}}(\infty)$, and by $\breve{\mathcal{B}}_{\widehat{w}}(\infty)$ the crystal base of the module $(U_{\widehat{w}}^-)_q(\breve{\mathfrak{g}})$ of Demazure type corresponding to $\widehat{w} \in \widehat{W}$.

By using results in [20] and [21], we prove the following theorem.

Theorem 2. *The set $\mathcal{B}^0(\infty) \cup \{0\}$ is stable under the ω-Kashiwara operators, defined in the same way as (2.2.2). Moreover, there exists a canonical bijection $P_\infty : \mathcal{B}^0(\infty) \xrightarrow{\sim} \breve{\mathcal{B}}(\infty)$ such that*

$$(P_\omega^*)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_\infty(b)) \quad \text{for } b \in \mathcal{B}^0(\infty),$$
$$P_\infty \circ \widetilde{e}_i = \widehat{e}_i \circ P_\infty \quad \text{and} \quad P_\infty \circ \widetilde{f}_i = \widehat{f}_i \circ P_\infty.$$

In addition, we have $P_\infty(\mathcal{B}_w^0(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\infty)$ for each $w \in \widetilde{W}$, where $\widehat{w} := \Theta^{-1}(w)$.

The twining character $\mathrm{ch}^\omega(U_q^-(\mathfrak{g}))$ of $U_q^-(\mathfrak{g})$ is defined to be the following formal sum:

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) e(\chi).$$

For each $w \in \widetilde{W}$, we define the twining character $\mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g}))$ of $(U_w^-)_q(\mathfrak{g})$ by

$$\mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g})) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{((U_w^-)_q(\mathfrak{g}))_\chi}\big) e(\chi).$$

As a corollary of Theorem 2, we obtain the following.

Corollary 3. *Let $w \in \widetilde{W}$, and set $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = P_\omega^*(\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})), \qquad \mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g})) = P_\omega^*(\mathrm{ch}\,(U_{\widehat{w}}^-)_q(\breve{\mathfrak{g}})).$$

This paper is organized as follows. In §1, we fix our notation for Kac–Moody algebras, and then recall some basic facts about diagram automorphisms and orbit Lie algebras. In §2, we recall the definition of an ω-root operator, and prove Theorem 1. In §3, we study the elements of some crystal bases fixed by a diagram automorphism, and show Theorem 2. In §4, we obtain Corollary 3 as an application of Theorem 2.

§1. Preliminaries.

1.1. Kac–Moody algebras and diagram automorphisms.

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{Q}$ associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, with Cartan subalgebra $\mathfrak{h}$, simple roots $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, simple coroots $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$, Chevalley generators $\{x_i,\ y_i \mid i \in I\}$, where $\mathfrak{g}_{\alpha_i} = \mathbb{Q}x_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{Q}y_i$, and Weyl group $W = \langle r_i \mid i \in I\rangle$.

Let $\omega : I \to I$ be a bijection of order N such that $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i,\ j \in I$, which we call a (Dynkin) diagram automorphism. Then ω naturally induces a Lie algebra automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ of order N such that $\omega(\mathfrak{h}) = \mathfrak{h}$, and $\omega(x_i) = x_{\omega(i)}$, $\omega(y_i) = y_{\omega(i)}$, $\omega(\alpha_i^\vee) = \alpha_{\omega(i)}^\vee$ for $i \in I$ (see [23, §1.1]). We define a $\mathbb{Q}$-linear automorphism $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ by $(\omega^*(\lambda))(h) := \lambda(\omega^{-1}(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and set

$$(1.1.1)\quad \mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}, \qquad (\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}.$$

We also set

$$(1.1.2)\qquad \widetilde{W} := \{w \in W \mid \omega^* w = w\omega^*\}.$$

Note that $\omega^* r_i (\omega^*)^{-1} = r_{\omega(i)}$ for every $i \in I$.

1.2. Orbit Lie algebras.

We set $c_{ij} := \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)}$ for $i,\ j \in I$, where $N_i := \#\{\omega^k(i) \mid k \geq 0\}$. We choose a complete set $\widehat{I}$ of representatives of the ω-orbits in I, and set $\check{I} := \{i \in \widehat{I} \mid c_{ii} > 0\}$.

Remark 1.2.1 (cf. [2, §2.2]). Assume that $c_{ii} > 0$. Then $c_{ii} = 1$ or 2. If $c_{ii} = 1$, then $a_{i,\omega^{N_i/2}(i)} = -1$ and $a_{i,\omega^k(i)} = 0$ for any other $1 \leq k \leq N_i - 1$, $k \neq N_i/2$, with N_i even. Hence the Dynkin diagram corresponding to the ω-orbit of the i is of type $A_2 \times \cdots \times A_2$ ($N_i/2$ times). If $c_{ii} = 2$, then $a_{i,\omega^k(i)} = 0$ for all $1 \leq k \leq N_i - 1$. Hence the Dynkin diagram corresponding to the ω-orbit of the i is of type $A_1 \times \cdots \times A_1$ (N_i times).

We set $\widehat{a}_{ij} := 2c_{ij}/c_j$ for $i, j \in \widehat{I}$, where $c_i := c_{ii}$ if $i \in \check{I}$, and $c_i := 2$ otherwise. We know from [1, Lemma 2.1] that a matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}}$ is a symmetrizable Borcherds-Cartan matrix, and its submatrix $\check{A} = (\widehat{a}_{ij})_{i,j\in\check{I}}$ is a symmetrizable generalized Cartan matrix. Let $\widehat{\mathfrak{g}} := \mathfrak{g}(\widehat{A})$ be the generalized Kac–Moody algebra over $\mathbb{Q}$ associated to $\widehat{A}$, with Cartan subalgebra $\widehat{\mathfrak{h}}$, simple roots $\widehat{\Pi} = \{\widehat{\alpha}_i\}_{i\in\widehat{I}}$, simple coroots $\widehat{\Pi}^\vee = \{\widehat{\alpha}_i^\vee\}_{i\in\widehat{I}}$, Chevalley generators $\{\widehat{x}_i, \widehat{y}_i \mid i \in \widehat{I}\}$, and Weyl group $\widehat{W} = \langle \widehat{r}_i \mid i \in \check{I}\rangle$. The orbit Lie algebra $\breve{\mathfrak{g}}$ is defined to be the subalgebra of $\widehat{\mathfrak{g}}$ generated by $\widehat{\mathfrak{h}} \cup \{\widehat{x}_i, \widehat{y}_i \mid i \in \check{I}\}$, which is a Kac–Moody algebra associated to $\check{A}$.

As in [1, §2], we obtain $\mathbb{Q}$-linear isomorphisms $P_\omega : \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ and $P_\omega^* : \widehat{\mathfrak{h}}^* \to (\mathfrak{h}^0)^* \cong (\mathfrak{h}^*)^0$ such that

$$(1.2.1) \qquad \begin{cases} P_\omega(\widetilde{\alpha}_i^\vee) = \widehat{\alpha}_i^\vee,\ P_\omega^*(\widehat{\alpha}_i) = \widetilde{\alpha}_i & \text{for each } i \in \widehat{I}, \\ \big(P_\omega^*(\widehat{\lambda})\big)(h) = \widehat{\lambda}(P_\omega(h)) & \text{for } \widehat{\lambda} \in \widehat{\mathfrak{h}}^* \text{ and } h \in \mathfrak{h}^0, \end{cases}$$

where

$$(1.2.2) \qquad \widetilde{\alpha}_i^\vee := \frac{1}{N_i}\sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}^\vee \in \mathfrak{h}^0, \qquad \widetilde{\alpha}_i := \frac{2}{c_i}\sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)} \in (\mathfrak{h}^*)^0.$$

We also know from [1, §3] that there exists a group isomorphism $\Theta : \widehat{W} \to \widetilde{W}$ such that $\Theta(\widehat{w}) = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for each $\widehat{w} \in \widehat{W}$.

§2. Properties of ω-root Operators.

2.1. Root operators.

In this subsection, we recall the definition of a root operator from [13]. A path is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathfrak{h}^*$ such that $\pi(0) = 0$, where $[0,1] := \{t \in \mathbb{Q} \mid 0 \le t \le 1\}$. We regard two paths π and π' as equivalent if there exist piecewise linear, nondecreasing, surjective, continuous maps $\psi, \psi' : [0,1] \to [0,1]$ (reparametrization) such that $\pi \circ \psi = \pi' \circ \psi'$. Denote by $\mathbb{P}$ the set of (representatives of) equivalence classes of all paths under this equivalence relation. For $\pi \in \mathbb{P}$ and $i \in I$, we set

$$(2.1.1) \qquad h_i^\pi(t) := \big(\pi(t)\big)(\alpha_i^\vee), \qquad m_i^\pi := \min\{h_i^\pi(t) \mid t \in [0,1]\}.$$

For convenience, we introduce an extra element θ (corresponding to "0" of a crystal).

For each $i \in I$, the raising root operator $e_i : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$ is defined as follows. We set $e_i\theta := \theta$, and $e_i\pi := \theta$ for $\pi \in \mathbb{P}$ with $m_i^\pi > -1$. If $m_i^\pi \leq 1$, then we can take the following points:

$$\begin{aligned} &t_1 = \min\{t \in [0,1] \mid h_i^\pi(t) = m_i^\pi\}, \\ &t_0 = \max\{t' \in [0,t_1] \mid h_i^\pi(t) \geq m_i^\pi + 1 \text{ for all } t \in [0,t']\}. \end{aligned} \tag{2.1.2}$$

Choose a partition $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ of $[t_0, t_1]$ such that either of the following holds:

$$\begin{array}{ll} (1) & h_i^\pi(s_{k-1}) = h_i^\pi(s_k) \text{ and } h_i^\pi(t) \geq h_i^\pi(s_{k-1}) \\ & \text{for } t \in [s_{k-1}, s_k]. \\ (2) & h_i^\pi(t) \text{ is strictly decreasing on } [s_{k-1}, s_k] \text{ and} \\ & h_i^\pi(t) \geq h_i^\pi(s_{k-1}) \text{ for } t \in [s_0, s_{k-1}]. \end{array} \tag{2.1.3}$$

Remark 2.1.1. We deduce from the definition of t_0 (resp. t_1) that $h_i^\pi(t)$ is strictly decreasing on $[s_0, s_1]$ (resp. $[s_{r-1}, s_r]$). Namely, $[s_0, s_1]$ and $[s_{r-1}, s_r]$ are of type (2).

We set

(2.1.4)
$$(e_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0 = s_0, \\ \pi(t) - (h_i^\pi(s_{k-1}) - m_i^\pi - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \alpha_i & \text{if } s_r = t_1 \leq t \leq 1. \end{cases}$$

The lowering root operator $f_i : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$ is defined in a similar way: We set $f_i\theta := \theta$, and $f_i\pi := \theta$ for $\pi \in \mathbb{P}$ with $h_i^\pi(1) - m_i^\pi < 1$. If $h_i^\pi(1) - m_i^\pi \geq 1$, then we can take the following points:

$$\begin{aligned} &t_0 = \max\{t \in [0,1] \mid h_i^\pi(t) = m_i^\pi\}, \\ &t_1 = \min\{t' \in [t_0,1] \mid h_i^\pi(t) \geq m_i^\pi + 1 \text{ for all } t \in [t',1]\}. \end{aligned} \tag{2.1.5}$$

Choose a partition $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ of $[t_0, t_1]$ such that either of the following holds:

$$\begin{array}{ll} (1) & h_i^\pi(s_{k-1}) = h_i^\pi(s_k) \text{ and } h_i^\pi(t) \geq h_i^\pi(s_{k-1}) \\ & \text{for } t \in [s_{k-1}, s_k]. \\ (2) & h_i^\pi(t) \text{ is strictly increasing on } [s_{k-1}, s_k] \text{ and} \\ & h_i^\pi(t) \geq h_i^\pi(s_k) \text{ for } t \in [s_k, s_1]. \end{array} \tag{2.1.6}$$

We set

(2.1.7)

$$(f_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h_i^{\pi}(s_{k-1}) - m_i^{\pi})\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h_i^{\pi}(t) - m_i^{\pi})\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) - \alpha_i & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

For $\pi \in \mathbb{P}$ and $r \in \mathbb{Q}$, we define a path $r\pi \in \mathbb{P}$ by $(r\pi)(t) := r\pi(t)$ for $t \in [0,1]$. Also, we define the "dual path" $\pi^{\vee} \in \mathbb{P}$ of a path $\pi \in \mathbb{P}$ by $\pi^{\vee}(t) := \pi(1-t) - \pi(1)$ for $t \in [0,1]$. Let us recall the following lemma from [13, Lemmas 2.1 and 2.4].

Lemma 2.1.2. (1) *We have* $(f_i\pi)^{\vee} = e_i\pi^{\vee}$ *and* $(e_i\pi)^{\vee} = f_i\pi^{\vee}$ *for all* $\pi \in \mathbb{P}$.

(2) *We have* $n(e_i\pi) = e_i^n(n\pi)$ *and* $n(f_i\pi) = f_i^n(n\pi)$ *for all* $\pi \in \mathbb{P}$ *and* $n \in \mathbb{Z}_{\ge 0}$.

2.2. ω-root operators.

For a path $\pi \in \mathbb{P}$, we define a path $\omega(\pi) \in \mathbb{P}$ by $(\omega(\pi))(t) := \omega^*(\pi(t))$ for $0 \le t \le 1$, and set $\omega(\theta) := \theta$. We set

$$\mathbb{P}^0 := \{\pi \in \mathbb{P} \mid \omega(\pi) = \pi\}. \tag{2.2.1}$$

Let us recall the following definition of the ω-root operators $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \breve{I}$ from [20, §3.1] (cf. Remark 1.2.1):

$$\widetilde{X}_i = \begin{cases} \displaystyle\prod_{k=1}^{N_i/2} (X_{\omega^k(i)} X^2_{\omega^{k+N_i/2}(i)} X_{\omega^k(i)}) & \text{if } c_{ii} = 1, \\ \displaystyle\prod_{k=1}^{N_i} X_{\omega^k(i)} & \text{if } c_{ii} = 2, \end{cases} \tag{2.2.2}$$

where X is either e or f.

Denote by $\widehat{\mathbb{P}}$ the set of all paths (modulo reparametrization) for the orbit Lie algebra $\breve{\mathfrak{g}}$. For $i \in \breve{I}$, we denote by $\widehat{e}_i : \widehat{\mathbb{P}} \cup \{\theta\} \to \widehat{\mathbb{P}} \cup \{\theta\}$ and $\widehat{f}_i : \widehat{\mathbb{P}} \cup \{\theta\} \to \widehat{\mathbb{P}} \cup \{\theta\}$ the raising root operator and the lowering root operator for $\breve{\mathfrak{g}}$, respectively. For a path $\widehat{\pi} \in \widehat{\mathbb{P}}$, we define a path $P_{\omega}^*(\widehat{\pi}) \in \mathbb{P}^0$ by $(P_{\omega}^*(\widehat{\pi}))(t) := P_{\omega}^*(\widehat{\pi}(t))$ for $t \in [0,1]$, and set $P_{\omega}^*(\theta) = \theta$.

In [20] and [22], we showed that the equalities $\widetilde{e}_i \circ P_{\omega}^* = P_{\omega}^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_{\omega}^* = P_{\omega}^* \circ \widehat{f}_i$ hold on a certain subset of $\widehat{\mathbb{P}}$. Here we extend this result to the whole of $\widehat{\mathbb{P}}$. The proof below essentially follows the same line as those of [20, Theorem 3.1.2] and [22, Theorem 2.1.2]; however, it is a little simplified by virtue of Lemma 2.1.2.

Theorem 2.2.1. *The set $\mathbb{P}^0 \cup \{\theta\}$ is stable under the ω-root operators. In addition, we have $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ for each $i \in \check{I}$.*

Proof. Let us show the following claim, which generalizes [20, Theorem 3.1.2].

Claim. Let $\pi \in \mathbb{P}^0$. If $m_i^\pi > -1$, then $\widetilde{e}_i\pi = \theta$. If $m_i^\pi \le -1$, then we have

(2.2.3)
$$(\widetilde{e}_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h_i^\pi(s_{k-1}) - m_i^\pi - 1)\widetilde{\alpha}_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi - 1)\widetilde{\alpha}_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \widetilde{\alpha}_i & \text{if } s_r = t_1 \le t \le 1, \end{cases}$$

where t_0, t_1 are the points given by (2.1.2) for $\pi \in \mathbb{P}^0$ and $i \in \check{I}$, and $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ is a partition of $[t_0, t_1]$ satisfying Condition (2.1.3) for $\pi \in \mathbb{P}^0$ and $i \in \check{I}$.

(Proof of Claim.) It is obvious that $\widetilde{e}_i\pi = \theta$ if $m_i^\pi > -1$. We will show Equality (2.2.3). If $c_{ii} = 2$, then Equality (2.2.3) immediately follows from the definition of root operators and Remark 1.2.1. Assume that $c_{ii} = 1$. For simplicity, we assume that the Dynkin diagram corresponding to the ω-orbit of the i is of type A_2 (cf. Remark 1.2.1). For $\pi \in \mathbb{P}^0$, we set $h(t) := h_i^\pi(t) = h_j^\pi(t)$ and $m := m_i^\pi = m_j^\pi$ with $j := \omega(i)$. Since $m \le -1$, it follows from the definition of the raising root operator e_i that

$$\eta_1(t) := (e_i\pi)(t) = \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h(s_{k-1}) - m - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h(t) - m - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \alpha_i & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

By definition, we have

(2.2.4)

$$h_j^{\eta_1}(t) = \begin{cases} h(t) & \text{if } 0 \le t \le t_0 = s_0, \\ h(t) + h(s_{k-1}) - m - 1 & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ 2h(t) - m - 1 & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ h(t) - 1 & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

Subclaim 1. We have $m_j^{\eta_1} = m - 1$ and $t_1 = \min\{t \in [0,1] \mid h_j^{\eta_1}(t) = m_j^{\eta_1}\}$.

It is obvious that $h_j^{\eta_1}(t) = h(t) - 1 \ge m - 1$ for $t \in [t_1, 1]$, and that $h_j^{\eta_1}(t_1) = m - 1$. So it suffices to show that $h_j^{\eta_1}(t_1) > m - 1$ for all $t \in [0, t_1)$. By the definition of t_0, we have $h_j^{\eta_1}(t) = h(t) \ge m+1 > m-1$ for $t \in [0, t_0]$. Suppose that $h_j^{\eta_1}(t) \le m - 1$ for some $t \in [t_0, t_1)$. If t is in $[s_{k-1}, s_k]$ of type (1), then we have $h(t) + h(s_{k-1}) - m - 1 \le m - 1$, and hence $h(s_{k-1}) \le m$, since $h(t) \ge h(s_{k-1})$ for all $t \in [s_{k-1}, s_k]$ (see (2.1.3)). This contradicts the definition of t_1 (notice that $s_{k-1} < s_r = t_1$). Similarly, if t is in $[s_{k-1}, s_k]$ of type (2), then we have $h(t) \le m$, which is a contradiction. Thus we conclude that $h_j^{\eta_1}(t) > m - 1$ for all $t \in [0, t_1)$.

Subclaim 2. We have $t_0 = \max\{t' \in [0, t_1] \mid h_j^{\eta_1}(t) \ge m_j^{\eta_1}+2 \text{ for all } t \in [0, t']\}$.

It is obvious from the definition of t_0 and Subclaim 1 that $h_j^{\eta_1}(t) = h(t) \ge m + 1 = m_j^{\eta_1} + 2$ for all $t \in [0, t_0]$. We deduce from Remark 2.1.1 and (2.2.4) that $h_j^{\eta_1}(t_0 + \varepsilon) < h_j^{\eta_1}(t_0) = m_j^{\eta_1} + 2$ for sufficiently small $\varepsilon > 0$. Now, Subclaim 2 immediately follows from these facts.

Set $\eta_1' := \frac{1}{2}\eta_1$. It follows from Subclaims 1 and 2 that t_0, t_1 are the points given by (2.1.2) for $\eta_1' \in \mathbb{P}$ and $j \in I$. In addition, we deduce from (2.2.4) that $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ is a partition of $[t_0, t_1]$ satisfying Condition (2.1.3) with $\pi = \eta_1'$ and $i = j$. Therefore, we have

$$(e_j\eta_1')(t) =$$

$$\begin{cases} \eta_1'(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \eta_1'(t) - (h_j^{\eta_1'}(s_{k-1}) - m_j^{\eta_1'} - 1)\alpha_j & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \eta_1'(t) - (h_j^{\eta_1'}(t) - m_j^{\eta_1'} - 1)\alpha_j & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \eta_1'(t) + \alpha_j & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

$$= \begin{cases} \frac{1}{2}\pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \frac{1}{2}\pi(t) - \frac{1}{2}(h(s_{k-1}) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \frac{1}{2}\pi(t) - \frac{1}{2}(h(t) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \frac{1}{2}\pi(t) + \frac{1}{2}\alpha_i + \alpha_j & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

Because $e_j^2\eta_1 = 2(e_j\eta_1')$ by Lemma 2.1.2 (2), we get

$$\eta_2(t) := (e_j^2\eta_1)(t) =$$
$$\begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h(s_{k-1}) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h(t) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \alpha_i + 2\alpha_j & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

Since $h_i^{\eta_2}(t) = h(t)$, we obtain

$$(\widetilde{e}_i\pi)(t) := (e_i\eta_2)(t) =$$
$$\begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - 2(h(s_{k-1}) - m - 1)(\alpha_i + \alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - 2(h(t) - m - 1)(\alpha_i + \alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + 2(\alpha_i + \alpha_j) & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

This completes the proof of Claim.

It immediately follows from the claim above that $\mathbb{P}^0 \cup \{\theta\}$ is stable under $\widetilde{e}_i$. Also, we deduce from Lemma 2.1.2 (1) that $\mathbb{P}^0 \cup \{\theta\}$ is stable under $\widetilde{f}_i$, since $\widetilde{f}_i\pi = (\widetilde{e}_i\pi^\vee)^\vee$ for all $\pi \in \mathbb{P}^0$ (remark that if $\pi \in \mathbb{P}^0$, then so is $\pi^\vee$). Moreover, because $((P_\omega^*(\widehat{\pi}))(t))(\alpha_i^\vee) = (\widehat{\pi}(t))(\widehat{\alpha}_i^\vee)$ for $\widehat{\pi} \in \widehat{\mathbb{P}}$ and $i \in \check{I}$ (cf. (1.2.1)), we can easily check that $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ by using (2.2.3). Since $P_\omega^*(\widehat{\pi}^\vee) = (P_\omega^*(\widehat{\pi}))^\vee$ for all $\widehat{\pi} \in \widehat{\mathbb{P}}$, we get by Lemma 2.1.2 (1) that for each $\widehat{\pi} \in \widehat{\mathbb{P}}$,

$$\begin{aligned} \widetilde{f}_i(P_\omega^*(\widehat{\pi})) &= (\widetilde{e}_i(P_\omega^*(\widehat{\pi}))^\vee)^\vee = (\widetilde{e}_i(P_\omega^*(\widehat{\pi}^\vee)))^\vee \\ &= (P_\omega^*(\widehat{e}_i\widehat{\pi}^\vee))^\vee = P_\omega^*((\widehat{e}_i\widehat{\pi}^\vee)^\vee) = P_\omega^*(\widehat{f}_i\widehat{\pi}). \end{aligned}$$

Therefore we get $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$. This completes the proof of the theorem. $\square$

Remark 2.2.2. We can easily check that

$$\omega \circ e_i = e_{\omega(i)} \circ \omega \quad \text{and} \quad \omega \circ f_i = f_{\omega(i)} \circ \omega \quad \text{on } \mathbb{P}. \tag{2.2.5}$$

Therefore we deduce from Theorem 2.2.1 that the ω-root operators on $\mathbb{P}^0$ do not depend on the choice of a representative of the ω-orbit of $i \in I$ with $c_{ii} > 0$.

We define $\widetilde{e}(n)_i$ and $\widetilde{f}(n)_i$ for $i \in \check{I}$ and $n \in \mathbb{Z}_{\geq 0}$ by

$$\widetilde{X}(n)_i := \begin{cases} \displaystyle\prod_{k=1}^{N_i/2} (X^n_{\omega^k(i)} X^{2n}_{\omega^{k+N_i/2}(i)} X^n_{\omega^k(i)}) & \text{if } c_{ii} = 1, \\ \displaystyle\prod_{k=1}^{N_i} X^n_{\omega^k(i)} & \text{if } c_{ii} = 2, \end{cases} \tag{2.2.6}$$

where X is either e or f. As an application of Theorem 2.2.1, we can give a shorter proof of (a generalization of) [22, Proposition 2.1.3].

Corollary 2.2.3. *On $\mathbb{P}^0$, we have $(\widetilde{e}_i)^n = \widetilde{e}(n)_i$ and $(\widetilde{f}_i)^n = \widetilde{f}(n)_i$ for each $n \in \mathbb{Z}_{\geq 0}$ and $i \in I$.*

Proof. Let $\pi \in \mathbb{P}^0$, and set $\pi' = \frac{1}{n}\pi \in \mathbb{P}^0$. We deduce that

$$\begin{aligned} \widetilde{e}(n)_i\pi &= n(\widetilde{e}_i\pi') && \text{by Lemma 2.1.2 (2)} \\ &= n(P^*_\omega \circ \widehat{e}_i \circ (P^*_\omega)^{-1}(\pi')) && \text{by Theorem 2.2.1} \\ &= P^*_\omega(n\widehat{e}_i((P^*_\omega)^{-1}(\pi'))) \\ &= P^*_\omega((\widehat{e}_i)^n(n(P^*_\omega)^{-1}(\pi'))) && \text{by Lemma 2.1.2 (2)} \\ &= P^*_\omega \circ (\widehat{e}_i)^n \circ (P^*_\omega)^{-1}(\pi) \\ &= (P^*_\omega \circ \widehat{e}_i \circ (P^*_\omega)^{-1})^n(\pi) \\ &= (\widetilde{e}_i)^n\pi && \text{by Theorem 2.2.1.} \end{aligned}$$

Therefore we get $\widetilde{e}(n)_i = (\widetilde{e}_i)^n$. The equality $\widetilde{f}(n)_i = (\widetilde{f}_i)^n$ can be shown similarly. □

Let $P \subset \mathfrak{h}^*$ be an ω^*-stable integral weight lattice such that $\alpha_i \in P$ for all $i \in I$, and set $P_+ := \{\lambda \in P \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$. For $\lambda \in P_+$, we denote by $\mathbb{B}(\lambda)$ the set of Lakshmibai–Seshadri paths of shape λ. Recall from [13, §4] that $\mathbb{B}(\lambda) \cup \{\theta\}$ is stable under the root operators, and that every element π of $\mathbb{B}(\lambda)$ is of the form $\pi = f_{i_1} f_{i_2} \cdots f_{i_k} \pi_\lambda$ for some $i_1, i_2, \ldots, i_k \in I$, where $\pi_\lambda(t) := t\lambda$ for $t \in [0, 1]$. Let $w \in W$, and $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. We put

$$\mathbb{B}_w(\lambda) := \{f^{m_1}_{i_1} f^{m_2}_{i_2} \cdots f^{m_k}_{i_k} \pi_\lambda \mid m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0}\} \setminus \{\theta\}. \tag{2.2.7}$$

We know that $\mathbb{B}_w(\lambda)$ does not depend on the choice of the reduced expression of w (cf. [12, §5] and [11, §6.1]).

If $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, then $\mathbb{B}(\lambda)$ is stable under ω (cf. (2.2.5)). Furthermore, we deduce from (2.2.5) that $\omega(\mathbb{B}_w(\lambda)) = \mathbb{B}_{\omega^* w (\omega^*)^{-1}}(\lambda)$. Hence, if $w \in \widetilde{W}$, then $\mathbb{B}_w(\lambda)$ is stable under ω. We set

$$(2.2.8) \qquad \begin{aligned} \mathbb{B}^0(\lambda) &:= \{\pi \in \mathbb{B}(\lambda) \mid \omega(\pi) = \pi\}, \\ \mathbb{B}^0_w(\lambda) &:= \{\pi \in \mathbb{B}_w(\lambda) \mid \omega(\pi) = \pi\}. \end{aligned}$$

We have the following theorem (see [20, Theorem 3.2.4] and [21, Theorem 4.2]).

Theorem 2.2.4. *Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P^*_\omega)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$.*

(1) *The set $\mathbb{B}^0(\lambda) \cup \{\theta\}$ is stable under the ω-root operators.*

(2) *Each element $\pi \in \mathbb{B}^0(\lambda)$ is of the form $\pi = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\pi_\lambda$ for some $i_1, i_2, \ldots, i_k \in \breve{I}$.*

(3) *We have $\mathbb{B}^0(\lambda) = P^*_\omega(\breve{\mathbb{B}}(\widehat{\lambda}))$ and $\mathbb{B}^0_w(\lambda) = P^*_\omega(\breve{\mathbb{B}}_{\widehat{w}}(\widehat{\lambda}))$, where $\breve{\mathbb{B}}(\widehat{\lambda})$ is the set of Lakshmibai–Seshadri paths of shape $\widehat{\lambda}$ for the orbit Lie algebra $\breve{\mathfrak{g}}$, and $\breve{\mathbb{B}}_{\widehat{w}}(\widehat{\lambda})$ is the subset of $\breve{\mathbb{B}}(\widehat{\lambda})$ corresponding to $\widehat{w}$ (cf. (2.2.7)).*

§3. Crystal Bases and Diagram Automorphisms.

3.1. Crystal bases $\mathcal{B}(\lambda)$ and $\mathcal{B}_w(\lambda)$.

Set $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$. Let $U_q(\mathfrak{g}) = \langle x_i, y_i, q^h \mid i \in I, h \in P^\vee\rangle$ be the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q, and $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{x_i \mid i \in I\}$ (resp. $\{y_i \mid i \in I\}$).

For $\lambda \in P_+$, let $V(\lambda) = \bigoplus_{\chi \in P} V(\lambda)_\chi$ be the integrable highest weight $U_q(\mathfrak{g})$-module of highest weight λ. Denote by e_i and f_i the raising Kashiwara operator and the lowering Kashiwara operator for $V(\lambda)$, respectively, by $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ the crystal base of $V(\lambda)$, and by $\{G_\lambda(b) \mid b \in \mathcal{B}(\lambda)\}$ the global base of $V(\lambda)$ (see [6]).

For $w \in W$, let $V_w(\lambda) = U_q^+(\mathfrak{g})V(\lambda)_{w(\lambda)}$ be the quantum Demazure module of lowest weight $w(\lambda)$. We know from [8, Proposition 3.2.3] that there exists a subset $\mathcal{B}_w(\lambda)$ of $\mathcal{B}(\lambda)$ such that $V_w(\lambda) = \bigoplus_{b \in \mathcal{B}_w(\lambda)} \mathbb{Q}(q) G_\lambda(b)$. We see from [8, Proposition 3.2.3] that if $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ is a reduced expression of w, then

$$(3.1.1) \quad \mathcal{B}_w(\lambda) = \{f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k} \overline{v}_\lambda \mid m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0}\} \setminus \{0\},$$

where $\overline{v}_\lambda$ is the image of a (nonzero) highest weight vector v_λ of $V(\lambda)$ in $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$.

Let $U_q(\breve{\mathfrak{g}}) = \langle \widehat{x}_i, \widehat{y}_i, q^{\widehat{h}} \mid i \in \breve{I}, \widehat{h} \in \widehat{P}^\vee \rangle$ be the quantized universal enveloping algebra of the orbit Lie algebra $\breve{\mathfrak{g}}$, where $\widehat{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\widehat{P}, \mathbb{Z})$ with $\widehat{P} := (P_\omega^*)^{-1}(P \cap (\mathfrak{h}^*)^0)$. Denote by $\breve{\mathcal{B}}(\widehat{\lambda})$ the crystal base of the integrable highest weight $U_q(\breve{\mathfrak{g}})$-module $\breve{V}(\widehat{\lambda})$ of dominant integral highest weight $\widehat{\lambda}$, and by $\widehat{e}_i$ (resp. $\widehat{f}_i$) the raising (resp. lowering) Kashiwara operator for $\breve{\mathcal{B}}(\widehat{\lambda})$. For $\widehat{w} \in \widehat{W}$, we denote by $\breve{\mathcal{B}}_{\widehat{w}}(\widehat{\lambda})$ the crystal base of the quantum Demazure module $\breve{V}_{\widehat{w}}(\widehat{\lambda}) \subset \breve{V}(\widehat{\lambda})$ of lowest weight $\widehat{w}(\widehat{\lambda})$.

3.2. Fixed point subsets of $\mathcal{B}(\lambda)$ and $\mathcal{B}_w(\lambda)$.

Since $P^\vee$ is ω-stable, we obtain a $\mathbb{Q}(q)$-algebra automorphism $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ such that $\omega(x_i) = x_{\omega(i)}$, $\omega(y_i) = y_{\omega(i)}$, and $\omega(q^h) = q^{\omega(h)}$ for $i \in I$ and $h \in P^\vee$ (cf. [23, Lemma 1.2]). Remark that $U_q^-(\mathfrak{g})$ is stable under ω. If $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, then we have a $\mathbb{Q}(q)$-linear automorphism $\omega : V(\lambda) \to V(\lambda)$ induced from $\omega : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$. Because

$$\omega \circ e_i = e_{\omega(i)} \circ \omega \quad \text{and} \quad \omega \circ f_i = f_{\omega(i)} \circ \omega \tag{3.2.1}$$

on $V(\lambda)$ (see [22, Lemma 2.3.2]), the crystal lattice $\mathcal{L}(\lambda)$ is stable under ω. Therefore, we have a $\mathbb{Q}$-linear automorphism $\omega : \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ induced from $\omega : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda)$. We deduce from (3.2.1) that the crystal base $\mathcal{B}(\lambda)$ is stable under ω. Moreover, we obtain by (3.1.1) and (3.2.1) that $\omega(\mathcal{B}_w(\lambda)) = \mathcal{B}_{\omega^* w (\omega^*)^{-1}}(\lambda)$. Hence, if $w \in \widetilde{W}$, then $\mathcal{B}_w(\lambda)$ is stable under ω. We set

$$\begin{aligned} \mathcal{B}^0(\lambda) &:= \{ b \in \mathcal{B}(\lambda) \mid \omega(b) = b \}, \\ \mathcal{B}^0_w(\lambda) &:= \{ b \in \mathcal{B}_w(\lambda) \mid \omega(b) = b \}. \end{aligned} \tag{3.2.2}$$

We see from [13] that $\mathbb{B}(\lambda)$ has a natural (normal) crystal structure for each $\lambda \in P_+$. We know from [3, Corollary 6.4.27] or [10, Theorem 4.1] that there exists an isomorphism $\Phi_\lambda : \mathbb{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda)$ of crystals, and from [11, §5.6] that $\Phi(\mathbb{B}_w(\lambda)) = \mathcal{B}_w(\lambda)$ for every $w \in W$. If $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, then we obtain the following commutative diagram (cf. (2.2.5) and (3.2.1)):

$$\begin{array}{ccc} \mathbb{B}(\lambda) & \xrightarrow{\omega} & \mathbb{B}(\lambda) \\ {\scriptstyle \Phi_\lambda}\downarrow & & \downarrow{\scriptstyle \Phi_\lambda} \\ \mathcal{B}(\lambda) & \xrightarrow{\omega} & \mathcal{B}(\lambda). \end{array} \tag{3.2.3}$$

Therefore, we obtain $\Phi_\lambda(\mathbb{B}^0(\lambda)) = \mathcal{B}^0(\lambda)$ and $\Phi_\lambda(\mathbb{B}^0_w(\lambda)) = \mathcal{B}^0_w(\lambda)$ for each $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Combining this fact with Theorems 2.2.1 and 2.2.4, we get the following proposition.

Proposition 3.2.1. *Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P^*_\omega)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$.*

(1) The set $\mathcal{B}^0(\lambda) \cup \{0\}$ is stable under the ω-Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$, defined in the same way as (2.2.2).

(2) Each element $b \in \mathcal{B}^0(\lambda)$ is of the form $b = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\lambda$ for some $i_1, i_2, \ldots, i_k \in \breve{I}$.

(3) There exists a canonical bijection $P_\lambda : \mathcal{B}^0(\lambda) \xrightarrow{\sim} \breve{\mathcal{B}}(\widehat{\lambda})$ such that

$$(3.2.4)\qquad \begin{aligned} &(P^*_\omega)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_\lambda(b)) \quad \textit{for each } b \in \mathcal{B}^0(\lambda), \\ &P_\lambda \circ \widetilde{e}_i = \widehat{e}_i \circ P_\lambda \quad \textit{and} \quad P_\lambda \circ \widetilde{f}_i = \widehat{f}_i \circ P_\lambda \quad \textit{for all } i \in \breve{I}. \end{aligned}$$

In addition, we have $P_\lambda(\mathcal{B}^0_w(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\widehat{\lambda})$.

3.3. Crystal bases $\mathcal{B}(\infty)$ and $\mathcal{B}_w(\infty)$.

We denote by e_i and f_i the raising Kashiwara operator and the lowering Kashiwara operator for $U_q^-(\mathfrak{g})$, respectively, and by $(\mathcal{L}(\infty), \mathcal{B}(\infty))$ the crystal base of $U_q^-(\mathfrak{g})$. Denote by $\{G(b) \mid b \in \mathcal{B}(\infty)\}$ the global base of $U_q^-(\mathfrak{g})$ (see [6]).

Let $Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\,\alpha_i$, and set $Q_+(n) := \{\alpha \in \mathbb{Q}_+ \mid \mathrm{ht}(\alpha) \leq n\}$ for each $n \in \mathbb{Z}_{\geq 0}$, where $\mathrm{ht}(\alpha) := \sum_{i\in I} k_i$ for $\alpha = \sum_{i\in I} k_i\alpha_i \in Q_+$. Let us recall the following theorem from [6, Theorem 5 and Corollary 4.4.5].

Theorem 3.3.1. *Let $\varphi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ be the canonical $U_q^-(\mathfrak{g})$-module homomorphism sending 1 to v_λ.*

(1) We have $\varphi_\lambda(\mathcal{L}(\infty)) = \mathcal{L}(\lambda)$. Hence we have a $\mathbb{Q}$-linear homomorphism

$$(3.3.1)\qquad \overline{\varphi}_\lambda : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$$

induced from $\varphi_\lambda : \mathcal{L}(\infty) \to \mathcal{L}(\lambda)$. The restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ is a bijection from $\mathcal{B}(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ to $\mathcal{B}(\lambda)$.

(2) We have $f_i \circ \overline{\varphi}_\lambda = \overline{\varphi}_\lambda \circ f_i$ for each $i \in I$. In addition, if $b \in \mathcal{B}(\infty)$ satisfies $\overline{\varphi}_\lambda(b) \neq 0$, then $e_i\overline{\varphi}_\lambda(b) = \overline{\varphi}_\lambda(e_ib)$ for each $i \in I$.

(3) Fix $n \in \mathbb{Z}_{\geq 0}$. If $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, then, for every $\xi \in Q_+(n)$, the restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}(\infty)_{-\xi}$ is a bijection from $\mathcal{B}(\infty)_{-\xi}$ to $\mathcal{B}(\lambda)_{\lambda-\xi}$. Here, for a crystal $\mathcal{B}$, we denote by $\mathcal{B}_\mu$ the set of elements of weight μ in $\mathcal{B}$.

Let $w \in W$, and $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. We define a module $(U_w^-)_q(\mathfrak{g})$ of Demazure type by

$$(U_w^-)_q(\mathfrak{g}) := \sum_{m_j \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) y_{i_1}^{m_1} y_{i_2}^{m_2} \cdots y_{i_k}^{m_k}. \tag{3.3.2}$$

We know from [8, Proposition 3.2.5] that $(U_w^-)_q(\mathfrak{g}) = \bigoplus_{b \in \mathcal{B}_w(\infty)} \mathbb{Q}(q) G(b)$, where

$$\mathcal{B}_w(\infty) := \left\{ f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k} \overline{v}_\infty \mid m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0} \right\}, \tag{3.3.3}$$

with $\overline{v}_\infty$ the image of $1 \in U_q^-(\mathfrak{g})$ in $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$. Furthermore, we can easily show the following theorem, by using [8, Proposition 3.2.5], Theorem 3.3.1, (3.1.1), and (3.3.3).

Theorem 3.3.2. (1) *The restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}_w(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ is a bijection from $\mathcal{B}_w(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ to $\mathcal{B}_w(\lambda)$.*
(2) *Fix $n \in \mathbb{Z}_{\geq 0}$. If $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, then, for every $\xi \in Q_+(n)$, the restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}_w(\infty)_{-\xi}$ is a bijection from $\mathcal{B}_w(\infty)_{-\xi}$ to $\mathcal{B}_w(\lambda)_{\lambda-\xi}$.*

Remark 3.3.3. It follows from Theorem 3.3.2 that $\mathcal{B}_w(\infty)$ (and hence $(U_w^-)_q(\mathfrak{g})$) does not depend on the choice of the reduced expression of w.

Denote by $\breve{\mathcal{B}}(\infty)$ the crystal base of $U_q^-(\breve{\mathfrak{g}}) := \langle \widehat{y}_i \mid i \in \breve{I} \rangle$, and by $\widehat{e}_i$ (resp. $\widehat{f}_i$) the raising (resp. lowering) Kashiwara operator for $\breve{\mathcal{B}}(\infty)$. For $\widehat{w} \in \widehat{W}$, we denote by $\breve{\mathcal{B}}_{\widehat{w}}(\infty)$ the crystal base of the module $(U_{\widehat{w}}^-)_q(\breve{\mathfrak{g}})$ of Demazure type corresponding to $\widehat{w}$.

3.4. Fixed point subsets of $\mathcal{B}(\infty)$ and $\mathcal{B}_w(\infty)$.

In a way similar to the case of $V(\lambda)$, we can show that $\omega \circ e_i = e_{\omega(i)} \circ \omega$ and $\omega \circ f_i = f_{\omega(i)} \circ \omega$ on $U_q^-(\mathfrak{g})$. Thus, $\mathcal{L}(\infty)$ is stable under ω, and hence we have a $\mathbb{Q}$-linear automorphism $\omega : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ induced from $\omega : \mathcal{L}(\infty) \to \mathcal{L}(\infty)$. It is obvious that $\mathcal{B}(\infty)$ is stable under ω. Moreover we deduce that $\omega(\mathcal{B}_w(\infty)) = \mathcal{B}_{\omega^* w (\omega^*)^{-1}}(\infty)$ for $w \in W$. Therefore, if $w \in \widetilde{W}$, then $\mathcal{B}_w(\infty)$ is stable under ω. We now set

$$\begin{aligned} \mathcal{B}^0(\infty) &:= \left\{ b \in \mathcal{B}(\infty) \mid \omega(b) = b \right\}, \\ \mathcal{B}_w^0(\infty) &:= \left\{ b \in \mathcal{B}_w(\infty) \mid \omega(b) = b \right\}. \end{aligned} \tag{3.4.1}$$

Theorem 3.4.1. (1) *The set $\mathcal{B}^0(\infty) \cup \{0\}$ is stable under the ω-Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$, defined in the same way as* (2.2.2).

(2) *Each element* $b \in \mathcal{B}^0(\infty)$ *is of the form* $b = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\infty$ *for some* $i_1, i_2, \dots, i_k \in \breve{I}$.

(3) *There exists a canonical bijection* $P_\infty : \mathcal{B}^0(\infty) \xrightarrow{\sim} \breve{\mathcal{B}}(\infty)$ *such that*

$$\begin{aligned} &(P_\omega^*)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_\infty(b)) \quad \textit{for each} \quad b \in \mathcal{B}^0(\infty), \\ &P_\infty \circ \widetilde{e}_i = \widehat{e}_i \circ P_\infty \quad \textit{and} \quad P_\infty \circ \widetilde{f}_i = \widehat{f}_i \circ P_\infty \quad \textit{for all} \quad i \in \breve{I}. \end{aligned} \tag{3.4.2}$$

In addition, we have $P_\infty(\mathcal{B}^0_w(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\infty)$ *for each* $w \in \widetilde{W}$, *where* $\widehat{w} := \Theta^{-1}(w)$.

Proof. Because $\omega \circ \varphi_\lambda = \varphi_\lambda \circ \omega$ for $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, we have the following commutative diagram (cf. Theorem 3.3.1):

$$\begin{tikzcd} \mathcal{B}(\infty) \arrow[r, "\overline{\varphi}_\lambda"] \arrow[d, "\omega"'] & \mathcal{B}(\lambda) \cup \{0\} \arrow[d, "\omega"] \\ \mathcal{B}(\infty) \arrow[r, "\overline{\varphi}_\lambda"] & \mathcal{B}(\lambda) \cup \{0\}. \end{tikzcd} \tag{3.4.3}$$

Thus we obtain $\overline{\varphi}_\lambda(\mathcal{B}^0(\infty)) = \mathcal{B}^0(\lambda) \cup \{0\}$ and $\overline{\varphi}_\lambda(\mathcal{B}^0_w(\infty)) = \mathcal{B}^0_w(\lambda) \cup \{0\}$ for each $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$.

(1) Let $b \in \mathcal{B}^0(\infty)$. Assume that $\widetilde{e}_i b \neq 0$. Take $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ such that $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$. Then we deduce from Theorem 3.3.1 (2) and (3) that $\widetilde{e}_i \overline{\varphi}_\lambda(b) = \overline{\varphi}_\lambda(\widetilde{e}_i b) \neq 0$. Since $\widetilde{e}_i \overline{\varphi}_\lambda(b) \in \mathcal{B}^0(\lambda)$ by Proposition 3.2.1 (1), we conclude that $\widetilde{e}_i b \in \mathcal{B}^0(\infty)$. Similarly, we can show that $\widetilde{f}_i b \in \mathcal{B}^0(\infty) \cup \{0\}$.

(2) Let $b \in \mathcal{B}^0(\infty)$. Since $\overline{\varphi}_\lambda(b) \in \mathcal{B}^0(\lambda)$ if $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, we see from Proposition 3.2.1 (2) that $\overline{\varphi}_\lambda(b) = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\lambda$ for some $i_1, i_2, \dots, i_k \in \breve{I}$. By Theorem 3.3.1 (1) and (2), we get $b = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\infty$. Thus we have proved part (2).

(3) Let $\xi \in Q_+ \cap (\mathfrak{h}^*)^0$, and set $\widehat{\xi} := (P_\omega^*)^{-1}(\xi)$. Take $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ such that $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, and set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$. We define a bijection $P_{\infty,\xi} : \mathcal{B}(\infty)_{-\xi} \to \breve{\mathcal{B}}(\infty)_{-\widehat{\xi}}$ as in the following commutative diagram:

$$\begin{tikzcd} \mathcal{B}^0(\infty)_{-\xi} \arrow[r, "\sim"] \arrow[d, "P_{\infty,\xi}"'] & \mathcal{B}^0(\lambda)_{\lambda-\xi} \arrow[d, "P_\lambda"] \\ \breve{\mathcal{B}}(\infty)_{-\widehat{\xi}} & \breve{\mathcal{B}}(\widehat{\lambda})_{\widehat{\lambda}-\widehat{\xi}}. \arrow[l, "\sim"'] \end{tikzcd} \tag{3.4.4}$$

We can easily check that $P_{\infty,\xi}$ does not depend on the choice of λ. Now we define $P_\infty : \mathcal{B}^0(\infty) \to \breve{\mathcal{B}}(\infty)$ by $P_\infty(b) := P_{\infty,\xi}(b)$ for $b \in \mathcal{B}^0(\infty)_{-\xi}$.

We can easily show by Proposition 3.2.1 (3) and Theorem 3.3.1 that P_∞ has the desired properties (3.4.2). The equality $P_\infty(\mathcal{B}_w^0(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\infty)$ immediately follows from the definition of P_∞ and the equality $\overline{\varphi}_\lambda(\mathcal{B}_w^0(\infty)) = \mathcal{B}_w^0(\lambda) \cup \{0\}$. $\square$

Remark 3.4.2. It immediately follows from Theorem 3.4.1 that there exists an injection from the global base of $U_q^-(\breve{\mathfrak{g}})$ to the global base of $U_q^-(\mathfrak{g})$. Therefore we have an embedding $U_q^-(\breve{\mathfrak{g}}) \hookrightarrow U_q^-(\mathfrak{g})$ of vector spaces.

§4. Twining Character Formulas.

4.1. Definitions.

The twining character $\mathrm{ch}^\omega(U_q^-(\mathfrak{g}))$ of $U_q^-(\mathfrak{g})$ is defined to be the following formal sum:

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) e(\chi). \tag{4.1.1}$$

For each $w \in \widetilde{W}$, we define the twining character $\mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g}))$ of $(U_w^-)_q(\mathfrak{g})$ by

$$\mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g})) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{((U_w^-)_q(\mathfrak{g}))_\chi}\big) e(\chi). \tag{4.1.2}$$

4.2. Twining character formulas.

Corollary 4.2.1. *Let $w \in \widetilde{W}$, and set $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\begin{aligned} \mathrm{ch}^\omega(U_q^-(\mathfrak{g})) &= P_\omega^*(\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})), \\ \mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g})) &= P_\omega^*(\mathrm{ch}\, (U_{\widehat{w}}^-)_q(\breve{\mathfrak{g}})). \end{aligned} \tag{4.2.1}$$

In order to prove this corollary, we need the following lemma, which can be shown in exactly the same way as [23, Lemma 3.4].

Lemma 4.2.2. *We have $\omega(G(b)) = G(\omega(b))$ for all $b \in \mathcal{B}(\infty)$. Therefore, we see that the global base $\{G(b) \mid b \in \mathcal{B}(\infty)\}$ of $U_q^-(\mathfrak{g})$ is stable under ω, and that $\omega(G(b)) = G(b)$ if and only if $b \in \mathcal{B}^0(\infty)$.*

Proof of Corollary 4.2.1. We give a proof only for the first equality of (4.2.1), since the proof for the second one is similar. Remark that for each $\chi \in (\mathfrak{h}^*)^0$, $\{G(b) \mid b \in \mathcal{B}(\infty)_\chi\}$ is a basis of $U_q^-(\mathfrak{g})_\chi$, which is stable under ω. Therefore we have

$$\mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) = \#\{G(b) \mid \omega(G(b)) = G(b),\, b \in \mathcal{B}(\infty)_\chi\}$$

for each $\chi \in (\mathfrak{h}^*)^0$ (note that if an endomorphism f on a finite-dimensional vector space V stabilizes a basis of V, then the trace of f on V is equal to the number of the basis elements fixed by f). By Lemma 4.2.2, we get

$$\mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) = \#\big(\mathcal{B}(\infty)_\chi \cap \mathcal{B}^0(\infty)\big),$$

and hence

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = \sum_{b \in \mathcal{B}^0(\infty)} e(\mathrm{wt}(b)) \tag{4.2.2}$$

Therefore we obtain

$$\begin{aligned}
\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) &= \sum_{b \in \mathcal{B}^0(\infty)} e(\mathrm{wt}(b)) && \text{by (4.2.2)} \\
&= P_\omega^*\Big(\sum_{\breve{b} \in \breve{\mathcal{B}}(\infty)} e(\mathrm{wt}(\breve{b}))\Big) && \text{by Theorem 3.4.1 (3)} \\
&= P_\omega^*(\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})),
\end{aligned}$$

as desired. □

Remark 4.2.3. Let $U_q^-(\mathfrak{g})_{\mathbb{Z}}$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q^-(\mathfrak{g})$ generated by the divided powers $\{y_i^{(n)} \mid i \in I,\, n \in \mathbb{Z}_{\geq 0}\}$ (see [6, §6.1]), and set

$$(U_w^-)_q(\mathfrak{g})_{\mathbb{Z}} := \sum_{m_j \geq 0} \mathbb{Z}[q, q^{-1}] y_{i_1}^{(m_1)} y_{i_2}^{(m_2)} \cdots y_{i_k}^{(m_k)} \subset (U_w^-)_q(\mathfrak{g}) \tag{4.2.3}$$

for $w \in W$ with $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. Now, for $\lambda \in P_+$ and $w \in \widetilde{W}$, we set $V(\lambda)_{\mathbb{Z}} := U_q^-(\mathfrak{g})_{\mathbb{Z}}\, v_\lambda \subset V(\lambda)$ and $V_w(\lambda)_{\mathbb{Z}} := (U_w^-)_q(\mathfrak{g})_{\mathbb{Z}}\, v_\lambda \subset V_w(\lambda)$ (cf. [8, Corollary 3.2.2]). Assume that $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. We can easily check that the $\mathbb{Z}[q, q^{-1}]$-forms $U_q^-(\mathfrak{g})_{\mathbb{Z}}$, $(U_w^-)_q(\mathfrak{g})_{\mathbb{Z}}$, $V(\lambda)_{\mathbb{Z}}$, and $V_w(\lambda)_{\mathbb{Z}}$ are stable under the action of ω. Hence we can define the twining characters of them in a way similar to (4.1.1) and (4.1.2). Using the fact that the global bases are $\mathbb{Z}[q, q^{-1}]$-bases of these $\mathbb{Z}[q, q^{-1}]$-forms, we can prove twining character formulas for these $\mathbb{Z}[q, q^{-1}]$-forms in exactly the same way as Corollary 4.2.1 (see also [23]).

Let $M(\lambda)$ be the Verma module of highest weight $\lambda \in \mathfrak{h}^*$ over $\mathfrak{g}$ with (nonzero) highest weight vector v_λ. For $w \in W$ with $w = r_{i_1} r_{i_2} \ldots r_{i_k}$

its reduced expression, we define a module $M_w(\lambda) \subset M(\lambda)$ of Demazure type by

$$(4.2.4) \qquad M_w(\lambda) := \sum_{m_j \in \mathbb{Z}} \mathbb{Q} y_{i_1}^{m_1} y_{i_2}^{m_2} \cdots y_{i_k}^{m_k} v_\lambda.$$

We see that $M_w(\lambda)$ does not depend on the choice of the reduced expression of w.

Assume that $\lambda \in (\mathfrak{h}^*)^0$. Then we have a $\mathbb{Q}$-linear automorphism $\omega : M(\lambda) \to M(\lambda)$ induced from the $\mathbb{Q}$-algebra automorphism $\omega \in \mathrm{Aut}(U(\mathfrak{g}))$ of the universal enveloping algebra of $\mathfrak{g}$ (cf. §1.1). We can easily check that $M_w(\lambda)$ is stable under ω if $w \in \widetilde{W}$. The twining characters $\mathrm{ch}^\omega(M(\lambda))$ and $\mathrm{ch}^\omega(M_w(\lambda))$ are defined in the same way as (4.1.1) and (4.1.2), respectively (see also [1, Definition 2.3]).

Corollary 4.2.4. *Let $\lambda \in (\mathfrak{h}^*)^0$, and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$, and $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$(4.2.5) \qquad \begin{aligned} \mathrm{ch}^\omega(M(\lambda)) &= P_\omega^*(\mathrm{ch}\, \breve{M}(\widehat{\lambda})), \\ \mathrm{ch}^\omega(M_w(\lambda)) &= P_\omega^*(\mathrm{ch}\, \breve{M}_{\widehat{w}}(\widehat{\lambda})), \end{aligned}$$

where $\breve{M}(\widehat{\lambda})$ is the Verma module of highest weight $\widehat{\lambda}$ over the orbit Lie algebra $\breve{\mathfrak{g}}$, and $\breve{M}_{\widehat{w}}(\widehat{\lambda}) \subset \breve{M}(\widehat{\lambda})$ is the module of Demazure type for $\breve{\mathfrak{g}}$ corresponding to $\widehat{w}$.

Proof. We give a proof only for the first equality $\mathrm{ch}^\omega(M(\lambda)) = P_\omega^*(\mathrm{ch}\, \breve{M}(\widehat{\lambda}))$ of (4.2.5), since the proof of the second one is similar. We see easily that $\mathrm{ch}^\omega(M(\lambda)) = e(\lambda)\,\mathrm{ch}^\omega(M(0))$ and $\mathrm{ch}\, \breve{M}(\widehat{\lambda}) = e(\widehat{\lambda})\,\mathrm{ch}\, \breve{M}(0)$. Hence we need only show that $\mathrm{ch}^\omega(M(0)) = P_\omega^*(\mathrm{ch}\, \breve{M}(0))$.

As in [23, §2.2], we deduce that the specialization "$q = 1$" of $\mathrm{ch}^\omega(U_q^-(\mathfrak{g}))$ is equal to $\mathrm{ch}^\omega(M(0))$. On the other hand, the specialization "$q = 1$" of $\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})$ is equal to $\mathrm{ch}\, \breve{M}(0)$. By combining these facts with Corollary 4.2.1, we obtain $\mathrm{ch}^\omega(M(0)) = P_\omega^*(\mathrm{ch}\, \breve{M}(0))$. Thus we have proved the corollary. □

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