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The representation theory of the Ariki-Koike and cyclotomic q-Schur algebras

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§1. Introduction

The Ariki–Koike algebras first appeared in the work of Cherednik [**30**] who discovered these algebras in his study of the q–analogue of Drinfeld's duality between the degenerate affine Hecke algebra and the Yangians for $\mathfrak{gl}_N$. Seven years later these algebras were rediscovered by Ariki and Koike [**8**] who were interested in them because they are a natural generalization of the Iwahori-Hecke algebras of types A and B. At almost the same time, Broué and Malle [**21**] also attached to each complex reflection group a *cyclotomic Hecke algebra* which, they conjectured, should play a role in the decomposition of the induced cuspidal representations of the finite groups of Lie type. The Ariki-Koike algebras are a special case of Broué and Malle's construction.

The deepest conjectures of Broué, Malle and Michel concerning the Ariki-Koike algebras have not yet been proved (see §2.5); however, many of the consequences of these conjectures have been established. Further, the representation theory of these algebras is beginning to be well understood. For example, the simple modules of the Ariki-Koike algebras have been classified; the blocks are known; there are analogues of Kleshchev's modular branching rules; and, in principle, the decomposition matrices of the Ariki-Koike algebras are known in characteristic zero. In many respects this theory looks much like that of the symmetric groups; in particular, there is a rich combinatorial mosaic underpinning these results which involves familiar objects like standard tableaux (indexed by multipartitions), Specht modules and so on.

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The cyclotomic Schur algebras were introduced by Dipper, James and the author [**42**]; by definition these algebras are endomorphism algebras of a direct sum of "permutation modules" for an Ariki-Koike algebra. This generalizes the Dipper-James definition [**39**] of the q-Schur algebras as endomorphism algebras of tensor space. We were interested in these algebras both as another tool for studying the Ariki-Koike algebras and because we hoped that there might be a cyclotomic analogue of the famous Dipper-James theory [**27**, **39**].

As with the Ariki-Koike algebras, the representation theory of the cyclotomic Schur algebras is now well developed. They are always cellular algebras; indeed, they are quasi-hereditary. The cellular basis of these algebras is indexed by a generalization of semistandard tableaux and their representation theory looks very much like the representation theory of the q-Schur algebras. In particular, they have a highest weight theory; there is a cyclotomic Schur functor and a double centralizer theorem; the Jantzen filtrations of the cyclotomic Weyl modules satisfy a generalization of the Jantzen sum formula; and the cyclotomic Schur algebras have Borel subalgebras and admit a triangular decomposition.

In the short time since its inception this theory has blossomed producing many interesting results; largely this is because it generalizes the representation theories of the symmetric groups, the Schur algebras and the q-analogues of these. Many of the results in this article have the flavour of results from Lie theory; however, as yet, there are no known connections between the representation theories of the cyclotomic Schur algebras and the finite groups of Lie type except in the case where the underlying complex reflection group is actually a Weyl group.

The aim of this article is to describe the representation theory of these algebras in detail. Throughout we have tried to give an indication of how the results are proved; unfortunately, in distilling one or more papers in to one or more paragraphs some of the finer details have inevitably been lost.

§2. The Ariki-Koike algebras

In this chapter we introduce the Ariki-Koike algebras by giving three different constructions of them. From the point of view of presentations it is clear that all three definitions agree; however, for motivation, and also for proving certain results, it is important to know the different contexts in which the Ariki-Koike algebras arise.

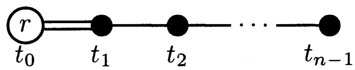
We begin with a brief discussion of the complex reflection groups which underpin the Ariki-Koike algebras. In the final section we give a brief account of the conjectures of Broué and Malle [**21**] which describe the role that the Ariki-Koike algebras should play in the representation theory of the finite groups of Lie type.

2.1. The complex reflection group of type $G(r, 1, n)$.

Fix integers $r \geq 1$ and $n \geq 0$ and let $W_{r,n} = \mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$ be the wreath product of a cyclic group of order r and a symmetric group of degree n. Then $W_{r,n}$ is the complex reflection group of type $G(r, 1, n)$ in the Shephard-Todd classification [**116**]; in particular, $W_{r,n}$ has a faithful representation on a complex vector space on which it acts as a group generated by reflections (see section (2.3)).

If $r = 1$ then $W_{1,n} \cong \mathfrak{S}_n$ is just the symmetric group $\mathfrak{S}_n$. If $r = 2$ then $W_{2,n} = \mathbb{Z}/2\mathbb{Z} \rtimes \mathfrak{S}_n$ is the hyperoctohedral group, or the group of signed permutations. In these two cases $W_{r,n}$ is a *Coxeter group* or *real reflection group*; in fact, they are the Weyl groups of type A_{n-1} and B_n respectively.

The group $W_{r,n}$ has the Coxeter like presentation given by the following diagram.



The circle around the r indicates that the corresponding generator t_0 has order r; otherwise, this should be read as a standard Dynkin diagram. Thus, as an abstract group, $W_{r,n}$ is generated by elements $t_0, t_1, \ldots, t_{n-1}$ which are subject to the relations

$$\begin{aligned} t_0^r &= 1, && \\ t_i^2 &= 1, && \text{for } 1 \leq i < n, \\ t_0 t_1 t_0 t_1 &= t_1 t_0 t_1 t_0, && \\ t_i t_j &= t_j t_i, && \text{for } 0 \leq j < i-1 < n-1, \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, && \text{for } 1 \leq i < n-1. \end{aligned}$$

In particular, the subgroup $\langle t_1, \ldots, t_{n-1} \rangle$ of $W_{r,n}$ is isomorphic to the symmetric group $\mathfrak{S}_n$; hereafter, we identify $\mathfrak{S}_n$ and $\langle t_1, \ldots, t_{n-1} \rangle$ via the map $(i, i+1) \longmapsto t_i$, for $1 \leq i < n$.

Let $l_1 = t_0$, $l_2 = t_1 t_0 t_1, \ldots, l_n = t_{n-1} \ldots t_1 t_0 t_1 \ldots t_{n-1}$. Then $l_1, \ldots, l_n$ generate a subgroup of $W_{r,n}$ isomorphic to $\mathbb{Z}/r\mathbb{Z} \times \cdots \times \mathbb{Z}/r\mathbb{Z}$ (n copies), which is just the base group when we consider $W_{r,n}$ as the semidirect product $(\mathbb{Z}/r\mathbb{Z} \times \cdots \times \mathbb{Z}/r\mathbb{Z}) \rtimes \mathfrak{S}_n$. Thus, as a set, $W_{r,n} = \{\, l_1^{a_1} \ldots l_n^{a_n} w \mid 0 \leq a_i < r \text{ and } w \in \mathfrak{S}_n \,\}$ and these elements are all distinct. In particular, $|W_{r,n}| = r^n n!$.

In general, $W_{r,n}$ is not a Coxeter group so the familiar combinatorics of root systems and length functions cannot be used in understanding $W_{r,n}$ and its representations. (Bremke and Malle [**16**] have defined a root system for $W_{r,n}$.) The theory of complex reflection groups is still very much in its infancy; the major tool being used to understand these groups is the geometry of their reflection representation.

2.2. The Ariki-Koike algebras

The Iwahori-Hecke algebras of Weyl groups play an important role in the representation theory of the groups of Lie type. Two important special cases of these algebras are the Iwahori-Hecke algebras of the Weyl groups of types A_{n-1} and B_n which are the groups $G(1,1,n)$ and $G(2,1,n)$, respectively. Ariki and Koike [**8**] observed that the definition of these algebras could be generalized to give a Hecke algebra, or deformation algebra, for each complex reflection group of type $G(r,1,n)$.

Let R be an integral domain with 1 and let $q, Q_1, \dots, Q_r$ be elements of R with q invertible. Let $\mathbf{Q} = \{Q_1, \dots, Q_r\}$.

Deforming the relations of $W_{r,n}$ we obtain the Ariki-Koike algebra.

Definition 2.1 (Ariki-Koike [**8**]). *The* Ariki-Koike algebra *is the unital associative R-algebra $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ with generators $T_0, T_1, \dots, T_{n-1}$ and relations*

$$\begin{aligned}
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0, && \\
(T_i - q)(T_i + 1) &= 0, && \textit{for } 1 \le i < n, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, && \\
T_iT_j &= T_jT_i, && \textit{for } 0 \le i < j-1 < n-1, \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && \textit{for } 1 \le i < n-1.
\end{aligned}$$

The three homogeneous relations are known as *braid relations*.

Typically, we write $\mathscr{H} = \mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$; when we wish to emphasize the ring of definition we will write $\mathscr{H} = \mathscr{H}_{R,q,\mathbf{Q}}(W_{r,n})$.

Notice that if R contains a primitive r^{th} root of unity ζ and we set $q = 1$ and $Q_s = \zeta^s$, for $1 \le s \le r$, then $\mathscr{H} \cong RW_{r,n}$, the group algebra of $W_{r,n}$ (because the relations collapse to give those of $W_{r,n}$ for this choice of parameters).

Let $w \in \mathfrak{S}_n$. Then $w = t_{i_1} \dots t_{i_k}$ for some i_j with $1 \le i_j < n$. If k is minimal we say that $t_{i_1} \dots t_{i_k}$ is a reduced expression for w and define $T_w = T_{i_1} \dots T_{i_k}$. Since the braid relations hold in $\mathscr{H}$ it follows from Matsumoto's monoid lemma (see, for example, [**103**, Theorem 1.8]), that T_w is independent of the choice of reduced expression for w.

Mimicking the definition of the elements l_k in $W_{r,n}$, for $k = 1, \dots, n$ set $L_k = q^{1-k} T_{k-1} \dots T_1 T_0 T_1 \dots T_{k-1}$. (The renormalization by the unit q^{1-k} is there to make the combinatorics more natural later on.) Using the relations it is straightforward to see that $L_1, \dots, L_n$ generate an abelian subalgebra of $\mathscr{H}$ and that the symmetric polynomials in $L_1, \dots, L_n$ belong to the centre of $\mathscr{H}$.

A priori there is no reason to expect that the presentation above will yield an interesting algebra. The first indication that $\mathscr{H}$ is worth studying is the following theorem.

Theorem 2.2 (Ariki-Koike [**8**]). *The Ariki-Koike algebra $\mathscr{H}$ is free as an R-module with basis $\{ L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_i < r$ and $w \in \mathfrak{S}_n \}$.*

In particular, notice that $\mathscr{H}$ is R-free of rank $r^n n! = |W_{r,n}|$ for any choice of R, q and $\mathbf{Q}$. Furthermore, the subalgebra of $\mathscr{H}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$. Hereafter, we identify the two algebras $\mathscr{H}_q(\mathfrak{S}_n)$ and $\langle T_1, \dots, T_{n-1}\rangle$.

Using the relations it is not hard to show that $\mathscr{H}$ is spanned by the elements $L_1^{a_1} \dots L_n^{a_n} T_w$; there are $|W_{r,n}|$ such elements. To prove linear independence Ariki and Koike explicitly constructed the simple $\mathscr{H}$-modules using a generalization of Young's seminormal form for the Ariki-Koike algebras when $R = \mathbb{C}(q, u_1, \dots , u_r)$; see Theorem 3.2 below. This shows that $\mathscr{H} / \operatorname{Rad} \mathscr{H}$ has dimension at most $|W_{r,n}|$. Hence, $\mathscr{H}$ is semisimple and Theorem 2.2 is proved when $R = \mathbb{C}(q, Q_1, \dots, Q_r)$. The general case now follows by a specialization argument.

There are now other proofs of Theorem 2.2 available. Broué, Malle and Rouquier [**24**, Theorem 4.24] have given a geometrical argument which results from thinking of $\mathscr{H}$ as a quotient of the group algebra of the braid group of $W_{r,n}$ and studying its monodromy representation; this is the topic of the next section. Sakamoto and Shoji [**113**] also proved Theorem 2.2 as a consequence of an analogue of Schur-Weyl reciprocity for $\mathscr{H}$ and a particular quantum group; we will return to this in §5.4 below. Another proof, using the affine Hecke algebra $\hat{H}_n$ below, can be extracted from related arguments of Brundan and Kleshchev; see the proof of [**28**, Theorem 3.6].

Finally, we remark that Shoji [**117**] has given a different presentation of $\mathscr{H}$ when $R = \mathbb{C}(q, Q_1, \dots, Q_r)$. Shoji's presentation is very interesting and deserves further study.

2.3. The braid group of $W_{r,n}$ and the Hecke algebra

At almost the same time that Ariki and Koike introduced their algebra, Broué and Malle [**21**] associated to each complex reflection group W a *cyclotomic Hecke algebra*; for the group $W_{r,n}$. Broué and Malle's cyclotomic Hecke algebra is precisely the Ariki-Koike algebra. Broué and Malle's motivation was that they expected that the cyclotomic Hecke algebras should play a role in the representation theory of the finite groups of Lie type similar to, but more complicated than, that played by the Iwahori-Hecke algebras (see §2.5).

In this section we briefly describe Broué and Malle's definition in the case of $W_{r,n}$ and some of its consequences.

Let V be the complex vector space with basis $\{\epsilon_1, \dots, \epsilon_n\}$ and let $\zeta \in \mathbb{C}$ be a primitive r^{th} root of unity. The symmetric group $\mathfrak{S}_n = \langle t_1, \dots, t_{n-1}\rangle$ acts on V in the natural way; extend this to an action of $W_{r,n}$ by letting t_0 act via the $n \times n$ matrix $\operatorname{diag}(\zeta, 1, \dots, 1)$. This defines a faithful representation of $W_{r,n}$. Observe that each of the generators of $W_{r,n}$ acts as a *reflection* (that is, fixes a space of codimension 1), so this shows that $W_{r,n}$ is a complex reflection group.

Let $\Omega = \{\, \epsilon_i - \zeta^k \epsilon_j \mid 1 \le j \le i \le n \text{ and } \max(j-i, -1) < k < r\,\}$. Then Ω is in one-to-one correspondence with the set of reflections in $W_{r,n}$, where the correspondence attaches to each reflection its unique eigenvector in Ω with non-trivial eigenvalue; see [**16**, §3]. For each $\omega \in \Omega$ let H_ω be the hyperplane orthogonal to ω, let $\mathscr{M} = V \setminus \bigcup_{\omega\in\Omega} H_\omega$ be the associated hyperplane complement and $\mathscr{M}/W_{r,n}$ its quotient by $W_{r,n}$.

Definition 2.3. *The* braid group *of $W_{r,n}$ is the group*

$$\mathfrak{B}_{r,n} = \pi_1(\mathscr{M}/W_{r,n}, x_0),$$

where $x_0 \in \mathscr{M}/W_{r,n}$.

Here, $\pi_1(\mathscr{M}/W_{r,n}, x_0)$ is the fundamental group of the quotient space $\mathscr{M}/W_{r,n}$ with base point x_0. Because $\mathscr{M}$ is connected $\mathfrak{B}_{r,n}$ is independent of the choice of x_0.

If $r > 1$ then $\mathfrak{B}_{r,n}$ is a braid group of type B_n and as an abstract group it is generated by elements $s_0, \dots, s_{n-1}$ subject to the relations

$$s_0s_1s_0s_1 = s_1s_0s_1s_0, \quad s_is_j = s_js_i, \quad \text{and} \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1},$$

where $1 \le i < n-1$, $0 \le j < n$ and $|i-j| > 1$. In particular, observe that $W_{r,n}$ is a quotient of $\mathfrak{B}_{r,n}$ (via the map which sends s_i to t_i for $0 \le i < n$).

The generators of $\mathfrak{B}_{n,r}$ can be chosen as generators of the monodromy around the hyperplanes. Bessis [**12**] has now given a general argument for the existence of such presentations for the braid groups of complex reflection groups.

Broué and Malle considered the algebra $R\mathfrak{B}_{r,n}/I_{q,\mathbf{Q}}$, where $I_{q,\mathbf{Q}}$ is the ideal of $R\mathfrak{B}_{r,n}$ generated by $(s_0-Q_1)\dots(s_0-Q_r)$ and $(s_i-q)(s_i+1)$, for $1 \le i < n$; evidently, $\mathscr{H} \cong R\mathfrak{B}_{r,n}/I_{q,\mathbf{Q}}$. One consequence of this definition is that we can use the monodromy representation of the braid group $\mathfrak{B}_{r,n}$ to analyze $\mathscr{H}$. This leads to a more conceptual proof of the fact that $\mathscr{H}$ is always free as an R-module of rank $|W_{r,n}|$ (a corollary of Theorem 2.2). Moreover, it yields the following important result.

Theorem 2.4 (Broué-Malle-Rouquier [**24**, Theorem 4.24]). *Let $\mathbb{K} = \mathbb{C}(q, Q_1, \dots, Q_r)$. Then the monodromy representation of $\mathfrak{B}_{r,n}$ induces an isomorphism of $\mathbb{K}$-algebras $\mathscr{H}_{\mathbb{K},q,\mathbf{Q}} \cong \mathbb{K}W_{r,n}$.*

Here, $\mathbb{K}W_{r,n}$ is the group algebra of $W_{r,n}$ over $\mathbb{K}$. That $\mathscr{H}_{\mathbb{K},q,\mathbf{Q}}$ and $\mathbb{K}W_{r,n}$ are isomorphic algebras can be established by a general Tits deformation theory argument (see, for example, [**34**, §66]). The main point of this result is that the isomorphism is canonically determined.

Lusztig [**96**] has proved a similar result for the Iwahori-Hecke algebras of Weyl groups; however, his argument is less elementary relying on a deep property of the cells of Weyl groups. For Weyl groups, Lusztig's isomorphism and that of Theorem 2.4 are different.

2.4. The affine Hecke algebra of type A

The Ariki-Koike algebras should really be considered as affine objects because they are quotients of the (extended) affine Hecke algebra of type A (i.e., the affine Hecke algebra of $\mathrm{GL}_n(\mathbb{C})$). The affine Hecke algebra $\hat{H}_n$ is the R-algebra with generators $T_1, \dots, T_{n-1}$ and $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and relations

$$\begin{array}{lll} (T_i - q)(T_i + 1) = 0, & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & T_iX_iT_i = qX_{i+1} \\ T_iT_k = T_kT_i, & X_iX_k = X_kX_i, & T_iX_k = X_kT_i \end{array}$$

and $X_iX_i^{-1} = 1 = X_i^{-1}X_i$ for all sensible values of i, j, k with $|i-k| > 1$. In particular, abusing notation slightly, notice that there is surjective algebra homomorphism $\hat{H}_n \twoheadrightarrow \mathscr{H}$ given by sending $T_i \longmapsto T_i$ and $X_j \longmapsto L_j$, for $1 \le i < n$ and $1 \le j \le n$ respectively. It is easy to see that $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n}) \cong \hat{H}_n / \langle (X_1 - Q_1) \dots (X_1 - Q_r) \rangle$.

It follows from the relations that $T_1, \dots, T_{n-1}$ generate a subalgebra of $\hat{H}_n$ isomorphic to $\mathscr{H}_q(\mathfrak{S}_n)$ and that $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ generate a Laurent polynomial ring. Therefore, as an R-module, $\hat{H}_n \cong \mathscr{H}_q(\mathfrak{S}_n) \otimes R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$; so, $\hat{H}_n$ is a twisted tensor product.

Let $\mathcal{P} = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ be the free $\mathbb{Z}$-module with basis $\epsilon_1, \dots, \epsilon_n$; so, $\mathcal{P}$ is the weight lattice of $\mathrm{GL}_n(\mathbb{C})$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{P}$ by permuting the ϵ_i.

If $\lambda \in \mathcal{P}$ set $X^\lambda = X_1^{\lambda_1} \dots X_n^{\lambda_n}$. Then the two commutation relations for the T_i and the X_j can be replaced by the relation

$$T_iX^\lambda = X^{t_i\lambda}T_i + (q-1)\frac{X^\lambda - X^{t_i\lambda}}{1 - X^{\alpha_i}},$$

where $\lambda \in \mathcal{P}$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $1 \leq i < n$. A quick calculation shows that $X^\lambda - X^{t_i\lambda}$ is divisible by $1 - X^{\alpha_i}$ so the right hand side does make sense. Notice that when $q = 1$ this relation becomes $t_i X^\lambda = X^{t_i\lambda} t_i$; this is what we expect because the extended affine Weyl group is the semidirect product $\mathcal{P} \rtimes \mathfrak{S}_n$.

Now suppose that R is an algebraically closed field. Bernstein showed that the centre of $\hat{H}_n$ is the set of symmetric polynomials in $X_1, \dots, X_n$ (Theorem 5.4). Consequently, $\hat{H}_n$ is finite dimensional over its centre; therefore, by Schur's Lemma, every irreducible $\hat{H}_n$-module is finite dimensional (with dimension at most $n!$ since $\dim_R \hat{H}_n / Z(\hat{H}_n) = (n!)^2$ by Theorem 5.4 below).

As remarked above, each Ariki-Koike algebra $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ is a quotient of $\hat{H}_n$, so every irreducible $\mathscr{H}$-module is also an irreducible $\hat{H}_n$-module. Conversely, suppose that R is algebraically closed and that M is an irreducible $\hat{H}_n$-module. If $c_M(X_1)$ is the characteristic polynomial for the action of X_1 on M then $\mathscr{H}_M := \hat{H}_n / \langle c_M(X_1) \rangle$ is an Ariki-Koike algebra (with parameters the eigenvalues for the action of X_1 on M) and M is an irreducible $\mathscr{H}_M$-module. (More generally, M is an irreducible module for any Ariki-Koike algebra obtained by quotienting out by the ideal generated by any polynomial in X_1 which is divisible by $c_M(X_1)$.) Thus the irreducible $\hat{H}_n$-modules are precisely the irreducible $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$-modules as $\mathbf{Q}$ ranges over the elements of $(R^\times)^r$ for $r \geq 1$.

2.5. The conjectures of Broué, Malle and Michel

The conjectures which we now discuss grew out of the attempts of Broué and others to understand Broué's [**18**] conjectures for blocks with abelian defect groups in the case of the finite reductive groups. We consider only a very special case of these conjectures; for references and further details see the original papers [**21, 22, 25**] and Broué's [**19**] comprehensive survey article.

Let $\mathbf{G}$ be an algebraic group defined over $\overline{\mathbb{F}}_q$, where q is a prime power, and let W be the Weyl group of $\mathbf{G}$. Let $F: \mathbf{G} \longrightarrow \mathbf{G}$ be a Frobenius map and let $G = \mathbf{G}^F$ be the F-fixed points of $\mathbf{G}$. We assume that W is F-split. The simplest example is to take $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F(a_{ij}) = (a_{ij}^q)$; then $G = \mathrm{GL}_n(q)$ and $W = \mathfrak{S}_n$.

Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$ and set $B = \mathbf{B}^F$. It is well-known that the irreducible constituents of $\mathrm{Ind}_B^G(1)$ are in one-to-one correspondence with the irreducible representations of W; see, for example, [**29**]. The Iwahori-Hecke algebras of Weyl groups play an important role in this theory; indeed, $\mathscr{H}_q(W) \cong \mathrm{End}_G\left(\mathrm{Ind}_B^G(1)\right)$ and this explains why the dimensions of the irreducible representations in the unipotent principal series, the constituents of $\mathrm{Ind}_B^G(1)$, are given by evaluating certain polynomials $D_\chi(x)$ at $x = q$. The conjectures which follow attempt to explain other "generic" features of the representation theory of finite reductive groups.

Let B_W be the Braid group of W and for $w \in W$ let $\underline{w} \in B_W$ be the lift of w (under the canonical embedding of W into the positive braid monoid B_W^+). Brieskorn and Saito [**17**] showed that the centre of B_W is generated by $\pi = \underline{w_0}^2$ (or $\underline{w_0}$ if w_0 is central in W), where w_0 is the unique element of maximal length in W.

Call an element $w \in W$ good if $\pi = \underline{w}^d$ for some d. Note that w has order d since $w_0^2 = 1$ in W. Every conjugacy class of regular elements in W contains a good element. Assume that w is good. Then Broué and Michel [**25**] have shown that every good element is regular; so $C_W(w)$ is a complex reflection group by Springer [**118**]. Let $B_w = B(C_W(w))$ be the braid group of $C_W(w)$. It is conjectured that $B_w = C_{B_W}(\underline{w})$; this has now been proved in almost all cases [**13, 14**].

Let X_w be the Deligne-Lusztig variety associated to w; so X_w is the variety of Borel subgroups $\mathbf{B}'$ of $\mathbf{G}$ such that $\mathbf{B}'$ and $F(\mathbf{B}')$ are in relative position w. Fix a prime ℓ not dividing q and consider the étale cohomology groups $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ of X_w. The finite group $G = \mathbf{G}^F$ acts on X_w and hence also on $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$. By [**25, 35**] there is also an action of $C_{B_W}(\underline{w})$ on $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ (this comes from an action of the positive braids in $C_{B_W}(\underline{w})$ on X_w). In many cases the action of $\overline{\mathbb{Q}}_\ell C_{B_W}(\underline{w})$ is known to factor through a cyclotomic Hecke algebra. Conjecturally, the action of $C_{B_W}(\underline{w})$ should generate $\mathrm{End}_{\overline{\mathbb{Q}}_\ell G}\big(H_c^i(X_w, \overline{\mathbb{Q}}_\ell)\big)$; this is one of the key unsolved problems and it appears to be very hard.

Let $\mathcal{H}(\mathbf{G}, F, W, w)$ be the image of $\overline{\mathbb{Q}}_\ell C_{B_W}(\underline{w})$ in the (graded) endomorphism algebra of $\bigoplus_{i\geq 0} H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$. Then $\mathcal{H}(\mathbf{G}, F, W, w)$ is a finite dimensional algebra and the following conjecture is expected to be true.

Conjecture 2.5 (Broué,Malle,Michel [**19, 21, 22, 25**]).
Suppose that w is a good element of order d.

(i) *If $i \neq j$ then the $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ and $H_c^j(X_w, \overline{\mathbb{Q}}_\ell)$ have no irreducible constituents in common.*

(ii) *There is a d-cyclotomic Hecke algebra $\mathscr{H}_x\big(C_W(w)\big)$ of the complex reflection group $C_W(w)$ such that*

$$\mathcal{H}(\mathbf{G}, F, W, w) \cong \mathscr{H}_{\overline{\mathbb{Q}}_\ell, q}\big(C_W(w)\big) \cong \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F}\Big(\bigoplus_{i\geq 0} H_c^i(X_w, \overline{\mathbb{Q}}_\ell)\Big).$$

(iii) *There is a one-to-one correspondence $\chi \longleftrightarrow \chi_q$ between the irreducible representations of $C_W(w)$ and the irreducible constituents* *of the $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-module $\bigoplus_{i\geq 0} H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$. Moreover, for each irreducible character χ of $C_W(w)$ there is a polynomial $D_\chi(x)$, depending only on χ, such that the degree of χ_q is equal to $D_\chi(q)$.*

We now explain the term *d-cyclotomic Hecke algebra* when $C_W(w) = W_{r,n}$. Let $\xi \in \mathbb{C}$ be a root of unity and x an indeterminate over $\mathbb{Z}[\xi]$ and let ζ_d be a primitive d^{th} root of unity. An Ariki-Koike algebra $\mathscr{H}_x(W_{r,n}) = \mathscr{H}_{R,v,\mathbf{Q}}(W_{r,n})$ is *d-cyclotomic* if $R = \mathbb{Z}[\xi][x, x^{-1}]$ and the parameters of $\mathscr{H}_x(W_{r,n})$ are of the form $v = \zeta^{a_v} x^{b_v}$ and $Q_s = \zeta^{a_s} x^{b_s}$, for some rational numbers a_v, b_v and a_s, b_s, such that:

(a) $\mathscr{H}_{\zeta_d} = \mathscr{H}_x(W_{r,n}) \otimes_R R/(x - \zeta_d) \cong \mathbb{Z}[\xi] W_{r,n}$; and,
(b) $\mathscr{H}_q = \mathscr{H}_x(W_{r,n}) \otimes_R R/(x - q)$ is semisimple over its field of fractions.

For example, take parameters $v = x^d$ and $Q_s = x^{s-1}$ (with $\xi = 1$); then $\mathscr{H}_{\zeta_d} \cong \mathbb{Z}[\zeta_d] W_{r,n}$.

Thus, part (ii) of the conjecture together with (b) implies that the irreducible representations occurring in $\bigoplus_i H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ are in one-to-one correspondence with the irreducible representations of $\mathscr{H}_q\big(C_W(w)\big)$; in turn, by (a) these representations are in one-to-one correspondence with the irreducible representations of $C_W(w)$. Importantly, nothing here depends upon the choice of q or ℓ. Conjecturally, these correspondences come from a derived equivalence, so they are really perfect isometries ("bijections with signs"). The polynomials $D_\chi(x)$ in part (iii) are the generic degrees of $\mathscr{H}_x\big(C_W(w)\big)$; see the remarks after Theorem 3.6.

In fact, part (iii) of the conjecture is already known. The key fact needed to establish this is that the virtual module $\bigoplus_{i\geq 0} (-1)^i H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ is a Deligne-Lusztig representation (specifically, it is $R^G_{T_w}(1)$, where T_w is the maximal torus associated to the conjugacy class of w in W), so its irreducible constituents are known. Parts (i) and (ii) of the conjecture are known in only a small number of cases.

We also mention that everything above is compatible with the decomposition of the unipotent characters of $\mathbf{G}^F$ into d-Harish-Chandra series [**22**]. For these details, and stronger forms of the conjecture, we refer the reader to Broué's article [**19**].

To conclude this section we remark that if $w = 1$ then $X_1 = G/B$ is the flag variety; so, $H^0_c(X_1, \overline{\mathbb{Q}}_\ell) \cong \operatorname{Ind}_B^G(1)$ and all higher cohomology groups are zero. Thus, in this case the conjectures recover the well-known results for the principal unipotent series of $\mathbf{G}^F$. (According to our definitions, $w = 1$ is not a good element of W; however, we have discussed only a special case of the general conjectures.)

§3. The representation theory of the Ariki-Koike algebras

3.1. The semisimple representation theory of $\mathscr{H}$

Because $W_{r,n}$ is the wreath product $\mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$, its ordinary irreducible representations are indexed by r-tuples of partitions of n. In this section we see that the same is true of the irreducible representations of $\mathscr{H}$ when $\mathscr{H}$ is semisimple.

A partition of n is a sequence $\sigma = (\sigma_1 \geq \sigma_2 \geq \cdots)$ of non-negative integers σ_i such that $|\sigma| = \sum_{i\geq 1} \sigma_i = n$; we write $\sigma = (\sigma_1, \ldots, \sigma_k)$ if $\sigma_i = 0$ for $i > k$. A multipartition of n is an ordered r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions with $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. We write $\lambda \vdash n$ if λ is a multipartition of n.

The multipartitions form a poset under dominance $\trianglerighteq$, where $\lambda \trianglerighteq \mu$ if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{j=1}^{i} \lambda_j^{(s)} \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{j=1}^{i} \mu_j^{(s)}$$

for $s = 1, 2, \ldots, r$ and all $i \geq 1$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$ we write $\lambda \triangleright \mu$.

The diagram of λ is $[\lambda] = \{ (i, j, s) \mid 1 \leq j \leq \lambda_i^{(s)} \text{ and } 1 \leq s \leq r \}$. The elements of $[\lambda]$ are called nodes; more generally, a node is any triple (i, j, s) where $1 \leq s \leq r$ and $i, j \geq 1$.

A λ-tableau is a bijection $\mathfrak{t}\colon [\lambda] \longrightarrow \{1, 2 \ldots, n\}$, which we consider as an r-tuple $\mathfrak{t} = (\mathfrak{t}^{(1)}, \ldots, \mathfrak{t}^{(r)})$ of labeled tableaux where $\mathfrak{t}^{(s)}$ is a $\lambda^{(s)}$-tableau for each s; the tableaux $\mathfrak{t}^{(s)}$ are the components of $\mathfrak{t}$. If $\mathfrak{t}$ is a λ-tableau we write $\operatorname{Shape}(\mathfrak{t}) = \lambda$.

A tableau $\mathfrak{t}$ is standard if, in each component, its entries increase from left to right along each row and from top to bottom down each column. For example,

$$\begin{aligned} \mathfrak{t}^\lambda &= \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 5 \\ \hline 6 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 7 & 8 \\ \hline 9 \\ \cline{1-1}\end{array} \right) \quad \text{and} \\ \mathfrak{t} &= \left(\begin{array}{|c|c|c|}\hline 4 & 7 & 9 \\ \hline 6 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 8 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \right) \end{aligned} \tag{3.1}$$

are two standard $((3,1),(1^2),(2,1))$-tableaux. Let $\mathcal{T}^{\mathrm{s}}(\lambda)$ be the set of standard λ-tableaux.

If $\mathfrak{t}$ is a λ-tableau and $w \in \mathfrak{S}_n$ let $\mathfrak{t}w = \mathfrak{t} \circ w$ be the tableau obtained from $\mathfrak{t}$ by replacing each entry in $\mathfrak{t}$ by its image under w. This defines a right action of $\mathfrak{S}_n$ on the set of all λ-tableaux. For example, $\mathfrak{t} = \mathfrak{t}^\lambda(1,4,6,8,5)(2,7)(3,9)$ in (3.1).

If $\mathfrak{t}$ is a tableau and k an integer, with $1 \leq k \leq n$, then the residue of k in $\mathfrak{t}$ is defined to be $\operatorname{res}_{\mathfrak{t}}(k) = q^{j-i}Q_s$, if k appears in row i and column j of $\mathfrak{t}^{(s)}$; that is, $\mathfrak{t}(i, j, s) = k$.

The last ingredient that we need is something like the Poincaré polynomial of a Coxeter group; however, be warned that it is not true that $|W_{r,n}| = P_{\mathscr{H}}(q,\mathbf{Q})$ when $R=\mathbb{C}$, $q=1$ and $Q_s=\zeta^{s-1}$, where $\zeta=\exp(2\pi i/e)$ (that is, when $\mathscr{H}=RW_{r,n}$). Let

$$P_{\mathscr{H}}(q,\mathbf{Q}) = \prod_{i=1}^{n}(1+q+\cdots+q^{i-1})\cdot\prod_{1\le i<j\le r}\ \prod_{-n<d<n}(q^dQ_i-Q_j).$$

We can now describe the irreducible representations of $\mathscr{H}$ when $P_{\mathscr{H}}(q,\mathbf{Q})$ is invertible. (Note that if R is a field then $P_{\mathscr{H}}(q,\mathbf{Q})$ is invertible if and only if $P_{\mathscr{H}}(q,\mathbf{Q})\ne 0$.)

Theorem 3.2 (Hoefsmit [**76**], Cherednik [**30**], Ariki-Koike [**8**]). *Suppose that $P_{\mathscr{H}}(q,\mathbf{Q})$ is invertible in R.*

(i) *For each multipartition λ let V^λ be the R-module with basis*

$$\{\, v_{\mathfrak{t}} \mid \mathfrak{t} \text{ a standard } \lambda\text{-tableau}\,\}.$$

Then V^λ becomes a right $\mathscr{H}$-module via $v_{\mathfrak{t}}T_0=\operatorname{res}_{\mathfrak{t}}(1)v_{\mathfrak{t}}$ and, for $1\le i<n$, if $\mathfrak{s}=\mathfrak{t}t_i$ is not standard then

$$v_{\mathfrak{t}}T_i=\begin{cases} qv_{\mathfrak{t}}, & \text{if } i \text{ and } i+1 \text{ are in the same row of } \mathfrak{t},\\ -v_{\mathfrak{t}}, & \text{if } i \text{ and } i+1 \text{ are in the same column of } \mathfrak{t},\end{cases}$$

and if $\mathfrak{s}$ is standard then

$$v_{\mathfrak{t}}T_i=\frac{(q-1)\operatorname{res}_{\mathfrak{t}}(i)}{\operatorname{res}_{\mathfrak{t}}(i)-\operatorname{res}_{\mathfrak{s}}(i)}v_{\mathfrak{t}}+\frac{q\operatorname{res}_{\mathfrak{t}}(i)-\operatorname{res}_{\mathfrak{s}}(i)}{\operatorname{res}_{\mathfrak{t}}(i)-\operatorname{res}_{\mathfrak{s}}(i)}v_{\mathfrak{s}}.$$

(ii) *If R is a field then V^λ is an irreducible $\mathscr{H}$-module for each multipartition λ.*

(iii) *If R is a field then $\{\,V^\lambda \mid \lambda\vdash n\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}$-modules.*

The general case follows from the type A case ($r=1$); this is due to Hoefsmit who, in turn built upon Young's seminormal form for the symmetric groups. Cherednik does not state the result in this form; it is necessary to do some work to see that his result is equivalent.

Part (i) is proved by a brute force calculation to show that the action of the generators on V^λ respects the relations in $\mathscr{H}$. The remaining parts can be proved by looking at how the commutative subalgebra $\mathscr{L}=\langle L_1,\dots,L_n\rangle$ of $\mathscr{H}$ acts on V^λ. From Theorem 3.2(i) it follows that $v_{\mathfrak{t}}L_k=\operatorname{res}_{\mathfrak{t}}(k)v_{\mathfrak{t}}$ for all standard tableaux $\mathfrak{t}$, for $1\le k\le n$. Hence $Rv_{\mathfrak{t}}$ is an irreducible $\mathscr{L}$-module; in fact, Ariki and Koike [**8**] show that every irreducible $\mathscr{L}$-module is of this form. Moreover, because $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$ if $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux then $\mathfrak{s} = \mathfrak{t}$ if and only if $\operatorname{res}_{\mathfrak{t}}(k) = \operatorname{res}_{\mathfrak{s}}(k)$, for $1 \leq k \leq n$; this implies that $Rv_{\mathfrak{s}} \cong Rv_{\mathfrak{t}}$ as $\mathscr{L}$-modules only if $\mathfrak{s} = \mathfrak{t}$. Therefore, V^λ and V^μ have a common composition factor only if $\lambda = \mu$; hence (ii). Part (iii) now follows by counting dimensions because

$$\dim \mathscr{H} \geq \dim(\mathscr{H}/\operatorname{Rad}\mathscr{H}) \geq \sum_{\lambda \vdash n} (\dim V^\lambda)^2 = r^n n! = |W_{r,n}| \geq \dim \mathscr{H}.$$

(The third equality follows from the Robinson-Schensted correspondence which implies that the sum of the squares of the number of standard λ-tableaux, as λ runs over all multipartitions of n, is equal to $|W_{r,n}|$.) As we have equality throughout, this also proves Theorem 2.2 (indeed, this is how Ariki and Koike first proved it).

Corollary 3.3 (Ariki [**2**]). *Suppose that R is a field. Then $\mathscr{H}$ is semisimple if and only if $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$.*

Sketch of proof. By Theorem 3.2 if $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$ then $\mathscr{H}$ is semisimple. For the converse, when $P_{\mathscr{H}}(q, \mathbf{Q}) = 0$ the ideal of $\mathscr{H}$ generated by

$$\Big(\prod_{k=1}^{n} \prod_{s=1}^{r-1} (L_k - Q_s)\Big)\Big(\sum_{w \in \mathfrak{S}_n} T_w\Big),$$

is nilpotent. (This ideal affords the "trivial" representation of $\mathscr{H}$.) □

Halverson and Ram [**75**] have generalized the Murnaghan-Nakayama rule of the symmetric groups to give a method for computing the characters of the irreducible representations V^λ. (In fact, they also compute the characters of the irreducible representations of the cyclotomic Hecke algebras of type $G(r, p, n)$; the irreducible representations of these algebras were constructed by Ariki [**3**].) See also Shoji [**117**].

As remarked earlier the symmetric polynomials in $L_1, \ldots, L_n$ belong to the centre of $\mathscr{H}$. In the semisimple case this is a complete description of the centre.

Theorem 3.4 (Ariki-Koike [**8**]). *Suppose that R is a field and that $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$. Then the centre of $\mathscr{H}$ is equal to the set of symmetric polynomials in $L_1, \ldots, L_n$.*

Graham [**63**] has recently announced that the centre of $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ is always equal to the set of symmetric polynomials in $L_1, \ldots, L_n$ when R is an integral domain (this is the case $r = 1$). Ariki [**4**] has given an example which shows that the centre of $\mathscr{H}$ can be larger than the set of symmetric polynomials when $r > 1$.

When $q \neq 1$ and $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$ the author [**104**] has explicitly described the primitive central idempotents as symmetric polynomials in $L_1, \dots, L_n$ (see also Shoji [**117**]); this gives a second proof of Theorem 3.4. In addition, [**104, 117**] construct the primitive idempotents and a Wedderburn basis of $\mathscr{H}$ in the semisimple case.

Define $\tau : \mathscr{H} \longrightarrow R$ to be the R-linear map determined by

$$\tau(L_1^{a_1} \dots L_n^{a_n} T_w) = \begin{cases} 1, & \text{if } a_1 = \dots = a_n = 0 \text{ and } w = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $0 \leq a_i < r$ and $w \in \mathfrak{S}_n$. Notice that if $q = 1$ and $Q_s = \zeta^s$, where $\zeta = \exp(2\pi i/r) \in \mathbb{C}$, then τ is the natural trace function on the group algebra $\mathbb{C}W_{r,n}$. The definition of τ looks quite ad hoc; however, as we explain below, τ is canonically determined.

Proposition 3.5. *Assume that R is an integral domain. Then the following hold.*

(i) (**Bremke-Malle** [**16**]) *τ is a trace form on $\mathscr{H}$.*

(ii) (**Malle-Mathas** [**101**]) *Suppose that $q, Q_1, \dots, Q_r$ are all invertible in R. Then τ is non-degenerate. Consequently, $\mathscr{H}$ is a symmetric algebra.*

Part (i) is straightforward; although we should mention that Bremke and Malle use a different (but, by [**101**], equivalent), definition of the trace form τ. For the Iwahori-Hecke algebras ($r \leq 2$), part (ii) is also routine (see, for example, [**103**, Prop. 1.16]); in contrast, whilst not difficult, the proof of (ii) is a laborious calculation when $r > 2$. As an indication of the difficulties here, no pair of dual bases for $\mathscr{H}$ is known when $r > 2$ (except in the semisimple case; see [**104**, Theorem 3.9]).

As we will describe, Proposition 3.5 provides the strongest known link between the representation theory of $\mathscr{H}$ and that of the finite groups of Lie type (when $r > 2$).

If R is a field and $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$ then $\mathscr{H}$ is semisimple. Let χ^λ be the character of V^λ. Since τ is a trace function we can write

$$\tau = \sum_{\lambda \vdash n} \frac{1}{s_\lambda(q, \mathbf{Q})} \chi^\lambda$$

for some $s_\lambda(q, \mathbf{Q}) \in R$. The rational functions $s_\lambda(q, \mathbf{Q})$ are the **Schur elements** of $\mathscr{H}$; to describe them we need some more notation. In fact, by general arguments (see [**61**, Prop. 7.3.9]), $s_\lambda(q, \mathbf{Q})$ in $\mathbb{Z}[q^{\pm}, \mathbf{Q}^{\pm}]$; this is by no means obvious from the explicit formula for $s_\lambda(q, \mathbf{Q})$ given below.

Define the length of a partition σ to be the smallest integer $\ell(\sigma)$ such that $\sigma_i = 0$ for all $i > \ell(\sigma)$; the **length** of a multipartition λ is

$\ell(\lambda) = \max\{\,\ell(\lambda^{(s)}) \mid 1 \le s \le r\,\}$. Suppose that $L \ge \ell(\lambda)$ and set $\beta_i^{(s)} = \lambda_i^{(s)} + L - i$ for $i = 1, \ldots, L$ and $1 \le s \le r$; also set $B_s = \{\beta_1^{(s)}, \ldots, \beta_L^{(s)}\}$, for $s = 1, \ldots, r$. The matrix $B = (\beta_i^{(s)})_{s,i}$ is the L-symbol of λ [**20**, **99**].

Theorem 3.6 (Geck-Iancu-Malle [**60**]). *Suppose that λ is a multipartition of n with L-symbol $B = (\beta_i^{(s)})_{s,i}$ such that $L \ge \ell(\lambda)$. Then the Schur element $s_\lambda(q, \mathbf{Q})$ is equal to*

$$(-1)^{a_{rL}} q^{b_{rL}} \frac{\displaystyle\prod_{1\le s<t\le r} (Q_s - Q_t)^L \cdot \prod_{1\le s,t\le r}\ \prod_{\alpha_s\in B_s}\ \prod_{1\le k\le \alpha_s} (q^k Q_s - Q_t)}{\displaystyle (q-1)^n (Q_1 \ldots Q_r)^n \prod_{1\le s\le t\le r}\ \prod_{\substack{(\alpha_s,\alpha_t)\in B_s\times B_t\\ \alpha_s>\alpha_t \text{ if } s=t}} (q^{\alpha_s} Q_s - q^{\alpha_t} Q_t)},$$

where $a_{rL} = n(r-1) + \binom{r}{2}\binom{L}{2}$ and $b_{rL} = \frac{rL(L-1)(2rL-r-3)}{12}$.

It is not hard to see that if f_λ is a primitive idempotent in $\mathscr{H}$ which generates the Specht module S^λ then $s_\lambda(q, \mathbf{Q}) = 1/\tau(f_\lambda)$; this observation is used in [**104**] to give a direct proof of Theorem 3.6. (Actually, [**60**] and [**104**] were both written at the same time; however, I obtained a different formula for $s_\lambda(q, \mathbf{Q})$. The final version of my paper shows that these two formulae coincide.)

For $r = 1, 2$ the Schur elements were first computed by Hoefsmit [**76**]. Murphy [**106**] gave a different argument for type A (that is, $r = 1$). For $r > 2$ this result was conjectured by Malle [**99**]. Geck, Iancu and Malle use a clever specialization argument due to Orellana [**108**] to compute the Schur elements using the Markov trace of the Hecke algebras $\mathscr{H}_q(\mathfrak{S}_m)$; in turn, this builds on work of Wenzl [**122**].

Theorem 3.6 is important because when combined with [**99**, 3.16 and 6.11] it implies that Φ_d-blocks [**22**] of the finite reductive groups satisfy a generalized Howlett–Lehrer theory [**77**]. More precisely, Conjecture 2.5(iii) is true and the dimensions of the irreducible representations in a unipotent Φ_d-block are given by specializations of the generic degrees of $\mathscr{H}$; these are the rational functions $D_\lambda(q) = s_\eta(q, \mathbf{Q})/s_\lambda(q, \mathbf{Q})$, where $\eta = ((n), (0), \ldots, (0))$. Remarkably, for "spetsial specializations" the generic degrees are actually polynomials; see [**100**]. (The significance of $s_\eta(q, \mathbf{Q})$ is that it is the Poincaré polynomial of the coinvariant algebra of $W_{r,n}$ when $Q_1 = q$ and $Q_s = \zeta^{s-1}$, for $2 \le s \le r$.)

As a second application of Theorem 3.6, it follows from [**60**, Theorem 5.2] and [**23**, Lemma 2.7] that the trace form τ is the unique trace form on $\mathscr{H}$ which, in a precise sense [**23**, Theorem 2.1], is compatible with the usual trace forms on both $W_{r,n}$ and on the braid group

$\mathfrak{B}_{r,n}$. In addition, Malle [**100**] uses Theorem 3.6 to define the notion of "spetsiality" for complex reflection groups; for more details see [**23**].

Finally, Broué and Kim [**20**] use Theorem 3.6, together with the block structure of $\mathscr{H}$, to show that the irreducible representations of $\mathscr{H}$ can be grouped according to a generalization of Lusztig's *families*; a key ingredient in their paper is a block theoretical characterisation of Lusztig's families due to Rouquier [**112**]. Again, the combinatorial description of the spetsial families of $\mathscr{H}$ had previously been conjectured by Malle [**99**].

3.2. The modular representation theory of $\mathscr{H}$

We now turn to the modular representation theory of $\mathscr{H}$; that is, the representation theory when $\mathscr{H}$ is not semisimple. In types A and B the irreducible modular representations were first constructed by Dipper and James [**37**] and Dipper, James and Murphy [**43**], respectively. Graham and Lehrer [**64**] considered the general case using cellular algebra techniques. Even though the papers [**43, 107**] predated Graham and Lehrer, the cellular approach is already implicit in them.

Graham and Lehrer constructed a cellular basis for $\mathscr{H}$ by building upon the Kazhdan-Lusztig basis of $\mathscr{H}_q(\mathfrak{S}_n)$ (which is itself cellular). We will describe a different cellular basis of $\mathscr{H}$ which comes from the work of Dipper, James and the author [**42**]. We prefer this basis because we know how to lift this basis to give a basis for the cyclotomic q-Schur algebras and because this basis has many nice combinatorial and representation theoretic properties.

Let $*$ be the anti-isomorphism of $\mathscr{H}$ determined by $T_i^* = T_i$, for $0 \le i < n$. Then $*$ is an involution and $T_w^* = T_{w^{-1}}$, $L_k^* = L_k$ and $(h_1h_2)^* = h_2^*h_1^*$ for $w \in \mathfrak{S}_n$, $1 \le k \le n$ and $h_1, h_2 \in \mathscr{H}$.

Fix a multipartition λ and let $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ be the associated Young subgroup of $\mathfrak{S}_n$. Equivalently, $\mathfrak{S}_\lambda$ is the row stabilizer of the λ-tableau $\mathfrak{t}^\lambda$ which has the numbers $1, \dots, n$ entered in order from left to right, top to bottom first along the rows of $\mathfrak{t}^{\lambda^{(1)}}$ and then $\mathfrak{t}^{\lambda^{(2)}}$ and so on (for example, see the first tableau in (3.1)).

If $\mathfrak{t}$ is a standard λ-tableau let $d(\mathfrak{t}) \in \mathfrak{S}_n$ be the unique permutation in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$. Define elements x_λ and u_λ^+ in $\mathscr{H}$ by

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \quad \text{and} \quad u_\lambda^+ = \prod_{s=2}^{r} \prod_{k=1}^{|\lambda^{(1)}|+\cdots+|\lambda^{(s-1)}|} (L_k - Q_s).$$

It follows easily from the relations in $\mathscr{H}$ that $x_\lambda u_\lambda^+ = u_\lambda^+ x_\lambda$. Although somewhat ungainly, the function of u_λ^+ is used to control the eigenvalues of the L_k on the modules below. Set $m_\lambda = x_\lambda u_\lambda^+$.

Definition 3.7. *Suppose that* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are standard* λ*-tableaux. Let* $m_{\mathfrak{st}} = T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}$.

Theorem 3.8 (The standard basis theorem [**42**]). *The Ariki-Koike algebra* $\mathscr{H}$ *is free as an* R*-module with cellular basis*

$$\{\, m_{\mathfrak{st}} \mid \mathfrak{s} \text{ and } \mathfrak{t} \text{ standard } \lambda\text{-tableaux}, \ \lambda \vdash n \,\}.$$

When $r = 1$ this result is due to Murphy [**107**] and when $r = 2$ it was proved by Dipper, James and Murphy [**43**]. The basis $\{m_{\mathfrak{st}}\}$ is called both the Murphy basis and the standard basis of $\mathscr{H}$. As mentioned above, Graham and Lehrer [**64**] were the first to produce a (different) cellular basis of $\mathscr{H}$.

The proof of this theorem starts by observing that $\mathscr{H}$ is spanned by a set of more general elements $m_{\mathfrak{st}}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are *row standard* tableaux of the same shape. (The entries in row standard tableaux increase along rows, but not necessarily down columns.) Next, one shows that if $\mathfrak{s}$ and $\mathfrak{t}$ are not standard tableaux then $m_{\mathfrak{st}}$ can be written as a linear combination of "higher terms" $m_{\mathfrak{uv}}$; so, by induction, $\mathscr{H}$ is spanned by standard basis elements (here, "higher" is essentially the Bruhat order on $\mathfrak{S}_n$). The rewriting rules involved in this step are essentially Garnir relations; in fact, they are a little bit easier than the classical Garnir relations because we work modulo a filtration. A counting argument now shows that we have a basis. In order to show that the basis is cellular some accounting details need to be carried through the argument; this adds only minor complications to the proof.

We will not describe the theory of cellular algebras here; instead the reader is referred to the beautiful paper of Graham and Lehrer [**64**] or to Chapter 2 of my book [**103**]. A different approach to cellular algebras can be found in [**93**].

The required indexing of a cellular basis is already implicit in our notation. The two properties that the basis $\{m_{\mathfrak{st}}\}$ must satisfy for it to be cellular are: (i) the R-linear map determined by $m_{\mathfrak{st}} \longmapsto m_{\mathfrak{ts}}$ must be an algebra anti-isomorphism — this is obvious for us because $m^*_{\mathfrak{st}} = m_{\mathfrak{ts}}$; and, (ii) for all λ-tableaux $\mathfrak{t}$ and all $h \in \mathscr{H}$ there exist scalars $r_{\mathfrak{v}} \in R$ such that for any standard λ-tableau $\mathfrak{s}$

$$m_{\mathfrak{st}} h \equiv \sum_{\mathfrak{v} \in \mathcal{T}^s(\lambda)} r_{\mathfrak{v}} m_{\mathfrak{sv}} \pmod{\mathcal{H}^\lambda}, \tag{3.9}$$

where $\mathcal{H}^\lambda$ is the R-module spanned by the elements $m_{\mathfrak{uv}}$ for Shape$(\mathfrak{u}) =$ Shape$(\mathfrak{v}) \rhd \lambda$. The point of this equation is that the scalars $r_{\mathfrak{v}}$ depend only on $\mathfrak{t}, \mathfrak{v}$ and h; importantly, $r_{\mathfrak{v}}$ does not depend on $\mathfrak{s}$.

Applying the anti-isomorphism $*$ to the last equation gives a left hand analogue of (3.9) for $hm_{\mathfrak{s}\mathfrak{t}}$. It follows that $\mathcal{H}^\lambda$ is a two-sided ideal of $\mathscr{H}$.

Definition 3.10. *Suppose that λ is a multipartition of n. The* Specht module S^λ *is the right $\mathscr{H}$-module generated by $m_\lambda + \mathcal{H}^\lambda$.*

Thus, S^λ is a submodule of the quotient module $\mathscr{H}/\mathcal{H}^\lambda$. Du and Rui [**55**] have shown how to construct the Specht modules as submodules of $\mathscr{H}$ (as distinct from subquotients as we have defined them here).

For each standard λ-tableau $\mathfrak{t}$ let $m_\mathfrak{t} = m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathcal{H}^\lambda = m_\lambda T_{d(\mathfrak{t})} + \mathcal{H}^\lambda$. It follows from Theorem 3.8 that S^λ is free as an R-module with basis $\{\, m_\mathfrak{t} \mid \mathfrak{t} \text{ a standard } \lambda\text{-tableau}\,\}$; moreover, by (3.9) the action of $\mathscr{H}$ on this basis is given by

$$m_\mathfrak{t} h = \sum_{\substack{\mathfrak{v} \text{ standard} \\ \lambda\text{-tableau}}} r_\mathfrak{v} m_\mathfrak{v},$$

where the scalars $r_\mathfrak{v} \in R$ are the same as those in (3.9). It follows from the left and right handed versions of (3.9) that there is a bilinear form on S^λ which is determined by

$$\langle m_\mathfrak{s}, m_\mathfrak{t}\rangle m_{\mathfrak{u}\mathfrak{v}} \equiv m_{\mathfrak{u}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{v}} \pmod{\mathcal{H}^\lambda}$$

for all standard λ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$. This form is $*$–associative; that is, $\langle xh, y\rangle = \langle x, yh^*\rangle$ for all $x, y \in S^\lambda$ and $h \in \mathscr{H}$. Hence, $\operatorname{Rad} S^\lambda = \{\, x \in S^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in S^\lambda \,\}$ is a submodule of S^λ and we may make the following definition.

Definition 3.11. *Suppose that λ is a multipartition of n. Then D^λ is the right $\mathscr{H}$-module $D^\lambda = S^\lambda / \operatorname{Rad} S^\lambda$.*

Everything that we have said since Theorem 3.8 is part of the general machinery of cellular algebras. Without too much work, the cellular theory now produces the following result.

Theorem 3.12 (Graham-Lehrer [**64**], Dipper-James-Mathas [**42**]). *Suppose that R is a field.*

(i) *For each multipartition μ, D^μ is either zero or absolutely irreducible.*
(ii) *$\{\, D^\mu \mid \mu \vdash n$ and $D^\mu \neq 0 \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}$-modules.*
(iii) *If $D^\mu \neq 0$ then the decomposition multiplicity $[S^\lambda : D^\mu] \neq 0$ only if $\lambda \trianglerighteq \mu$; further, $[S^\mu : D^\mu] = 1$.*

Graham and Lehrer proved this result for a different collection of modules; but this should really be considered their result. Again, for the cases $r = 1, 2$ see [**37**, **43**].

In particular, note that every field is a splitting field for $\mathscr{H}$. The reader might be concerned with the claim that any field R is a splitting field for $\mathscr{H}$ because, for example, $R = \mathbb{Q}$ is not a splitting field for $W_{r,n}$ when $r > 2$; however, this is OK because by definition all of the eigenvalues of T_0 automatically belong to R.

The multiplicities $d_{\lambda\mu} = [S^\lambda{:}D^\mu]$ are the decomposition numbers of $\mathscr{H}$ and the matrix $(d_{\lambda\mu})$ is the decomposition matrix of $\mathscr{H}$. Part (iii) of Theorem 3.12 says that the decomposition matrix of $\mathscr{H}$ is unitriangular when its rows and columns are ordered in a way that is compatible with the dominance order.

Corollary 3.3 and the theory of cellular algebras also gives us the following result.

Theorem 3.13. *Suppose that R is a field. Then the following are equivalent.*

(i) $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$;
(ii) *$\mathscr{H}$ is semisimple;*
(iii) *$\mathscr{H}$ is split semisimple; and,*
(iv) *$S^\lambda = D^\lambda$ for all multipartitions λ of n.*

If $1 \le k \le n$ let $\mathfrak{t}{\downarrow}k$ be the subtableau of $\mathfrak{t}$ containing the integers $1, 2, \ldots, k$; so, if $\mathfrak{t}$ is standard then $\mathrm{Shape}(\mathfrak{t}{\downarrow}k)$ is a multipartition of k. We extend the dominance ordering to the set of standard tableaux by defining $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $\mathrm{Shape}(\mathfrak{s}{\downarrow}k) \trianglerighteq \mathrm{Shape}(\mathfrak{t}{\downarrow}k)$ for $k = 1, \ldots, n$. Again we write $\mathfrak{s} \rhd \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$. In fact, this partial order coincides with the Chevalley–Bruhat order $\le$ on $\mathfrak{S}_n$: $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if and only if $d(\mathfrak{s}) \le d(\mathfrak{t})$. This result really goes back to Ehresmann and, independently, Dipper and James [**37**]; see also [**103**, Theorem 3.8].

A useful fact about the standard basis of $\mathscr{H}$ is the following.

Proposition 3.14 ([**81**, Prop. 3.7]). *Suppose that $1 \le k \le n$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be standard tableaux of the same shape. Then, there exist scalars $r_{\mathfrak{v}} \in R$ such that*

$$m_{\mathfrak{s}\mathfrak{t}} L_k = \mathrm{res}_{\mathfrak{t}}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \pmod{\mathcal{H}^\lambda}.$$

As shown in [**81**], the general case can be reduced to the case $r = 1$ where it is a theorem of Dipper and James [**38**]. When $r = 1$ the result can be proved by induction on n and k using the fact that $L_1 + \cdots + L_n$ belongs to the centre of $\mathscr{H}$; see [**103**].

As an application of Proposition 3.14, if R is a field and $P_{\mathscr{H}}(q,\mathbf{Q})\neq 0$ then we can construct the irreducible $\mathscr{H}$-modules either as the modules V^λ of Theorem 3.2 or as the Specht modules S^λ. By Proposition 3.14 the modules V^λ and S^λ have the same $\mathscr{L}$-module composition factors; this implies that $V^\lambda\cong S^\lambda$ as $\mathscr{H}$-modules.

We close this section with a reduction theorem which shows that, up to Morita equivalence, the only important Ariki-Koike algebras are those with parameters of the form (i) $Q_s=q^{a_s}$ for some integers a_s with $|a_s|<n$, for $1\le s\le r$, or (ii) $Q_s=0$ for $1\le s\le r$. The result actually says that we can reduce to the case where there exists a constant $c\in R$ and integers a_s such that $Q_s=cq^{a_s}$, for all s; However, if $c\neq 0$ then we can renormalize the generator T_0 as $\tilde{T}_0=c^{-1}T_0$ and then the order relation for $\tilde{T}_0$ becomes $(\tilde{T}_0-q^{a_1})\dots(\tilde{T}_0-q^{a_r})=0$, so we are back in case (i).

Recall that $\mathbf{Q}=\{Q_1,\dots,Q_r\}$ and fix a partition $\mathbf{Q}=\mathbf{Q}_1\coprod\dots\coprod\mathbf{Q}_\kappa$ (disjoint union) of $\mathbf{Q}$ and let

$$P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa)=\prod_{1\le\alpha<\beta\le\kappa}\prod_{\substack{Q_i\in\mathbf{Q}_\alpha\\ Q_j\in\mathbf{Q}_\beta}}\prod_{-n<d<n}(q^dQ_i-Q_j).$$

Observe that $P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa)$ is a factor of the polynomial $P_{\mathscr{H}}(q,\mathbf{Q})$.

Theorem 3.15 (Dipper-Mathas [**44**]). *Suppose that R is an integral domain and that $\mathbf{Q}=\mathbf{Q}_1\coprod\dots\coprod\mathbf{Q}_\kappa$ is a partition of $\mathbf{Q}$ such that the polynomial $P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa)$ is invertible in R. For $\alpha=1,\dots,\kappa$ let $r_\alpha=|\mathbf{Q}_\alpha|$. Then $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ is Morita equivalent to the R-algebra*

$$\bigoplus_{\substack{n_1,\dots,n_\kappa\ge 0\\ n_1+\dots+n_\kappa=n}}\mathscr{H}_{q,\mathbf{Q}_1}(W_{r_1,n_1})\otimes\dots\otimes\mathscr{H}_{q,\mathbf{Q}_\kappa}(W_{r_\kappa,n_\kappa}).$$

If $r=2$ then $|\mathbf{Q}_1|=|\mathbf{Q}_2|=1$ and this is a result of Dipper and James [**40**]. Du and Rui [**54**] extended the argument of [**40**] to prove the special case of Theorem 3.15 when $|\mathbf{Q}_\alpha|=1$ for $1\le\alpha\le\kappa$; notice that in this case $\mathscr{H}$ is Morita equivalent to a direct sum of tensor products of Iwahori-Hecke algebras of type A.

For the proof of Theorem 3.15 observe that by induction it is enough to consider the special case $\kappa=2$. Without loss of generality we may assume that $\mathbf{Q}_1=\{Q_1,\dots,Q_s\}$ and $\mathbf{Q}_2=\{Q_{s+1},\dots,Q_r\}$ for some s. The trick is to consider the right ideals $V^b=v_b\mathscr{H}$, for $0\le b\le n$, where

$$v_b=\prod_{t=1}^{s}(L_1-Q_t)\dots(L_{n-b}-Q_t)\cdot T_{w_b}\cdot\prod_{t=s+1}^{r}(L_1-Q_t)\dots(L_b-Q_t)$$

and $w_b = (n, \dots, 2, 1)^b$. It turns out that the standard basis of $\mathscr{H}$ can be adapted to give a 'standard' basis of V^b. With this basis in hand one sees that V^b is a projective $\mathscr{H}$-module, that

$$\operatorname{End}_{\mathscr{H}}(V^b) \cong \mathscr{H}_{q,\mathbf{Q}_1}(W_{s,b}) \otimes \mathscr{H}_{q,\mathbf{Q}_2}(W_{r-s,n-b})$$

and $\operatorname{Hom}_{\mathscr{H}}(V^b, V^c) = 0$ for $b \neq c$. These results imply that $\bigoplus_{b=0}^{n} V^b$ is a projective generator for $\mathscr{H}$ which gives the result. The Morita equivalence can be described very explicitly; consequently, when R is a field it is easy to compare the dimensions of the simple modules under the equivalence.

3.3. Ariki's theorem

This section discusses a very deep result of Ariki [**4**] which gives a way to compute the decomposition numbers of the Ariki-Koike algebras $\mathscr{H}_{\mathbb{C},q,\mathbf{Q}}(W_{r,n})$ when $q \neq 1$ and $Q_s \neq 0$ for all s. Throughout we assume that R is a field (we won't restrict ourselves to characteristic zero until we have to). For convenience write $\mathscr{H}_n = \mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ and let $\mathscr{H}_n$-**mod** be the category of finite dimensional right $\mathscr{H}_n$-modules. We begin with some motivation.

If M is an $\mathscr{H}_n$-module let $\operatorname{Res} M$ be the restriction of M to $\mathscr{H}_{n-1}$. Then Res is an exact functor from $\mathscr{H}_n$-**mod** to $\mathscr{H}_{n-1}$-**mod**. Since $\mathscr{H}_n$ is free as an $\mathscr{H}_{n-1}$-module Res has a right adjoint; namely, the induction functor which sends a right $\mathscr{H}_{n-1}$-module N to $\operatorname{Ind} N = N \otimes_{\mathscr{H}_{n-1}} \mathscr{H}_n$.

If λ is a multipartition of $n-1$ and μ is a multipartition of n write $\lambda \longrightarrow \mu$ if the diagrams of λ and μ differ by only one node. From the definition of the Specht modules it is clear that the action of $\mathscr{H}_{n-1}$ on $\operatorname{Res} S^\mu$ is given by ignoring the node in the tableaux with label n. With only a small amount of work this implies the following result.

Proposition 3.16 (Ariki [**4**, Lemma 2.1]). *Suppose that μ is a multipartition of n. Then $\operatorname{Res} S^\mu$ has a filtration with composition factors isomorphic to the Specht modules S^λ, where λ runs over the multipartitions of $n-1$ such that $\lambda \longrightarrow \mu$.*

Let $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$ be the Grothendieck group of $\mathscr{H}_n$-**mod**. Thus, $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$ is the free abelian group generated by all isomorphism classes of finitely generated right $\mathscr{H}_n$-modules where the relations are given by short exact sequences. If M is a right $\mathscr{H}_n$-module let $[M]$ be the corresponding equivalence class in $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$. By Theorem 3.12 $\{\,[S^\mu] \mid D^\mu \neq 0\,\}$ and $\{\,[D^\mu] \mid D^\mu \neq 0\,\}$ are both bases of $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$ and the transition matrix between these bases is the decomposition matrix of $\mathscr{H}$.

The functors Res and Ind induce homomorphisms of Grothendieck groups which, by abuse of notation, we also denote by Res and Ind. Thus, $\operatorname{Res}: K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow K_0(\mathscr{H}_{n-1}\text{-}\mathbf{mod})$ and $\operatorname{Ind}: K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow K_0(\mathscr{H}_{n+1}\text{-}\mathbf{mod})$ are the maps given by $\operatorname{Res}[M] = [\operatorname{Res} M]$ and $\operatorname{Ind}[M] = [\operatorname{Ind} M]$. These homomorphisms are completely determined by their actions on the Specht modules and this is given by Proposition 3.16 and Frobenius reciprocity.

Corollary 3.17. *Suppose that λ is a multipartition of n. Then*

$$\operatorname{Res}[S^\lambda] = \sum_{\nu \longrightarrow \lambda} [S^\nu] \quad \textit{and} \quad \operatorname{Ind}[S^\lambda] = \sum_{\lambda \longrightarrow \mu} [S^\mu].$$

Let $c_n = L_1 + \cdots + L_n$; then c_n belongs to the centre of $\mathscr{H}_n$. If M is any $\mathscr{H}_n$-module let $M_\alpha = \{\, m \in M \mid (c_n - \alpha)^k m = 0 \text{ for } k \gg 0 \,\}$ be the corresponding generalized eigenspace for c_n acting on M, for $\alpha \in R$. Then M_α is an $\mathscr{H}_n$-module since $c_n \in Z(\mathscr{H}_n)$; so $M = \oplus_{\alpha\in R} M_\alpha$ as an $\mathscr{H}_n$-module.

Until further notice we assume that $q \neq 1$ and that $Q_s = q^{a_s}$ for some integers a_s, for $1 \leq s \leq r$. In particular, this implies that the eigenvalues of c_n are always linear combinations of powers of q. Let e be the multiplicative order of q; then $e \in \mathbb{N} \cup \{\infty\}$.

Now the Specht module S^λ is irreducible when $R = \mathbb{C}(q)$; therefore, it follows from Proposition 3.14, and a specialization argument, that c_n acts on the Specht module S^λ as multiplication by the scalar $c(\lambda) = \sum_{k=1}^n \operatorname{res}_{\mathfrak{t}^\lambda}(k)$. Therefore, $S^\lambda = (S^\lambda)_{c(\lambda)}$ is a single generalized eigenspace and, by the Corollary, $\operatorname{Res} S^\lambda = \oplus_{i\in\mathbb{Z}} (\operatorname{Res} S^\lambda)_{c(\lambda)-q^i}$ and $\operatorname{Ind} S^\lambda = \oplus_{i\in\mathbb{Z}} (\operatorname{Ind} S^\lambda)_{c(\lambda)+q^i}$. Therefore the eigenvalues of c_n on an arbitrary $\mathscr{H}_n$-module change by $\pm q^i$, for some $i \in \mathbb{Z}/e\mathbb{Z}$, under the functors Res and Ind respectively. Accordingly, we define new functors i-Res and i-Ind on $\mathscr{H}_n$-**mod** by

$$i\text{-}\operatorname{Res} M = \bigoplus_\alpha (\operatorname{Res} M_\alpha)_{\alpha - q^i} \quad \text{and} \quad i\text{-}\operatorname{Ind} M = \bigoplus_\alpha (\operatorname{Ind} M_\alpha)_{\alpha + q^i},$$

for $i = 0, 1, \dots, e-1$. Then $\operatorname{Res} = \sum_{i=0}^{e-1} i\text{-}\operatorname{Res}$ and $\operatorname{Ind} = \sum_{i=0}^{e-1} i\text{-}\operatorname{Ind}$. These functors also induce group homomorphisms $K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow K_0(\mathscr{H}_{n\pm1}\text{-}\mathbf{mod})$ and these maps are completely determined by their actions on the Specht modules.

Write $\lambda \xrightarrow{i} \mu$ if $\lambda \longrightarrow \mu$ and the node in $[\mu] \setminus [\lambda]$ has residue q^i. Then we have the following refinement of Corollary 3.17.

Corollary 3.18. *Suppose that $0 \leq i < e$ and let λ be a multipartition of n. Then*

$$i\text{-}\operatorname{Res}[S^\lambda] = \sum_{\nu \xrightarrow{i} \lambda} [S^\nu] \quad \text{and} \quad i\text{-}\operatorname{Ind}[S^\lambda] = \sum_{\lambda \xrightarrow{i} \mu} [S^\mu].$$

Let $\mathscr{H}_n$-**proj** be the category of finitely generated *projective* $\mathscr{H}_n$-modules and let $K_0(\mathscr{H}_n\text{-}\mathbf{proj})$ be its Grothendieck group. If P is a projective $\mathscr{H}_n$-module let $[\![P]\!]$ denote its image in $K_0(\mathscr{H}_n\text{-}\mathbf{proj})$. Observe that there is a natural non-degenerate paring

$$\langle\ ,\ \rangle : K_0(\mathscr{H}_n\text{-}\mathbf{proj}) \times K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow \mathbb{Z}$$

given by $\langle [\![P]\!], [M] \rangle = \dim_R \operatorname{Hom}_{\mathscr{H}_n}(P, M)$; hence, $K_0(\mathscr{H}_n\text{-}\mathbf{proj}) \cong K_0(\mathscr{H}_n\text{-}\mathbf{mod})^*$. Consequently, if P^μ is the projective cover of D^μ then $\{ [\![P^\mu]\!] \mid \mu \vdash n \text{ and } D^\mu \neq 0 \}$ is a basis of $K_0(\mathscr{H}_n\text{-}\mathbf{proj})$ and we have induced maps $i\text{-}\operatorname{Res}^*, i\text{-}\operatorname{Ind}^* : K_0(\mathscr{H}_n\text{-}\mathbf{proj}) \longrightarrow K_0(\mathscr{H}_{n\pm 1}\text{-}\mathbf{proj})$.

We are almost ready to state Ariki's theorem. Let $U(\widehat{\mathfrak{sl}}_e)$ be the Kac-Moody Lie algebra of type $A^{(1)}_{e-1}$. Thus, $U(\widehat{\mathfrak{sl}}_e)$ is the $\mathbb{C}$-algebra generated by d, e_i, f_i and h_i, for $0 \leq i < r$, subject to a well-known set of relations; see [**7**, **86**]. Let $\Lambda_0, \dots, \Lambda_{e-1}$ be the fundamental weights of $U(\widehat{\mathfrak{sl}}_e)$ and recall that for each dominant weight $\Lambda \in \sum_{i=0}^{e-1} \mathbb{N}\Lambda_i$ there is a unique integrable highest weight $U(\widehat{\mathfrak{sl}}_e)$-module $L(\Lambda)$ with highest weight Λ.

Theorem 3.19 (Ariki [**4**, **9**]). *Suppose that R is a field and fix $q, Q_1 = q^{a_1}, \dots, Q_r = q^{a_r}$ in R such that $q \neq 1$ is a primitive e^{th} root of unity and integers $a_1, \dots, a_r$ (with $0 \leq a_i < e$ if $e < \infty$). Finally, let $\Lambda = \sum_{i=0}^{e-1} a_i \Lambda_i$ and set $V_{q,\mathbf{Q}}(R) = \bigoplus_{n \geq 0} K_0(\mathscr{H}_{R,n}\text{-}\mathbf{proj}) \otimes_{\mathbb{Z}} \mathbb{C}$.*

(i) *$V_{q,\mathbf{Q}}(R)$ is an integrable $U(\widehat{\mathfrak{sl}}_e)$-module upon which the Chevalley generators e_i and f_i act as follows:*

$$e_i[\![M]\!] = i\text{-}\operatorname{Res}^*[\![M]\!] \quad \text{and} \quad f_i[\![M]\!] = i\text{-}\operatorname{Ind}^*[\![M]\!],$$

for all $[\![M]\!] \in V_{q,\mathbf{Q}}(R)$. Moreover, $V_{q,\mathbf{Q}}(R) \cong L(\Lambda)$ as a $U(\widehat{\mathfrak{sl}}_e)$-module.

(ii) *If R is a field of characteristic zero then the canonical basis of $V_{q,\mathbf{Q}}(R)$ coincides with the basis*

$$\{ [\![P^\mu]\!] \mid D^\mu \neq 0 \text{ for some } \mu \vdash n \geq 0 \}$$

given by the projective indecomposable $\mathscr{H}_n$-modules.

Some remarks are in order. First, the hard part of this theorem is the case where $R = \mathbb{C}$; this is proved in [**4**]. The result for an arbitrary field follows from the complex case by a modular reduction argument; see [**9**]. Next, by the canonical basis of $L(\Lambda)$ we mean the specialization at $v = 1$ of the Kashiwara-Lusztig canonical basis[1] of $L_v(\Lambda)$, the corresponding integrable highest weight representation of the quantum group $U_v(\widehat{\mathfrak{sl}}_e)$.

Theorem 3.19 is a very deep result which relies upon the topological K-theory of Kazhdan and Lusztig [**88**] and Ginzburg's equivariant K-theory [**31**]; these theories give different constructions of the standard modules of the affine Hecke algebras in characteristic zero. For details of the proof see Ariki's original paper [**4**] and also his forthcoming book [**7**]. Geck [**58**] has also written an excellent survey article on the modular representation theory of Hecke algebras; he includes a detailed account of Ariki's paper.

The special case of Theorem 3.19 with $r = 1$ proves the conjecture of Lascoux, Leclerc and Thibon [**94**] for computing the decomposition matrices of the Iwahori-Hecke algebras $\mathscr{H}_q(\mathfrak{S}_n)$ of the symmetric groups. The main point of [**94**] is that they gave an elementary combinatorial algorithm for computing the canonical basis of the integrable highest weight module $L_v(\Lambda_0)$ for $U_v(\widehat{\mathfrak{sl}}_e)$ — and hence the decomposition matrices of $\mathscr{H}_q(\mathfrak{S}_n)$. This and similar algorithms are described in [**7, 62, 94, 95, 103**]. In contrast to the difficulty of Theorem 3.19, these algorithms involve only basic linear algebra; they amount to computing certain parabolic affine Kazhdan-Lusztig polynomials of type A and evaluating them at 1. This is described explicitly in [**62, 95, 103**].

Uglov [**120**], extending the ideas of Leclerc and Thibon [**95**], has given an algorithm for computing the canonical basis of any integrable highest weight module for $U_v(\widehat{\mathfrak{sl}}_e)$; see also [**119**]. Hence, combining Theorem 3.19(ii) with Uglov's work and Theorem 3.15 we have the following.

Corollary 3.20. *Suppose that R is a field of characteristic zero and that $q \neq 1$ and $Q_s \neq 0$ for $1 \leq s \leq r$. Then the decomposition matrix of $\mathscr{H}_{R,q,\mathbf{Q}}(W_{r,n})$ is known.*

[1] Canonical bases of quantum groups were introduced independently by Lusztig [**97**] and Kashiwara [**87**]. Jantzen [**83**] has given an excellent treatment of this theory; unfortunately, he only considers quantum groups of finite type which is insufficient for our purposes. Ariki [**7**] gives a largely self-contained account of the canonical bases of $U_v(\widehat{\mathfrak{sl}}_e)$, which is exactly what we need. See also Lusztig's book [**98**].

In practice there is a bit of work to be done to use this result to compute the decomposition numbers of $\mathscr{H}$. First, Uglov's algorithm computes a canonical basis for a larger space which contains $L_v(\Lambda)$ as a submodule; this is less efficient than the LLT algorithm and its variants. Next, Uglov's indexing of the canonical basis of $L_v(\Lambda)$ is not compatible with Theorem 3.12(ii) and Theorem 3.24 below; a bijection between the different indexing sets for the irreducibles is given by the paths in the associated crystal graphs. Finally, the effect of the Morita equivalence of Theorem 3.15 on the decomposition numbers must be taken into account; this last step is straightforward and is described in [**44**].

3.4. The irreducible $\mathscr{H}$-modules

In principle, the simple $\mathscr{H}_n$-modules are completely determined by Theorem 3.12; that is, the simple $\mathscr{H}_n$-modules are precisely the non-zero modules D^μ for μ a multipartition of n. Unfortunately, it is non-trivial to determine when D^μ is zero and when it is non-zero.

We begin the classification of the simple modules of the Ariki-Koike algebras with the case $r = 1$; that is, when $\mathscr{H} = \mathscr{H}_q(\mathfrak{S}_n)$. Let e be the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$. A partition is e-restricted if $\mu_i - \mu_{i+1} < e$ for $i \geq 1$. (This is compatible with our previous definition of e: if $q \neq 1$ then e is the multiplicative order of q in R; otherwise, e is the characteristic of R.)

Theorem 3.21 (Dipper and James [**37**]). *Suppose that R is a field. Then the $\mathscr{H}_q(\mathfrak{S}_n)$–module D^μ is non–zero if and only if μ is e-restricted.*

Dipper and James actually showed that the simple $\mathscr{H}_q(\mathfrak{S}_n)$–modules are indexed by e-regular partitions (that is, a partition with no e non-zero parts being equal). Our statement is different from theirs because our Specht modules are isomorphic to the duals of the Dipper-James Specht modules [**107**].

Using the $\mathscr{L}$-module structure of the Specht modules it is straightforward to see that the $\mathscr{H}_q(\mathfrak{S}_n)$–module D^μ is non–zero whenever μ is e-restricted (recall that $\mathscr{L} = \langle L_1, \ldots, L_n \rangle$). The converse is harder and follows from showing that if μ is not e-restricted then $[e]_q! = \prod_{k=1}^{e}(1 + q + \cdots + q^{k-1})$ divides the Gram determinant of the Specht module defined over $\mathbb{Z}[q, q^{-1}]$. For the proof see [**37, 103, 107**].

Returning to the general case where $r \geq 1$, the next result follows easily from Theorem 3.21. The statement is misleading because two separate, but similar, arguments are needed. For the proof when $q = 1$ see Mathas [**102**]; for the case where $Q_s = 0$, for all s, see Ariki-Mathas [**9**].

Corollary 3.22 (Mathas [**102**], Ariki-Mathas [**9**]). *Suppose that R is a field and that either (i) $q = 1$, or (ii) $Q_1 = \cdots = Q_r = 0$. Let $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ be a multipartition of n. Then $D^\mu \neq 0$ if and only if the following two conditions are satisfied.*

(i) *$\mu^{(s)}$ is e-restricted for $1 \leq s \leq r$.*

(ii) *$\mu^{(s)} = (0)$ whenever $Q_s = Q_t$ for some $t > s$.*

In the case $Q_1 = \cdots = Q_r = 0$ the last result simplifies to saying that $D^\mu \neq 0$ if and only if $\mu = \big((0), \ldots, (0), \mu^{(r)}\big)$ for some e-restricted partition $\mu^{(r)}$.

It remains to treat the cases where $q \neq 1$ and $Q_s \neq 0$ for all s.

Given two nodes $x = (a, b, s)$ and $y = (c, d, t)$ we say that y is below x if either $s < t$, or $s = t$ and $a < c$. Further, $x \in [\lambda]$ is removable if $[\lambda] \setminus \{x\}$ is the diagram of a multipartition; similarly, $y \notin [\lambda]$ is addable if $[\lambda] \cup \{y\}$ is the diagram of a multipartition. If $i = \operatorname{res}(x)$ we call x an i-node.

An i-node x is normal if (i) whenever y is a removable i-node below x then there are more removable i-nodes between x and y than there are addable i-nodes, and (ii) there are at least as many removable i-nodes below x as addable i-nodes below x. In addition, a normal i-node x is good if there are no normal i-nodes above x. If $[\mu] = [\lambda] \cup \{x\}$ for some good node x we write $\lambda \xrightarrow{\text{good}} \mu$.

Definition 3.23. *A multipartition μ is* Kleshchev *if either $\mu = \big((0), , \ldots, (0)\big)$ or $\lambda \xrightarrow{\text{good}} \mu$ for some Kleshchev multipartition λ.*

The origin of the definition of the Kleshchev multipartitions is that they are the vertices of the crystal graph of an integrable $U_v(\widehat{\mathfrak{sl}}_e)$-module. (When $Q_s = q^{a_s}$, for all s, then the Kleshchev multipartitions are the vertices of the crystal graph of $L_v(\Lambda)$, where $\Lambda = \sum_{s=1}^r \Lambda_{a_s}$. In general, we take a direct sum of tensor products of crystal graphs in accordance with Theorem 3.15.) There is an edge in the crystal graph between two Kleshchev multipartitions if $\lambda \xrightarrow{\text{good}} \mu$; the label of the edge is the residue of the node in $[\mu] \setminus [\lambda]$. For more details see [**9**, **85**].

When $r = 1$ a partition μ is Kleshchev if and only if μ is e-restricted; consequently, as it must, the next result agrees with Theorem 3.21 when $r = 1$.

Theorem 3.24 (Ariki [**6**]). *Suppose that R is a field, $q \neq 1$, $Q_s \neq 0$ for $1 \leq s \leq r$, and that μ is a multipartition of n. Then $D^\mu \neq 0$ if and only if μ is a Kleshchev multipartition.*

The first step towards Theorem 3.24 is to observe that Theorem 3.15 allows us to reduce to the crucial case where $q \neq 1$ and $Q_s = q^{a_s}$ for some integers a_s (a different argument is given in [**9**]). Using Theorem 3.19(ii), Ariki [**6**] is able to complete the classification of the irreducible $\mathscr{H}$-modules over $\mathbb{C}$. To complete the argument, [**9**] shows that the number of simple modules depends only on the integers a_s and the multiplicative order of q in R.

Finally, we remark that by combining these techniques with results of Ginzburg [**31**], Ariki and the author [**9**] classified the simple modules of the affine Hecke algebras over an algebraically closed field of positive characteristic; again, the hard work is done by Ariki's paper [**4**]. When $R = \mathbb{C}$ and q is not a root of unity the simple $\hat{H}_n$-modules were classified by Zelevinsky [**124**]; see also [**88, 111**]. When $q \in \mathbb{C}^\times$ is a root of unity the simple $\hat{H}_n$-modules were classified by Lusztig and Ginzburg; see [**4, 31**].

3.5. The modular branching rules

One of the most significant results in modular representation theory from the nineties is Kleshchev's modular branching rule for the symmetric groups [**89–92**]. Using a streamlined version of the same techniques Brundan [**26**] extended these results to the Iwahori-Hecke algebra of the symmetric group. Using completely different methods, Grojnowski [**71**] and Grojnowski-Vazirani [**73**] generalized Kleshchev's modular branching rules to the Ariki-Koike algebras and the affine Hecke algebra of type A. (Brundan and Kleshchev [**28**] have also applied Grojnowski's methods to the projective representations of the symmetric groups.)

Grojnowski was mainly interested in representations of the affine Hecke algebra $\hat{H}_n$; however, as remarked in §2.4 every irreducible representation of the affine Hecke algebra is an irreducible representation for a family of Ariki-Koike algebras. He studies the functors given by induction and restriction (from $\hat{H}_n$ to $\hat{H}_{n\pm1}$), followed by the taking of socles by analyzing the effect of these functors on the central characters of $\hat{H}_n$. Grojnowski shows that these functors can be described in terms of the crystal graphs of integral highest weight modules for the quantum group $U_v(\widehat{\mathfrak{sl}}_e)$; cf. Theorem 3.19(i).

Theorem 3.25 (Grojnowski [**71**], Grojnowski-Vazirani [**73**]). *Suppose that R is a field, $q \neq 1$ and $Q_s \neq 0$, for $1 \leq s \leq r$. Then, for each m, there is an (unknown) permutation π_m of the set Kleshchev multipartitions of m such that if μ is a Kleshchev multipartition of n then*

$$\operatorname{Soc}(\operatorname{Res}(D^\mu)) \cong \bigoplus_{\pi_{n-1}(\lambda) \xrightarrow{\text{good}} \pi_n(\mu)} D^\lambda.$$

In [**73**] Grojnowski-Vazirani prove that $\mathrm{Soc}(\mathrm{Res}(D^\mu))$ is multiplicity free. In [**71**] Grojnowski shows that there exists a set of irreducible $\mathscr{H}$–modules which are indexed by the Kleshchev multipartitions and for which the modular branching rule is given by removing good nodes; Grojnowski does not give an explicit construction of these modules. Conjecturally, π_m is trivial for all m.

Notice that Theorem 3.25 implies that there are at most e direct summands of $\mathrm{Soc}(\mathrm{Res}(D^\mu))$ and that they all belong to different blocks.

As Grojnowski remarks, the assumption that $q \neq 1$ is not essential and can be removed (at the expense of some additional notation). Du and Rui [**55**] also obtained the modular branching rule in the special case where $q^d Q_s \neq Q_t$, for $1 \leq s < t \leq r$ and $|d| < n$. In fact, in their case they obtain the stronger result that $\pi_m = 1$, for all m. By Theorem 3.15 such Ariki-Koike algebras are Morita equivalent to direct sums of tensor products of Iwahori-Hecke algebras $\mathscr{H}_q(\mathfrak{S}_m)$; Du and Rui use this to deduce the result from Brundan's theorem [**26**] for $\mathscr{H}_q(\mathfrak{S}_m)$.

Grojnowski [**71**] shows that the number of irreducible $\mathscr{H}_n$–modules is equal to the number of Kleshchev multipartitions of n; this gives a more elementary proof of part of Theorem 3.24. Grojnowski also counts the number of irreducible modules of the affine Hecke algebra $\hat{H}_n$ over an arbitrary algebraically closed field. In §5.2 below we discuss the application of Theorem 3.25 to classifying the blocks of $\mathscr{H}$.

§4. The cyclotomic q-Schur algebra

This chapter introduces the cyclotomic q-Schur algebras. These algebras are defined as endomorphism algebras

$$\mathscr{S}(\Lambda) = \mathrm{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big),$$

where Λ is a finite set of multicompositions and M^μ is a certain $\mathscr{H}$-module. In the special case where $r = 1$ the cyclotomic q-Schur algebras are the q-Schur algebras of Dipper and James [**39**]; see [**27**,**46**,**65**,**103**]. This was one of the motivations for introducing the cyclotomic q-Schur algebras.

Prior to [**42**] several authors [**41**,**51**,**56**,**59**,**69**,**74**] had studied Schur algebras of type B; these algebras are either subalgebras or special cases of the cyclotomic q–Schur algebras. See [**57**, **70**] for Schur algebras of other types.

4.1. Permutation modules.

We begin by describing the $\mathscr{H}$-modules M^μ.

A composition of n is a sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ of non-negative integers σ_i such that $|\sigma| = \sum_{i\geq 1}\sigma_i = n$; we will sometimes write $\sigma = (\sigma_1, \dots, \sigma_k)$ if $\sigma_i = 0$ for $i > k$. A multicomposition of n is an ordered r-tuple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of compositions with $|\mu^{(1)}| + \dots + |\mu^{(r)}| = n$.

Definition 4.1. *Suppose that μ is a multicomposition of n. Then M^μ is the right ideal $M^\mu = m_\mu \mathscr{H}$ of $\mathscr{H}$ (where $m_\mu = x_\mu u_\mu^+$ as before)*

Given a multicomposition μ let $\vec{\mu} = (\vec{\mu}^{(1)}, \dots, \vec{\mu}^{(r)})$ be the multipartition where $\vec{\mu}^{(s)}$ is the partition obtained by ordering the parts of the composition $\mu^{(s)}$. It is not hard to see that $M^{\vec{\mu}} \cong M^\mu$; indeed, if $d \in \mathfrak{S}_n$ is a right coset representative of $\mathfrak{S}_\mu$ of minimal length such that $\mathfrak{S}_{\vec{\mu}} = d^{-1}\mathfrak{S}_\mu d$ then $T_d x_{\vec{\mu}} = x_\mu T_d$; hence, $T_d m_{\vec{\mu}} = m_\mu T_d$ and an isomorphism $M^{\vec{\mu}} \cong M^\mu$ is given by $h \longmapsto T_d h$, for $h \in M^{\vec{\mu}}$.

When $r = 1$ the module M^μ is the induced trivial representation of the parabolic subalgebra

$$\mathscr{H}_q(\mathfrak{S}_\mu) = \langle T_i \mid t_i \in \mathfrak{S}_\mu \rangle = \sum_{w\in\mathfrak{S}_\mu} RT_w.$$

More precisely, let $\mathbf{1}_\mu$ be the trivial representation of the subalgebra $\mathscr{H}_q(\mathfrak{S}_\mu)$; so $\mathbf{1}_\mu$ is a free R-module of rank 1 on which T_w acts as multiplication by $q^{\ell(w)}$ for all $w \in \mathfrak{S}_\mu$. Then

$$M^\mu \cong \mathbf{1}_\mu \otimes_{\mathscr{H}_q(\mathfrak{S}_\mu)} \mathscr{H}_q(\mathfrak{S}_n).$$

(Note that $\mathscr{H}_q(\mathfrak{S}_n)$ is free as a right $\mathscr{H}_q(\mathfrak{S}_\mu)$-module.)

If $r > 1$ then, in general, the modules M^μ are not obviously induced from subalgebras (except in the case considered by Shoji [**117**]). Even so, the M^μ behave very much like permutation modules, so it is not a bad idea to think of them as such.

In order to describe a basis of M^μ we need to introduce some more notation. Let $\underline{\mathbf{nr}} = \{\, (i,s) \mid i \geq 1 \text{ and } 1 \leq s \leq r \,\}$. If $(i,s), (j,t)$ are elements of $\underline{\mathbf{nr}}$ write $(i,s) \preceq (j,t)$ if either $s < t$, or $s = t$ and $i \leq j$.

Let μ be a multicomposition. Then a λ-tableau of type μ is a map $\mathsf{T}\colon [\lambda] \longrightarrow \underline{\mathbf{nr}}$ such that $\mu_i^{(s)} = \#\{\, x \in [\lambda] \mid \mathsf{T}(x) = (i,s) \,\}$, for $1 \leq s \leq r$ and all $i \geq 1$; we write $\mathrm{Type}(\mathsf{T}) = \mu$. Again, we will think of a tableau of type μ as being an r-tuple of tableaux. For example, two tableaux of type $\big((3,1), (1^2), (2,1)\big)$ are

$$\left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 1_1 \\ \hline 2_1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1_2 \\ \hline 2_2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1_3 & 1_3 \\ \hline 2_3 \\ \cline{1-1}\end{array}\right) \text{ and } \left(\begin{array}{|c|c|c|c|c|}\hline 1_1 & 1_1 & 1_1 & 2_1 & 1_3 \\ \hline 1_2 & 2_3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|}\hline 2_2 & 1_3 \\ \hline\end{array}, \begin{array}{|c|}\hline 2_3 \\ \hline\end{array}\right),$$

where we write i_s instead of the ordered pair (i,s).

If $\mathfrak{s}$ is a standard λ-tableau let $\mu(\mathfrak{s})$ be the tableau of type μ obtained by replacing each entry k in $\mathfrak{s}$ by (i,s) if k appears in row i of component s of $\mathfrak{t}^\mu$ — as for multipartitions, we define $\mathfrak{t}^\mu$ to be the μ-tableau with the integers $1,\dots,n$ entered from left to right and then top to bottom along the rows of the components of $[\mu]$.

Definition 4.2. *Let λ be a multipartition and μ a multicomposition. A* semistandard *λ-tableau is a λ-tableau $\mathsf{T}=(\mathsf{T}^{(1)},\dots,\mathsf{T}^{(r)})$ such that*

(i) *the entries in each row of T are non-decreasing in each component (when ordered by $\preceq$); and,*
(ii) *the entries in each column of T are strictly increasing in each component; and,*
(iii) *if $(a,b,c)\in[\lambda]$ and $\mathsf{T}(a,b,c)=(i,s)$ then $s\ge c$.*

Let $\mathcal{T}^{ss}_\mu(\lambda)$ be the set of semistandard λ-tableaux of type μ and let $\mathcal{T}^{ss}_\Lambda(\lambda)=\bigcup_{\mu\in\Lambda}\mathcal{T}^{ss}_\mu(\lambda)$.

When $r=1$ condition (iii) is redundant and Definition 4.2 becomes the familiar definition of semistandard tableaux from the representation theory of the general linear and symmetric groups.

Write $\operatorname{comp}_\mathfrak{t}(k)=s$ if k appears in component s of $\mathfrak{t}$. For $r>1$ condition (iii) is unexpected; it has its origin in the fact [**42**, Prop. 3.23] that if $h\in M^\mu$ and $h=\sum_{\mathfrak{s},\mathfrak{t}}r_{\mathfrak{s}\mathfrak{t}}m_{\mathfrak{s}\mathfrak{t}}$ for some $r_{\mathfrak{s}\mathfrak{t}}\in R$ then $\operatorname{comp}_\mathfrak{s}(k)\le\operatorname{comp}_{\mathfrak{t}^\mu}(k)$, for $k=1,\dots,n$. Observe that $\mu(\mathfrak{s})$ satisfies condition (iii) if and only if $\operatorname{comp}_\mathfrak{s}(k)\le\operatorname{comp}_{\mathfrak{t}^\mu}(k)$ for all k.

For example, if λ is a multipartition then $\mathsf{T}^\lambda=\lambda(\mathfrak{t}^\lambda)$ is the unique semistandard λ-tableau of type λ. The first of the two tableaux in the example above is T^λ for $\lambda=\big((3,1),(1^2),(2,1)\big)$; the second tableau there is also semistandard. Finally, let $\omega=\big((0),\dots,(0),(1^n)\big)$. Then it is easy to see that the map

$$\omega:\mathcal{T}^{s}(\lambda)\xrightarrow{\sim}\mathcal{T}^{ss}_\omega(\lambda);\ \mathfrak{s}\longmapsto\omega(\mathfrak{s})\tag{4.3}$$

is a bijection between the set of standard λ-tableaux and the set of semistandard λ-tableaux of type ω. Hereafter, we identity $\mathcal{T}^s(\lambda)$ and $\mathcal{T}^{ss}_\omega(\lambda)$ via (4.3).

Definition 4.4. *Suppose that S is a semistandard λ-tableau of type μ and that $\mathfrak{t}$ is a standard λ-tableau. Define*

$$m_{\mathsf{S}\mathfrak{t}}=\sum_{\substack{\mathfrak{s}\in\mathcal{T}^s(\lambda)\\ \mathsf{S}=\mu(\mathfrak{s})}}m_{\mathfrak{s}\mathfrak{t}}.$$

The point of all of this notation is the following useful theorem.

Theorem 4.5. *Suppose that μ is a multicomposition of n. Then M^μ is free as an R-module with basis*

$$\left\{\, m_{\mathsf{St}} \;\middle|\; \begin{array}{c} \mathsf{S} \in \mathcal{T}^{ss}_\mu(\lambda) \text{ and } \mathfrak{t} \in \mathcal{T}^{s}(\lambda) \text{ for} \\ \text{some multipartition } \lambda \text{ of } n \end{array} \right\}.$$

When $r = 1$ this result was first proved by Murphy [**107**]; the general case can be found in Dipper-James-Mathas [**42**].

The proof of this result is straightforward. A small calculation shows that m_{St} is an element of M^μ. Next, the elements in the statement of Theorem 4.5 are linearly independent by Theorem 3.8. Finally, if $h \in M^\mu$ then h can be written as a linear combination of standard basis elements; in turn, these are a linear combination of the m_{St}.

The importance of Theorem 4.5 stems from the following applications.

Corollary 4.6. *Suppose that μ is a multicomposition of n. Then there exists a filtration $M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0$ of M^μ such that*

(i) *$M_i/M_{i+1} \cong S^{\lambda_i}$ for some multipartition λ_i for $i = 1, \dots, k$; and,*
(ii) *for each multipartition λ the number of i with $\lambda = \lambda_i$ is equal to the number of semistandard λ-tableaux of type μ.*

Sketch of proof. Fixing S and varying $\mathfrak{t}$ in the basis $\{m_{\mathsf{St}}\}$ of M^μ gives a Specht module modulo higher terms. □

For each semistandard λ-tableau S of type μ and each semistandard λ-tableau T of type ν define

$$m_{\mathsf{ST}} = \sum_{\substack{\mathfrak{t} \in \mathcal{T}^s(\lambda) \\ \mathsf{T} = \nu(\mathfrak{t})}} m_{\mathsf{St}}.$$

By definition, $m_{\mathsf{ST}} = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{st}}$ where the sum is over the standard λ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$ such that $\mu(\mathfrak{s}) = \mathsf{S}$ and $\nu(\mathfrak{t}) = \mathsf{T}$.

Corollary 4.7. *Suppose that μ and ν are multicompositions of n. Then*

$$\left\{\, m_{\mathsf{ST}} \;\middle|\; \begin{array}{c} \mathsf{S} \in \mathcal{T}^{ss}_\mu(\lambda) \text{ and } \mathsf{T} \in \mathcal{T}^{ss}_\nu(\lambda) \text{ for} \\ \text{some multipartition } \lambda \text{ of } n \end{array} \right\}$$

is a basis of $\mathscr{H} m_\nu \cap m_\mu \mathscr{H}$.

We are now ready to tackle the cyclotomic q-Schur algebras.

4.2. The semistandard basis theorem

We give a slightly more general definition for the cyclotomic q-Schur algebras than appeared in [**42**] in that we allow the set Λ to be an arbitrary finite set of multicompositions. We invite the reader to check that the arguments from [**42**] go through without change.

Extend the dominance ordering $\trianglerighteq$ to the set of all multicompositions; by restriction we consider any set of multicompositions as a poset.

Definition 4.8. *Suppose that Λ is a finite set of multicompositions of n. The* cyclotomic q-Schur algebra *is the endomorphism algebra*

$$\mathscr{S}(\Lambda) = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big).$$

Let $\Lambda^+ = \{\,\lambda \vdash n \mid \lambda \trianglerighteq \mu \text{ for some } \mu \in \Lambda\,\}$.

We should really write $\mathscr{S}(\Lambda) = \mathscr{S}_{R,q,\mathbf{Q}}(\Lambda)$ since $\mathscr{S}(\Lambda)$ depends on Λ, R, q and $\mathbf{Q}$.

Part of the original definition of the cyclotomic q-Schur algebras in [**42**] was the requirement that $\Lambda^+ \subseteq \Lambda$. Following Donkin [**46**], we say that Λ is saturated if $\Lambda^+ \subseteq \Lambda$. In analogy with representations of Lie groups, Λ^+ should be thought of as the set of dominant weights and Λ the set of weights. Note that Λ^+ is not necessarily a subset of Λ.

Let $\Lambda = \Lambda(d;n)$ be the set of all compositions $\mu = (\mu_1,\dots,\mu_d)$ of n of length at most d (so $\mu_i = 0$ whenever $i > d$). Then $\mathscr{S}_q(d;n) = \mathscr{S}\big(\Lambda(d;n)\big)$ is a q-Schur algebra in the sense of Dipper and James [**39**].

As an R-module we see that

$$\mathscr{S}(\Lambda) = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big) = \bigoplus_{\mu,\nu\in\Lambda} \operatorname{Hom}_{\mathscr{H}}\Big(M^\nu, M^\mu\Big);$$

so we need to understand the R-modules $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$.

Proposition 4.9. *Suppose that μ and ν are multicompositions of n. Then an R-linear map $\varphi\colon M^\nu \longrightarrow M^\mu$ belongs to $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ if and only if*

$$\varphi(m_\nu) = \sum_{\substack{\mathsf{S}\in\mathcal{T}^{ss}_\mu(\lambda)\\ \mathsf{T}\in\mathcal{T}^{ss}_\nu(\lambda)}} r_{\mathsf{ST}} m_{\mathsf{ST}}$$

for some $r_{\mathsf{ST}} \in R$.

Sketch of proof. If $Q_1,\dots,Q_r$ are invertible elements of R then $\mathscr{H}$ is a symmetric algebra by Proposition 3.5(ii); therefore, $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ and $m_\mu\mathscr{H} \cap \mathscr{H} m_\nu$ are canonically isomorphic R-modules (via the map $\varphi \longmapsto \varphi(m_\nu)$), so the proposition follows by Corollary 4.7.

For the general case, an intricate induction (see [**42**, §5]), which is independent of Proposition 3.5, shows that the double annihilator of m_μ,

$$\{\, x \in \mathscr{H} \mid xh = 0 \text{ whenever } m_\mu h = 0 \text{ for some } h \in \mathscr{H} \,\},$$

is $\mathscr{H} m_\mu$. Hence, it again follows that $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu) \cong m_\nu \mathscr{H} \cap \mathscr{H} m_\mu$, so we can complete the proof using the argument of the last paragraph. □

Definition 4.10. *Suppose that $\lambda \in \Lambda^+$ is a multipartition and that $\mu, \nu \in \Lambda$ are multicompositions. For each pair of standard λ-tableaux $\mathsf{S} \in \mathcal{T}^{ss}_\mu(\lambda)$ and $\mathsf{T} \in \mathcal{T}^{ss}_\nu(\lambda)$ let φ_{ST} be the R-linear endomorphism of $\bigoplus_{\mu\in\Lambda} M^\mu$ determined by*

$$\varphi_{\mathsf{ST}}(m_\alpha h) = \delta_{\alpha\nu} m_{\mathsf{ST}} h,$$

for all $\alpha \in \Lambda$ and $h \in \mathscr{H}$ (here $\delta_{\alpha\nu}$ is the Kronecker delta).

By Proposition 4.9 φ_{ST} is an element of $\mathscr{S}(\Lambda)$.

Let $\mathcal{S}^\lambda$ be the R-submodule of $\mathscr{S}(\Lambda)$ spanned by the φ_{UV}, for some $\mathsf{U}, \mathsf{V} \in \mathcal{T}^{ss}_\Lambda(\rho)$ where $\rho \in \Lambda^+$ and $\rho \rhd \lambda$. From the definitions, $\mathcal{S}^\lambda$ consists of those elements of $\mathscr{S}(\Lambda)$ whose image is contained in $\mathcal{H}^\lambda$.

Observe that a map $\varphi \in \operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ is completely determined by $\varphi(m_\nu)$ since $\varphi(m_\nu h) = \varphi(m_\nu)h$ for all $h \in \mathscr{H}$. Therefore, we can lift the involution $*$ of $\mathscr{H}$ to give an involutory anti-isomorphism of $\mathscr{S}(\Lambda)$ by defining $\varphi^* \in \operatorname{Hom}_{\mathscr{H}}(M^\mu, M^\nu)$ by $\varphi^*(m_\mu h) = \big(\varphi(m_\nu)\big)^* h$ for all $h \in \mathscr{H}$. In particular, note that $\varphi^*_{\mathsf{ST}} = \varphi_{\mathsf{TS}}$.

We can now state the semistandard basis theorem for the cyclotomic Schur algebras.

Theorem 4.11 (Dipper-James-Mathas [**42**, Theorem 6.6]). *Let Λ be a finite set of multicompositions. Then the cyclotomic q-Schur algebra $\mathscr{S}(\Lambda)$ is free as an R-module with basis*

$$\{\, \varphi_{\mathsf{ST}} \mid \textit{for some } \mathsf{S}, \mathsf{T} \in \mathcal{T}^{ss}_\Lambda(\lambda) \textit{ and } \lambda \in \Lambda^+ \,\}.$$

Moreover, this basis is a cellular basis of $\mathscr{S}(\Lambda)$; more precisely, if S *and* T *are semistandard λ-tableaux, for some $\lambda \in \Lambda^+$, then*

(i) $\varphi^*_{\mathsf{ST}} = \varphi_{\mathsf{TS}}$*; and,*

(ii) *for all $\varphi \in \mathscr{S}(\Lambda)$ there exist scalars $r_{\mathsf{V}} = r_{\mathsf{TV}}(\varphi) \in R$, which do not depend on* S*, such that*

$$\varphi_{\mathsf{ST}}\varphi \equiv \sum_{\mathsf{V}\in\mathcal{T}^{ss}_\Lambda(\lambda)} r_{\mathsf{V}} \varphi_{\mathsf{SV}} \pmod{\mathcal{S}^\lambda}.$$

Sketch of proof. Proposition 4.9 implies that these elements give a basis of $\mathscr{S}(\Lambda)$. Using Theorem 3.8 it is not hard to see that the semistandard basis is cellular. □

In particular, notice that $\mathscr{S}(\Lambda)$ is always free as an R-module and that its rank is independent of R, q and $\mathbf{Q}$. The semistandard basis of $\mathscr{S}(\Lambda)$ really comes from Theorem 4.5 and the basis element φ_{ST} really comes from a Specht filtration of M^μ.

It is worthwhile explaining how the multiplication in $\mathscr{S}(\Lambda)$ is determined. Suppose that $\mathsf{S}, \mathsf{T}, \mathsf{U}$ and V are semistandard tableaux and suppose that $\nu = \mathrm{Type}(\mathsf{V})$ and $\mu = \mathrm{Type}(\mathsf{U})$. Then $m_{\mathsf{UV}} = m_\mu h_{\mathsf{UV}}$, for some $h_{\mathsf{UV}} \in \mathscr{H}$, and

$$\varphi_{\mathsf{ST}}\varphi_{\mathsf{UV}} = \sum_{\mathsf{A},\mathsf{B}} r_{\mathsf{AB}}\varphi_{\mathsf{AB}},$$

where the scalars $r_{\mathsf{AB}} \in R$ are determined by $m_{\mathsf{ST}}h_{\mathsf{UV}} = \sum r_{\mathsf{AB}}m_{\mathsf{AB}}$; this makes sense by Proposition 4.7 and is proved by evaluating the functions on both sides at m_ν. Note, in particular, that $r_{\mathsf{AB}} = 0$ unless $\mathrm{Type}(\mathsf{U}) = \mathrm{Type}(\mathsf{T})$, $\mathrm{Type}(\mathsf{A}) = \mathrm{Type}(\mathsf{S})$ and $\mathrm{Type}(\mathsf{B}) = \mathrm{Type}(\mathsf{V})$. In Theorem 4.11(ii), $r_{\mathsf{V}} = r_{\mathsf{SV}}$.

With some work it is possible to show that when $r = 1$ this basis agrees with Richard Green's codeterminant basis of the q-Schur algebra [**68**]; see also [**67, 123**]. When $r = 2$ and $\mathscr{H}$ is symmetric Theorem 4.11 is equivalent to a theorem of Du and Scott [**56**].

4.3. Weyl modules for cyclotomic q-Schur algebras

By the semistandard basis theorem $\mathscr{S}(\Lambda)$ is a cellular algebra. Therefore, exactly as in Definition 3.10 we can write down a collection of *cell modules* for $\mathscr{S}(\Lambda)$ and, up to isomorphism, every irreducible $\mathscr{S}(\Lambda)$-module is a quotient of one of these modules.

Definition 4.12. *Suppose that $\lambda \in \Lambda^+$ is a multipartition. The* Weyl module W^λ *is the free R-module with basis $\{\, \varphi_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{T}_\Lambda^{ss}(\lambda)\,\}$ on which $\varphi \in \mathscr{S}(\Lambda)$ acts via*

$$\varphi_{\mathsf{T}}\varphi = \sum_{\mathsf{V} \in \mathcal{T}_\Lambda^{ss}(\lambda)} r_{\mathsf{V}}\varphi_{\mathsf{V}},$$

where the scalars $r_{\mathsf{V}} \in R$ are as in Theorem 4.11(ii).

It follows from Theorem 4.11 that W^λ is a right $\mathscr{S}(\Lambda)$-module. As with the Specht modules we define a bilinear form on W^λ by requiring that $\langle \varphi_{\mathsf{S}}, \varphi_{\mathsf{T}}\rangle \varphi_{\mathsf{UV}} \equiv \varphi_{\mathsf{US}}\varphi_{\mathsf{TV}} \pmod{\mathcal{S}^\lambda}$ for semistandard tableaux

$\mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V} \in \mathcal{T}^{ss}_{\Lambda}(\lambda)$. Then the radical of this form, $\operatorname{Rad} W^\lambda$, is a submodule of W^λ and we define $L^\lambda = W^\lambda / \operatorname{Rad} W^\lambda$.

Exactly as in Theorem 3.12, the theory of cellular algebras now gives us the following.

Theorem 4.13. *Suppose that R is a field.*

(i) *For each $\lambda \in \Lambda^+$, L^λ is either zero or an absolutely irreducible $\mathscr{S}(\Lambda)$-module.*
(ii) *$\{ L^\lambda \mid \lambda \in \Lambda^+ \text{ and } L^\lambda \neq 0 \}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{S}(\Lambda)$-modules.*
(iii) *$\mathscr{S}(\Lambda)$ is semisimple if and only if $L^\lambda = W^\lambda$ for all $\lambda \in \Lambda^+$.*
(iv) *Suppose that $\mu, \lambda \in \Lambda^+$ and $L^\lambda \neq 0$. Then $[W^\mu{:}L^\lambda] \neq 0$ only if $\mu \trianglerighteq \lambda$; moreover, $[W^\lambda{:}L^\lambda] = 1$.*

At this level of generality, determining exactly when L^λ is non-zero is a difficult task. To see this notice that if $\Lambda = \{\omega\}$ then $\mathscr{S}(\Lambda) = \operatorname{End}_{\mathscr{H}}(\mathscr{H}) \cong \mathscr{H}$ and Λ^+ is the set of all partitions of n; so Theorem 3.24 is a special case of Theorem 4.13. When the poset Λ is saturated (that is, $\Lambda^+ \subseteq \Lambda$) we can say much more.

Assume now that $\Lambda^+ \subseteq \Lambda$ and let λ be a multipartition of n. Then M^λ is a summand of $\oplus_{\mu\in\Lambda} M^\mu$ and so the identity map $\varphi_\lambda : M^\lambda \longrightarrow M^\lambda$ is an element of $\mathscr{S}(\Lambda)$. Indeed, looking at the definitions, $\varphi_\lambda = \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda}$, where $\mathsf{T}^\lambda = \lambda(\mathsf{t}^\lambda)$ is the unique semistandard λ-tableau of type λ. It follows that the Weyl module W^λ is isomorphic to the submodule of $\mathscr{S}(\Lambda)/\mathcal{S}^\lambda$ generated by $\varphi_\lambda + \mathcal{S}^\lambda$, the isomorphism being given by

$$\varphi_{\mathsf{T}} \longmapsto \varphi_{\mathsf{T}^\lambda\mathsf{T}} + \mathcal{S}^\lambda = (\varphi_\lambda + \mathcal{S}^\lambda)\varphi_{\mathsf{T}^\lambda\mathsf{T}},$$

for all $\mathsf{T} \in \mathcal{T}^{ss}_{\Lambda}(\lambda)$.

Theorem 4.14. *Suppose that R is a field and that $\Lambda^+ \subseteq \Lambda$.*

(i) *L^λ is a non-zero absolutely irreducible $\mathscr{S}(\Lambda)$-module for all $\lambda \in \Lambda^+$.*
(ii) *$\mathscr{S}(\Lambda)$ is a quasi-hereditary algebra.*

Sketch of proof. To prove (i) observe that $\varphi_\lambda = \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda}$ is an element of $\mathscr{S}(\Lambda)$ because $\lambda \in \Lambda$. Therefore, $\varphi_{\mathsf{T}^\lambda} \in W^\lambda$ and so

$$\langle \varphi_{\mathsf{T}^\lambda}, \varphi_{\mathsf{T}^\lambda} \rangle \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda} \equiv \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda}\varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda} = \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda} \pmod{\mathcal{S}^\lambda};$$

hence, $\langle \varphi_{\mathsf{T}^\lambda}, \varphi_{\mathsf{T}^\lambda} \rangle = 1$ and $\varphi_{\mathsf{T}^\lambda} \notin \operatorname{Rad} W^\lambda$; so $L^\lambda \neq 0$. Part (ii) follows from (i) and the structure of cellular algebras. □

Parshall and Wang [**109**] were the first to show that the q-Schur algebras are quasi-hereditary. More generally, the argument above shows that the q-Schur algebras and the cyclotomic Schur algebras are integrally quasi-hereditary in the sense of [**50**].

As the example $\Lambda = \{\omega\}$ indicates, when Λ is not saturated the classification of the simple $\mathscr{S}(\Lambda)$-modules is non-trivial. Nor are there obvious necessary and sufficient conditions for when $\mathscr{S}(\Lambda)$ is quasi-hereditary. The answers to these questions will depend on Λ, R and the parameters $q, Q_1, \dots, Q_r$.

The final result of this section is the analogue of Theorem 3.15 for the cyclotomic Schur algebras. In [**44**] a general version of the result below is proved for an arbitrary finite set of (saturated) multicompositions; we state only a special case in order to avoid introducing additional notation.

Let $\Lambda_{r,n}$ be the set of all multicompositions of n of length at most n and let $\Lambda_{r,n}^+ \subseteq \Lambda_{r,n}$ be the set of multipartitions of n. We write $\mathscr{S}(\Lambda_{r,n}) = \mathscr{S}_{q,\mathbf{Q}}(\Lambda_{r,n})$ to emphasize the choice of parameters.

Theorem 4.15 (Dipper-Mathas [**44**]). *Suppose that R is an integral domain and let $\mathbf{Q} = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_\kappa$ be a partition of $\mathbf{Q}$ and suppose that the polynomial $P_n(q, \mathbf{Q}_1, \dots, \mathbf{Q}_\kappa)$ is invertible in R. Let $r_\alpha = |\mathbf{Q}_\alpha|$, for $1 \le \alpha \le \kappa$. Then $\mathscr{S}(\Lambda_{r,n})$ is Morita equivalent to the R-algebra*

$$\bigoplus_{\substack{n_1,\dots,n_\kappa \ge 0\\ n_1+\dots+n_\kappa = n}} \mathscr{S}_{q,\mathbf{Q}_1}(\Lambda_{r_1,n_1}) \otimes \cdots \otimes \mathscr{S}_{q,\mathbf{Q}_\kappa}(\Lambda_{r_\kappa,n_\kappa}).$$

This result is deduced from Theorem 3.15 using the theory of Young modules for Ariki-Koike algebras [**105**]. In the special case when $|\mathbf{Q}_\alpha| = 1$, for all α, this was first proved by Ariki [**5**]; see also [**54**].

§5. The representation theory of cyclotomic q-Schur algebras

This chapter gives a summary of the main results in the representation theory of the cyclotomic q-Schur algebras. All of these results are generalizations of theorems for the q-Schur algebras.

5.1. A Schur functor and double centralizer property

Throughout this section we assume that $\omega \in \Lambda$; because of this $\varphi_\omega = \varphi_{\mathsf{T}^\omega\mathsf{T}^\omega}$ is an element of $\mathscr{S}(\Lambda)$. Now, φ_ω is the identity map on $\mathscr{H}$; in particular, it is an idempotent. Moreover, it is easy to see that $\mathscr{H} \cong \varphi_\omega\mathscr{S}(\Lambda)\varphi_\omega$. Hence, by general nonsense (see, for example [**27, 65**]), φ_ω gives rise to a functor Φ_ω from the category of $\mathscr{S}(\Lambda)$-modules to the category of $\mathscr{H}$-modules; explicitly, if M is a right $\mathscr{S}(\Lambda)$-module then $\Phi_\omega(M) = M\varphi_\omega$ is a right $\mathscr{H}$-module.

Notice that the condition $\omega \in \Lambda$ is the analogue for the cyclotomic Schur algebras of the familiar requirement that $d \geq n$ for the q-Schur algebra $\mathscr{S}_q(d;n)$.

Theorem 5.1 (The cyclotomic Schur functor [**81**]). *Suppose that R is a field and that $\omega \in \Lambda^+ \subseteq \Lambda$. Let $\lambda \in \Lambda^+$. Then, as right $\mathscr{H}$-modules,*

(i) $\Phi_\omega(W^\lambda) \cong S^\lambda$;
(ii) $\Phi_\omega(L^\lambda) \cong D^\lambda$.

Furthermore, if $D^\mu \neq 0$ then $[W^\lambda{:}L^\mu] = [S^\lambda{:}D^\mu]$.

Sketch of proof. This can be proved either by general arguments as in [**65**]. Alternatively, from the definitions and the semistandard basis theorem it is clear that $\Phi_\omega(W^\lambda) \cong S^\lambda$ (if $\mathsf{T} \in \mathcal{T}^{ss}_\mu(\lambda)$ then $\varphi_\mathsf{T}\varphi_\omega = \delta_{\mu\omega}\varphi_\mathsf{T}$). Next observe that if $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux then the definition of the inner product on W^λ is that

$$\langle \varphi_\mathfrak{s}, \varphi_\mathfrak{t} \rangle \varphi_\lambda \equiv \varphi_{\mathfrak{t}^\lambda\mathfrak{s}} \varphi_{\mathfrak{t}\mathfrak{t}^\lambda} \pmod{\mathcal{S}^\lambda}.$$

Evaluating the functions on both sides at m_λ we find that

$$\langle \varphi_\mathfrak{s}, \varphi_\mathfrak{t} \rangle m_\lambda \equiv m_{\mathfrak{t}^\lambda\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda} \equiv \langle m_\mathfrak{s}, m_\mathfrak{t} \rangle m_\lambda \pmod{\mathcal{H}^\lambda}.$$

Hence, $\langle \varphi_\mathfrak{s}, \varphi_\mathfrak{t} \rangle = \langle m_\mathfrak{s}, m_\mathfrak{t} \rangle$ and the remaining claims follow. □

An important consequence of Theorem 5.1 is that the decomposition matrix of $\mathscr{H}$ is a submatrix of the decomposition matrix of $\mathscr{S}(\Lambda)$.

Corollary 5.2. *Suppose that R is a field and that $\omega \in \Lambda^+ \subseteq \Lambda$. Then the decomposition matrix of $\mathscr{H}$ is the submatrix of the decomposition matrix of $\mathscr{S}(\Lambda)$ obtained by deleting those columns indexed by the multipartitions μ such that $D^\mu = 0$.*

Observe that $\bigoplus_{\lambda\in\Lambda} M^\lambda$ is an $(\mathscr{S}(\Lambda), \mathscr{H})$-bimodule. In fact, each algebra is the full centralizer algebra for the other and we have a cyclotomic analogue of Schur-Weyl duality.

Theorem 5.3 (Double centralizer property). *Suppose that $\omega \in \Lambda$ and that $\Lambda^+ \subseteq \Lambda$. Then*

$$\mathscr{S}(\Lambda) \cong \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\lambda\in\Lambda} M^\lambda\Big) \quad \textit{and} \quad \mathscr{H} \cong \operatorname{End}_{\mathscr{S}(\Lambda)}\Big(\bigoplus_{\lambda\in\Lambda} M^\lambda\Big).$$

Sketch of proof. The first isomorphism is just the definition of $\mathscr{S}(\Lambda)$ so there is nothing to prove here. For the second isomorphism for each $\lambda \in \Lambda$ let φ_λ be the identity map on M^λ and let $\mathscr{M}^\lambda = \varphi_\lambda\mathscr{S}(\Lambda)$. (So $\mathscr{M}^\lambda$ is an $\mathscr{S}(\Lambda)$–module and M^λ is an $\mathscr{H}$–module.) Then there an isomorphism of $\mathscr{H}$-modules

$$\bigoplus_{\lambda\in\Lambda} M^\lambda \cong \bigoplus_{\lambda\in\Lambda} \Phi_\omega(\mathscr{M}^\lambda) = \bigoplus_{\lambda\in\Lambda} \varphi_\lambda\mathscr{S}(\Lambda)\varphi_\omega.$$

By definition $\sum_\lambda \varphi_\lambda$ is the identity of $\mathscr{S}(\Lambda)$, so $\mathscr{S}(\Lambda) = \bigoplus_\lambda \varphi_\lambda\mathscr{S}(\Lambda)$ and $\bigoplus_{\lambda\in\Lambda} M^\lambda \cong \mathscr{S}(\Lambda)\varphi_\omega$ as a left $\mathscr{S}(\Lambda)$-module. Therefore,

$$\operatorname{End}_{\mathscr{S}(\Lambda)}\big(\bigoplus_{\lambda\in\Lambda} M^\lambda\big) \cong \operatorname{End}_{\mathscr{S}(\Lambda)}\big(\mathscr{S}(\Lambda)\varphi_\omega\big) \cong \varphi_\omega\mathscr{S}(\Lambda)\varphi_\omega.$$

As $\varphi_\omega\mathscr{S}(\Lambda)\varphi_\omega \cong \mathscr{H}$, this completes the proof. □

5.2. The blocks of the cyclotomic Schur algebras

The centre of the affine Hecke algebra $\hat{H}_n$ is given by the following well-known result of Bernstein.

Theorem 5.4 (Bernstein). *Suppose that R is an algebraically closed field. Then the centre of $\hat{H}_n$ is equal to $R[X_1^\pm,\dots,X_n^\pm]^{\mathfrak{S}_n}$, the R-algebra of symmetric Laurent polynomials in $X_1,\dots,X_n$.*

This is quite straightforward to prove given the Bernstein presentation of $\hat{H}_n$.

Now X_k maps to L_k under the natural surjection $\hat{H}_n \longrightarrow \mathscr{H}_n$ so this implies that any symmetric polynomial in $L_1,\dots,L_n$ belongs to the centre of the Ariki-Koike algebra $\mathscr{H}_n$. As we remarked earlier, in the semisimple case the centre of $\mathscr{H}_n$ is always the algebra of symmetric polynomials in $L_1,\dots,L_n$; however, when $\mathscr{H}_n$ is not semisimple the centre of $\mathscr{H}_n$ can be larger than this. Because of this Theorem 5.5 below is a little surprising. First, some notation.

Given a multipartition λ let $\operatorname{res}(\lambda) = \{\operatorname{res}_{\mathfrak{t}^\lambda}(k) \mid 1 \le k \le n\}$, which we consider as a *multiset.* By the remarks above and Proposition 3.14 if two simple $\mathscr{H}_n$-modules D^λ and D^μ are in the same block then $\operatorname{res}(\lambda) = \operatorname{res}(\mu)$ as multisets.

We also note that because $\mathscr{H}_n$ is a cellular algebra all of the composition factors of S^λ belong to the same block; hence, D^λ and D^μ are in the same block if and only if S^λ and S^μ are in the same block. The same remark applies to the simple modules and the Weyl modules of the cyclotomic q-Schur algebras.

Theorem 5.5. *Suppose that R is an algebraically closed field and that λ and μ are multipartitions of n. Then the following are equivalent.*

(i) $\operatorname{res}(\lambda) = \operatorname{res}(\mu)$ *as multisets.*
(ii) S^λ *and* S^μ *are in the same block as* $\mathscr{H}_n$*-modules.*
(iii) S^λ *and* S^μ *are in the same block as* $\hat{H}_n$*-modules.*
(iv) W^λ *and* W^μ *are in the same block as* $\mathscr{S}(\Lambda_{r,n})$*-modules.*

Sketch of proof. As noted above, the implication (ii)$\Rightarrow$(i) follows from Proposition 3.14; this is was first proved by Graham and Lehrer [**64**] who also conjectured that the converse was true. That (i) and (iii) are equivalent follows from Theorem 5.4.

The hard part is proving that (iii) implies (ii); this was done by Grojnowski [**72**] using his modular branching rule. The key point is that if λ and μ are distinct multipartitions with $D^\lambda \neq 0$, $D^\mu \neq 0$ and $\operatorname{res}(\lambda) = \operatorname{res}(\mu)$ then $\operatorname{Hom}_{\hat{H}_{n-1}}(\operatorname{Res} D^\lambda, \operatorname{Res} D^\mu) = 0$ by Theorem 3.25; here Res is the functor $\operatorname{Res} : \hat{H}_n\text{-}\mathbf{mod} \longrightarrow \hat{H}_{n-1}\text{-}\mathbf{mod}$. Grojnowski shows that this implies that whenever $0 \longrightarrow D^\lambda \longrightarrow X \longrightarrow D^\mu \longrightarrow 0$ is an exact sequence of $\hat{H}_n$-modules then it is still exact when considered as a sequence of $\mathscr{H}_n$-modules (for any $\mathscr{H}_n$-module X). This implies (ii).

Finally, by the double centralizer property (Theorem 5.3), $\mathscr{S}(\Lambda_{r,n})$ and $\mathscr{H}_n$ have the same number of blocks (see [**103**, Cor. 5.38]), so it follows that (ii) and (iv) are equivalent. □

Theorem 5.5 does not classify the blocks of an arbitrary cyclotomic Schur algebra; rather it classifies the blocks of $\mathscr{S}(\Lambda)$ for any Λ with $\Lambda_{r,n}^+ \subseteq \Lambda$ (by standard arguments, all of these algebras are Morita equivalent). When $r = 1$ the blocks for the q-Schur algebras $\mathscr{S}_q(d; n)$ with $d \geq n$ were classified by Dipper, James and the author [**38, 80**]; the general case was settled by Cox [**33**]. The classification of the blocks of the cyclotomic q–Schur algebras is still open when $r > 1$ and $\omega \notin \Lambda$.

5.3. The Jantzen sum formula

Throughout this section assume that R is a field and that Λ is saturated. Let t be an indeterminate over R and let $\mathfrak{p}$ be the maximal ideal of $R[t, t^{-1}]$ generated by $t - q$. The localization $\mathcal{O} = R[t, t^{-1}]_\mathfrak{p}$ of $R[t, t^{-1}]$ at $\mathfrak{p}$ is a discrete valuation ring and $R \cong \mathcal{O}/\mathfrak{p}$. Let $\nu_\mathfrak{p}$ be the $\mathfrak{p}$-adic valuation on $\mathcal{O}$.

Let $\mathscr{H}_\mathcal{O}$ be the Hecke algebra over $\mathcal{O}$ with parameters qt and $U_s = Q_s t^{ns}$ if $Q_s \neq 0$ and $U_s = (t^{ns} - 1)$ if $Q_s = 0$. Then $\mathscr{H}_{R(t)} = \mathscr{H}_\mathcal{O} \otimes R(t)$ is semisimple by Corollary 3.3 and $\mathscr{H}_R = \mathscr{H}_{q,\mathbf{Q}}(W_{r,n}) \cong \mathscr{H}_\mathcal{O} \otimes_\mathcal{O} R$ is the reduction of $\mathscr{H}_\mathcal{O}$ modulo $\mathfrak{p}$. Let $\mathscr{S}_\mathcal{O}(\Lambda)$, $\mathscr{S}_{R(t)}(\Lambda)$ and $\mathscr{S}_R(\Lambda)$ be the corresponding cyclotomic q-Schur algebras.

Define the $\mathcal{O}$-residue of a node $x = (i, j, s)$ to be $\operatorname{res}_{\mathcal{O}}(x) = (qt)^{j-i}U_s$, an element of $\mathcal{O}$. The connection with our previous definition of residue is that $\operatorname{res}(x) = \operatorname{res}_{\mathcal{O}}(x) \otimes_{\mathcal{O}} 1_R$.

Let λ be a multipartition and for each node $x = (i, j, s) \in [\lambda]$ let $r_x \subseteq [\lambda]$ be the corresponding rim hook (so r_x is a rim hook in $[\lambda^{(s)}]$); then $[\lambda] \setminus r_x$ is the diagram of a multipartition. Let $\ell\ell(r_x)$ be the leg length of r_x and define $\operatorname{res}_{\mathcal{O}}(r_x) = \operatorname{res}_{\mathcal{O}}(f_x)$ where f_x is the foot node of r_x. These definitions can be found in [**79, 103**].

Suppose that λ and μ are multipartitions of n. If $\lambda \not\trianglerighteq \mu$ let $g_{\lambda\mu} = 1$; otherwise set

$$g_{\lambda\mu} = \prod_{x\in[\lambda]} \prod_{\substack{y\in[\mu] \\ [\mu]\setminus r_y=[\lambda]\setminus r_x}} \big(\operatorname{res}_{\mathcal{O}}(r_x) - \operatorname{res}_{\mathcal{O}}(r_y)\big)^{\varepsilon_{xy}},$$

where $\varepsilon_{xy} = (-1)^{\ell\ell(r_x)+\ell\ell(r_y)}$. The scalars $g_{\lambda\mu} \in \mathcal{O}$ have a combinatorial interpretation in terms of moving rim hooks in the diagram of a multipartition; see [**81**, Example 3.39].

Finally, let $W_{\mathcal{O}}^{\lambda}$ and W_R^{λ} be the Weyl modules for $\mathscr{S}_{\mathcal{O}}(\Lambda)$ and $\mathscr{S}_R(\Lambda)$ respectively; note that $W_R^{\lambda} \cong W_{\mathcal{O}}^{\lambda} \otimes_{\mathcal{O}} R$ as R-modules. For each $i \geq 0$ define

$$W_{\mathcal{O}}^{\lambda}(i) = \{\, x \in W_{\mathcal{O}}^{\lambda} \mid \langle x, y\rangle \in \mathfrak{p}^i \text{ for all } y \in W_{\mathcal{O}}^{\lambda} \,\}$$

and set $W_R^{\lambda}(i) = \big(W_{\mathcal{O}}^{\lambda}(i) + \mathfrak{p}W_{\mathcal{O}}^{\lambda}\big)/\mathfrak{p}W_{\mathcal{O}}^{\lambda}$. The Jantzen filtration of W_R^{λ} is

$$W_R^{\lambda} = W_R^{\lambda}(0) \geq W_R^{\lambda}(1) \geq W_R^{\lambda}(2) \geq \cdots .$$

In particular, note that $\operatorname{Rad} W_R^{\lambda} = W_R^{\lambda}(1)$; consequently, $W_R^{\lambda}(1)$ is a proper submodule of $W_R^{\lambda}(0)$ and $W_R^{\lambda}(0)/W_R^{\lambda}(1) \cong L_R^{\lambda}$. Note also that $W_R^{\lambda}(k) = 0$ for $k \gg 0$.

Actually, what we have just given is a special case of the definition of a Jantzen filtration. More generally, the same construction gives a Jantzen filtration for any suitable modular system $(K, \mathcal{O}, \mathfrak{p})$ (with parameters). The point of this remark is that the Jantzen filtration of W_R^{λ} depends upon a non-canonical choice of modular system.

We can now state the analogue of Jantzen's sum formula for $\mathscr{S}_R(\Lambda)$.

Theorem 5.6 (James-Mathas [**81**, Theorem 4.6]). *Let λ be a multipartition of n. Then*

$$\sum_{i>0} [W_R^{\lambda}(i)] = \sum_{\mu:\lambda\triangleright\mu} \nu_{\mathfrak{p}}(g_{\lambda\mu})[W_R^{\mu}].$$

in the Grothendieck group $K_0(\mathscr{S}_R(\Lambda)\text{-}\mathbf{mod})$ of $\mathscr{S}_R(\Lambda)$.

When $r = 1$ this result describes the Jantzen filtration of the Weyl modules of the q-Schur algebra. The Weyl modules of the q-Schur algebra coincide with the Weyl modules of quantum $\mathfrak{gl}_d$; therefore, when $r = 1$ Theorem 5.6 is a special case of a result of Andersen, Polo and Wen [**1**] who proved the analogue of the Jantzen sum formula for the quantum groups of finite type as a consequence of Kempf's vanishing theorem. For a combinatorial proof which takes place inside the q-Schur algebra see [**80, 103**]. When $r > 1$ there is no geometry to work with. The argument of [**81**] generalizes that of [**80**].

The idea behind the proof of Theorem 5.6 is quite simple. First, for each μ compute the determinant of the Gram matrix $G^\lambda_\mu = (\langle \varphi_\mathsf{S}, \varphi_\mathsf{T} \rangle)$, $\mathsf{S}, \mathsf{T} \in \mathcal{T}^{ss}_\mu(\lambda)$, of the μ-weight space $W^\lambda_\mathcal{O}\varphi_\mu$ of $W^\lambda_\mathcal{O}$. It turns out that $\det G^\lambda_\mu = g_{\lambda\mu}$. Now, the inner product $\langle\ ,\ \rangle$ on $W^\lambda_\mathcal{O}$ is non-degenerate; so Jantzen's elementary, yet fundamental, lemma says that

$$\sum_{i>0} \dim_R W^\lambda_R(i)\varphi_\mu = \nu_\mathfrak{p}(\det G^\lambda_\mu).$$

This is enough to deduce the result because, by Theorem 4.13(iv), any $\mathscr{S}(\Lambda)$-module is uniquely determined by the dimensions of its weight spaces since $\dim W^\nu\varphi_\nu = 1$ and $W^\mu\varphi_\nu \neq 0$ only if $\mu \trianglerighteq \nu$, for all $\nu \in \Lambda^+$.

Of course, computing $\det G^\lambda_\mu$ is not so easy. This is accomplished using an orthogonal basis of W^λ_R when $P_\mathscr{H}(q, \mathbf{Q}) \neq 0$. With this basis the Gram determinant is easier to calculate because almost all inner products are zero (we are really computing inner products in $W^\lambda_{R(t)}$). The orthogonal basis is constructed using a family of operators which act in a triangular fashion on the semistandard basis of the Weyl modules; intuitively, these operators belong to something like a Cartan subalgebra of $\mathscr{S}(\Lambda)$ — in fact, they are 'lifts' of the elements L_k to $\mathscr{S}(\Lambda)$.

The definition of the Jantzen filtration only requires a finitely generated $\mathcal{O}$-module which possesses a non-degenerate bilinear form. The same construction gives a Jantzen filtration $S^\lambda = S^\lambda_R(0) \geq S^\lambda_R(1) \geq \cdots$ for each Specht module; equivalently, by the proof of Theorem 5.1, we can set $S^\lambda_R(i) = \Phi_\omega(W^\lambda_R(i))$. Applying the Schur functor to Theorem 5.6 yields the following.

Corollary 5.7 (James-Mathas [**81**]). *Let λ be a multipartition of n. Then*

$$\sum_{i>0}[S^\lambda_R(i)] = \sum_{\mu:\lambda\rhd\mu} \nu_\mathfrak{p}(g_{\lambda\mu})[S^\mu_R].$$

*in the Grothendieck group $K_0(\mathscr{H}_R$-**mod**$)$ of $\mathscr{H}_R$.*

For the symmetric groups (that is, $r = 1$ and $q = 1$) this is a result of long standing known as Schaper's Theorem [**114**]. Schaper's argument is a translation of the Jantzen sum formula for the Weyl modules of the general linear group [**82**] (phrased in terms of the dot action of the symmetric group upon the weight lattice of GL_n), into the combinatorial language of the symmetric group. It is worth remarking that the corresponding result for the Weyl groups of type B (i.e., $r = 2$ and $q = 1$), was obtained only relatively recently [**81**].

The main application of the cyclotomic sum formula has been a classification of the irreducible Weyl modules and the irreducible Specht modules with $S^\lambda = D^\lambda$; see [**81**]. When $r = 1$ the sum formula was used to complete the classification of the blocks of the q-Schur algebras and the Iwahori-Hecke algebras of type A and also to classify the ordinary irreducible $\mathrm{GL}_n(q)$-modules which remain irreducible when reduced mod p when $p \nmid q$; see [**80**]. Ariki and the author [**10**] have also used the Jantzen sum formula to classify the representation type of the Iwahori-Hecke algebras of type B.

5.4. Connections with quantum groups

For this section only we renormalize the basis of the Ariki-Koike algebras so as to be consistent with the notation in [**5, 113**]. We assume that q has a square root in R and let $q = v^2$. As every field is a splitting field for $\mathscr{H}$ and $\mathscr{S}(\Lambda)$ we are free to extend R so that it contains a square root of q if necessary.

Let $\tilde{T}_i = v^{-1}T_i$ for $1 \le i < n$. Then $T_0, \tilde{T}_1, \dots, \tilde{T}_{n-1}$ still generate $\mathscr{H}$ and they are subject to the same relations as before except that the quadratic relation for the T_i becomes $(\tilde{T}_i - v)(\tilde{T}_i + v^{-1}) = 0$, for $1 \le i < n$. Observe that $L_k = \tilde{T}_{k-1} \dots \tilde{T}_1 T_0 \tilde{T}_1 \dots \tilde{T}_{k-1}$ for $k = 1, \dots, n$.

Fix an integer $d \ge 1$ and let $U_v(\mathfrak{gl}_d)$ be the quantized enveloping algebra of $\mathfrak{gl}_d$. Thus, $U_v(\mathfrak{gl}_d)$ is an associative $\mathbb{Q}(v)$-algebra which is generated by elements $E_i, F_i, K_j^{\pm}$, where $1 \le i < n$ and $1 \le j \le n$, which are subject to the quantum Serre relations.

Let V be a d dimensional $\mathbb{Q}(v)$-vector space with basis $\{e_1, \dots, e_d\}$. Then V is naturally a $U_v(\mathfrak{gl}_d)$-module, where the action of $U_v(\mathfrak{gl}_d)$ on V is determined by

$$E_i e_a = \delta_{a,i+1} e_{a-1}, \qquad F_i e_a = \delta_{a,i} e_{a+1}, \qquad \text{and} \qquad K_j e_a = v^{\delta_{j,a}} e_a,$$

for $1 \le i < n$, $1 \le j \le n$ and $1 \le a \le n$. Now, $U_v(\mathfrak{gl}_d)$ is a Hopf algebra with coproduct Δ given by $\Delta(K_j) = K_j \otimes K_j$,

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \qquad \text{and} \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

for $1 \le i < n$ and $1 \le j \le n$. Therefore, $V^{\otimes n}$ is a $U_v(\mathfrak{gl}_d)$-module; let $\rho_n : U_v(\mathfrak{gl}_d) \longrightarrow \operatorname{End}(V^{\otimes n})$ be the corresponding representation.

Let $I(d;n) = \{ (a_1, \dots, a_n) \mid 1 \le a_1, \dots, a_n \le d \}$. If $\mathbf{a} \in I(d;n)$ let $e_\mathbf{a} = e_{a_1} \otimes \cdots \otimes e_{a_n}$. Then $\{ e_\mathbf{a} \mid \mathbf{a} \in I(d;n) \}$ is a basis of $V^{\otimes n}$.

The symmetric group $\mathfrak{S}_n$ also acts on $V^{\otimes n}$ by place permutations and it acts on $I(d;n)$ by permuting components; indeed, $e_\mathbf{a} w = e_{\mathbf{a}w}$ for $\mathbf{a} \in I(d;n)$ and $w \in \mathfrak{S}_n$. Jimbo showed how to deform the action of $\mathfrak{S}_n$ to give an action of $\mathscr{H}_q(\mathfrak{S}_n)$ on $V^{\otimes n}$.

Recall that $\mathfrak{S}_n$ is generated by $t_1, \dots, t_{n-1}$, where $t_i = (i, i+1)$ for $i = 1, \dots, n-1$. Let $\Lambda^+(d;n)$ be the set of partitions in $\Lambda(d;n)$.

Theorem 5.8 (Jimbo [**84**]). *Assume that $\mathscr{H}_q(\mathfrak{S}_n)$ and $U_v(\mathfrak{gl}_d)$ are defined over $\mathbb{Q}(v)$.*

(i) *There is a unique $\mathscr{H}_q(\mathfrak{S}_n)$-module structure on $V^{\otimes n}$ such that*

$$e_\mathbf{a}\tilde{T}_j = \begin{cases} v e_\mathbf{a}, & \text{if } a_j = a_{j+1}, \\ e_{\mathbf{a}t_j}, & \text{if } a_j > a_{j+1}, \\ e_{\mathbf{a}t_j} + (v - v^{-1}) e_\mathbf{a}, & \text{if } a_j < a_{j+1} \end{cases}$$

for $j = 1, \dots, n-1$ and $\mathbf{a} \in I(d;n)$.

(ii) *The algebras $\mathscr{H}_q(\mathfrak{S}_n)$ and $\rho_n\big(U_v(\mathfrak{gl}_d)\big)$ are mutually the full centralizer algebras for each other for their actions on $V^{\otimes n}$. Moreover,*

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda^+(d;n)} W^\lambda \otimes S^\lambda$$

as an $\big(\mathscr{S}_q(d;n), \mathscr{H}_q(\mathfrak{S}_n)\big)$-bimodule.

It is not hard to see that there is an isomorphism $V^{\otimes n} \cong \oplus_{\lambda \in \Lambda(d;n)} M^\lambda$ of $\mathscr{H}$-modules; consequently, $\rho_n\big(U_v(\mathfrak{gl}_d)\big) \cong \mathscr{S}_q(d;n)$. In part (ii), W^λ is a Weyl module for $U_v(\mathfrak{gl}_d)$; by what we have just said this is the same as a Weyl module for the q-Schur algebra $\mathscr{S}(d;n)$.

Actually, this is a slight modification of Jimbo's original action of $\mathscr{H}_{v^2}(\mathfrak{S}_n)$ on $V^{\otimes n}$; this action comes from Du-Parshall-Wang [**52**].

The proof of Theorem 5.8 is straightforward. Checking the relations it is easy to see that $V^{\otimes n}$ is an $\mathscr{H}$-module and that the actions of $\mathscr{H}$ and $U_v(\mathfrak{gl}_d)$ commute. The double centralizer property can be proved using a highest weight argument to decompose $V^{\otimes n}$ as a $U_v(\mathfrak{gl}_d)$-module.

Notice that Theorem 5.8 is stated over the rational function field $\mathbb{Q}(v)$. Using the work of Beĭlinson, Lusztig and MacPherson [**15**], Du [**48**] showed that when $d \ge n$ Theorem 5.8 holds over the Laurent polynomial ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, where we replace $U_v(\mathfrak{gl}_d)$ with its Lusztig

$\mathcal{A}$-form $U_{\mathcal{A}}(\mathfrak{gl}_d)$; for the case $d < n$ see [**50**]. Further, if $d \ge n$ then $\mathscr{H}_q(\mathfrak{S}_n) \cong \mathrm{End}_{U_{\mathcal{A}}(\mathfrak{gl}_d)}(V^{\otimes n})$.

We remark that Doty and Giaquinto [**47**] have recently used the surjection $U(\mathfrak{gl}_d) \longrightarrow \mathscr{S}_q(d;n)$ to give a presentation of the q-Schur algebras over $\mathbb{Q}(v)$; see also [**49**]. No such presentation is known for the cyclotomic q-Schur algebras.

Now we indicate how Sakamoto and Shoji [**113**] have generalized Theorem 5.8 to the cyclotomic case. We extend the coefficient ring for all our algebras to the rational function field $\mathbb{Q}(v, Q_1, \dots, Q_r)$, where $v, Q_1, \dots, Q_r$ are indeterminates.

Fix positive integers $d_1, \dots, d_r$ with $d = d_1 + \cdots + d_r$ and let $\gamma : \{1, \dots, d\} \longrightarrow \{1, \dots, r\}$ be the map such that $\gamma(a) = s$ if s is minimal such that $a \le d_1 + \cdots + d_s$ and let V_s be the subspace of V with basis $\{ e_a \mid \gamma(a) = s \}$, for $1 \le s \le r$, and let $\mathfrak{g} = \mathfrak{gl}_{d_1}(V_1) \oplus \cdots \oplus \mathfrak{gl}_{d_r}(V_r)$. We consider $U_v(\mathfrak{g})$ as a Levi subalgebra of $U_v(\mathfrak{gl}_d)$ in the natural way. Then $V^{\otimes n}$ is a $U_v(\mathfrak{g})$-module by restriction; let $\rho_{n,r} : U_v(\mathfrak{g}) \longrightarrow \mathrm{End}(V^{\otimes n})$ be the corresponding representation of $U_v(\mathfrak{g})$.

In order to extend the action of $\mathscr{H}_q(\mathfrak{S}_n)$ on $V^{\otimes n}$ to an action of $\mathscr{H}$ define linear operators ϖ and S_j on $V^{\otimes n}$ by

$$e_{\mathbf{a}}\varpi = Q_{\gamma(a_1)} e_{\mathbf{a}} \quad \text{and} \quad e_{\mathbf{a}} S_j = \begin{cases} e_{\mathbf{a}} \tilde{T}_j, & \text{if } \gamma(a_{j-1}) = \gamma(a_j), \\ e_{\mathbf{a}t_j}, & \text{otherwise,} \end{cases}$$

for $\mathbf{a} \in I(d;n)$ and $1 \le j < n$.

Let $\Lambda(d_1, \dots, d_r; n)$ be the set of multicompositions λ of n such that $|d_s| = |\lambda^{(s)}|$ for $1 \le s \le r$ and let $\Lambda^+(d_1, \dots, d_r; n)$ be the set of multipartitions in $\Lambda(d_1, \dots, d_r; n)$. We warn the reader that $\Lambda^+(d_1, \dots, d_r; n) \ne \big(\Lambda(d_1, \dots, d_r; n)\big)^+$, in the sense of Definition 4.8 — unless $d_s \ge n$ for $1 \le s < r$.

The irreducible representations of $U_v(\mathfrak{g})$ can be parametrized by multipartitions in $\Lambda^+(d_1, \dots, d_r; n)$ for $n \ge 0$. If $\lambda \in \Lambda^+(d_1, \dots, d_r; n)$ let $W(\lambda)$ be the corresponding Weyl module for $U_v(\mathfrak{g})$. If $r = 1$ then $W(\lambda) \cong W^\lambda$ as $\mathscr{S}_q(d;n)$-modules; however, in general, W^λ and $W(\lambda)$ are not isomorphic even as vector spaces.

Theorem 5.9 (Sakamoto and Shoji [**113**]). *Assume that $\mathscr{H}$ and $U_v(\mathfrak{g})$ are defined over the field $\mathbb{Q}(v, Q_1, \dots, Q_r)$.*

(i) *The action of $\mathscr{H}_q(\mathfrak{S}_n)$ on $V^{\otimes n}$ extends to give an action of $\mathscr{H}$ on $V^{\otimes n}$ via*

$$e_{\mathbf{a}} T_0 = e_{\mathbf{a}} \varpi S_1 \dots S_{n-1} \tilde{T}_{n-1}^{-1} \dots \tilde{T}_1^{-1}$$

for all $\mathbf{a} \in I(d;n)$.

(ii) *The algebras $\mathscr{H}$ and $\mathscr{S}_n^{(r)} \cong \rho_{n,r}\big(U_v(\mathfrak{g})\big)$ are mutually the full centralizer algebras for the others action on $V^{\otimes n}$. Moreover,*

$$V^{\otimes n} \cong \bigoplus_{\lambda\in\Lambda^+(d_1,\ldots,d_r;n)} W(\lambda)\otimes S^\lambda$$

as an $\big(\mathscr{S}_n^{(r)}, \mathscr{H}\big)$-bimodule.

Sakamoto and Shoji were guided in part by Ariki, Terasoma and Yamada [**11**] who considered the special case when $d_1 = \cdots = d_r = 1$. The proof of part (i) of the theorem is a long calculation building on Theorem 5.8(i). Once again, part (ii) is a highest weight computation.

Sakamoto and Shoji also note that part (i) of Theorem 5.9 is true over an arbitrary integral domain. Using this observation they gave another proof that $\mathscr{H}$ is free of rank $|W_{r,n}|$ (Theorem 2.2).

Ariki [**5**] asked whether Theorem 5.9(ii) is true over an arbitrary field; he was particularly interested in knowing when the dimension of $\rho_{r,n}\big(U_v(\mathfrak{g})\big) = \mathrm{End}_{\mathscr{H}}(V^{\otimes n})$ is independent of R, q and $\mathbf{Q}$. Ariki found an example which showed that in general the dimension of $\rho_{r,n}\big(U_v(\mathfrak{g})\big)$ does depend upon these choices; nonetheless, he was able to prove the result below.

Let $U_{\mathcal{A}}(\mathfrak{g})$ be the Kostant-Lusztig $\mathcal{A}$-form of $U_v(\mathfrak{g})$ and set $U_{R,v}(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_{\mathcal{A}} R$, where R is an integral domain. We also consider V to be the free R-module with basis $\{e_1, \ldots, e_d\}$. Finally, define $\mathscr{S}_n^{(r)} = \mathrm{End}_{\mathscr{H}}(V^{\otimes n})$.

Theorem 5.10 (Ariki [**5**]). *Suppose that R is an integral domain and that $q = v^2, Q_1, \ldots, Q_r$ are elements of R such that*

$$P_n(q,\mathbf{Q}) = \prod_{1\le i<j\le r}\ \prod_{-n<d<n} (q^d Q_i - Q_j)$$

is invertible in R. Then the following hold.

(i) *Suppose that $d_s \ge n$ for all s. Then there is an isomorphism of R-algebras*

$$\mathscr{S}_n^{(r)} \cong \bigoplus_{\substack{n_1,\ldots,n_r\ge 0\\ n_1+\cdots+n_r=n}} \mathscr{S}_q(d_1;n_1)\otimes\cdots\otimes\mathscr{S}_q(d_r;n_r).$$

In particular, $\mathscr{S}_n^{(r)}$ is free as an R-module and its rank is independent of the choice of R or the parameters $v, Q_1, \ldots, Q_r$.

(ii) *The algebra $\mathscr{S}_n^{(r)}$ is a quotient of $U_{R,v}(\mathfrak{g})$.*

(iii) *Assume that* $d_s \geq n$ *for* $1 \leq s \leq r$. *Then* $\operatorname{End}_{U_{R,v}(\mathfrak{g})}(V^{\otimes n})$ *is Morita equivalent to the algebra*

$$\bigoplus_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\dots+n_r=n}} \mathscr{H}_q(\mathfrak{S}_{n_1})\otimes\cdots\otimes\mathscr{H}_q(\mathfrak{S}_{n_r}).$$

Observe that $P_n(q,\mathbf{Q})$ is equal to the polynomial $P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_r)$ of Theorem 3.15 (with $\mathbf{Q}_\alpha = \{Q_\alpha\}$, for all α). Assume that $d_s \geq n$ for all s. Then, by part (i) and Theorem 4.15, the algebra $\mathscr{S}_n^{(r)}$ is Morita equivalent to the cyclotomic Schur algebra $\mathscr{S}(\Lambda_{r,n})$. Similarly, by part (iii) and Theorem 3.15, if $d_s \geq n$ for all s then $\operatorname{End}_{U_{R,v}}(V^{\otimes n})$ is Morita equivalent to $\mathscr{H}$. Hence, up to Morita equivalence, we have a complete analogue of Schur-Weyl duality linking $U_v(\mathfrak{g})$, $\mathscr{S}(\Lambda_{r,n})$ and $\mathscr{H}$ in this setting; however, note that this is really a type A phenomenon and is not genuinely 'cyclotomic'.

Ariki also uses this result to compute the decomposition matrices of the algebras $\mathscr{S}_n^{(r)}$ when $R=\mathbb{Q}$, $q\neq 1$ and $P_n(q,\mathbf{Q})\neq 0$. To do this he uses part (i) and the LT-conjecture [**95**] which gives an extension of Theorem 3.19(ii) to the q-Schur algebras. The LT-conjecture was proved by Varagnolo and Vasserot [**121**]; see also Schiffmann [**115**].

Combining these results, the decomposition matrices of the cyclotomic Schur algebras are known whenever R is a field of characteristic zero, $q\neq 1$ and $P_n(q,\mathbf{Q})\neq 0$. Actually, we do not need Ariki's work to do this as we already obtain this result from Theorem 4.15 and [**95,121**]. (Note that Ariki's paper appeared before [**44**], the source of Theorem 4.15.)

5.5. Borel subalgebras

In this section we show that the cyclotomic Schur algebras admit a "triangular decomposition". For the Schur algebras this is a result of J.A. Green [**66**]; the cyclotomic case is due to Du and Rui [**53**].

For simplicity we consider the case where $\Lambda=\Lambda_{r,n}$ is the set of all multicompositions of n of length at most n. Du and Rui note that the general case can be deduced from this because if Λ is a saturated set of multicompositions then $\mathscr{S}(\Lambda)$ is Morita equivalent to the subalgebra $e\mathscr{S}(\Lambda_{r,n})e$ of $\mathscr{S}(\Lambda_{r,n})$, where e is the idempotent $\sum_{\lambda\in\Lambda^+}\varphi_\lambda$.

Recall that $I(rn;n)=\{\,(a_1,\dots,a_n)\mid 1\leq a_i\leq rn\,\}$. Then $\mathfrak{S}_n$ acts on $I(rn;n)$ by place permutations. Given a multicomposition λ in $\Lambda_{r,n}$ let $\overline{\lambda}=(\overline{\lambda}_1,\dots,\overline{\lambda}_{rn})$ be the composition in $\Lambda(rn;n)$ with $\overline{\lambda}_i=\lambda_j^{(s)}$

if $i = (s-1)n + j$. Define

$$\mathbf{i}_\lambda = (i_{\lambda,1}, \dots, i_{\lambda,n}) = (\underbrace{1, \dots, 1}_{\overline{\lambda}_1}, \underbrace{2, \dots, 2}_{\overline{\lambda}_2} \dots, \underbrace{rn, \dots, rn}_{\overline{\lambda}_{rn}}) \in I(rn; n).$$

Let $\succeq$ be the partial order on $I(rn;n)$ given by $\mathbf{a} \succeq \mathbf{b}$ if $a_k \geq b_k$ for $1 \leq k \leq rn$. Note that for any $d \in \mathfrak{S}_n$ if λ and μ are multicompositions with $\mathbf{i}_\lambda d \succeq \mathbf{i}_\mu$ then $\mu \trianglerighteq \lambda$.

Recall that for each multicomposition $\lambda \in \Lambda_{r,n}$ we have a Young subgroup $\mathfrak{S}_\lambda$ and that $\mathcal{D}_\lambda$ is the set of minimal length coset right representatives for $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Moreover, if μ is another multicomposition then $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ is a set of minimal length $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-double coset representatives. For each $d \in \mathcal{D}_{\lambda\mu}$ define $\varphi^d_{\lambda\mu}$ to be the R-linear endomorphism of $\bigoplus_\alpha M^\alpha$ determined by

$$\varphi^d_{\lambda\mu}(m_\alpha h) = \delta_{\alpha\mu}\Big(\sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w\Big) u^+_\mu h$$

for all $\alpha \in \Lambda_{r,n}$ and all $h \in \mathscr{H}$. If $\mathbf{i}_\lambda d \succeq \mathbf{i}_\mu$ then $\varphi^d_{\lambda\mu} \in \mathscr{S}(\Lambda_{r,n})$ by [**53**, Lemma 5.6]. In particular, if $\nu \in \Lambda_{r,n}$ then $\varphi^1_{\nu\nu} = \varphi_\nu$ restricts to the identity map on M^ν (and is zero on M^α for $\alpha \neq \nu$).

Finally, given multicompositions λ and μ in $\Lambda_{r,n}$ let

$$\Omega_{\lambda\mu} = \{\, d \in \mathcal{D}_{\lambda\mu} \mid \mathbf{i}_\lambda d \succeq \mathbf{i}_\mu \,\}.$$

Define $\mathscr{S}^\pm(\Lambda_{r,n})$ to be the two R-submodules of $\mathscr{S}(\Lambda_{r,n})$ spanned by $\{\, \varphi^d_{\lambda\mu} \mid d^\mp \in \Omega_{\lambda\mu} \,\}$. We can now state the main result.

Theorem 5.11 (Du and Rui [**53**]). *Suppose that R is an integral domain.*

(i) *The two R-modules $\mathscr{S}^\pm(\Lambda_{r,n})$ are subalgebras of $\mathscr{S}(\Lambda_{r,n})$.*
(ii) *$\mathscr{S}^\pm(\Lambda_{r,n})$ is free as an R-module with basis*

$$\{\, \varphi^d_{\lambda\mu} \mid \lambda, \mu \in \Lambda_{r,n} \text{ and } d^\mp \in \Omega_{\lambda\mu} \,\}.$$

(iii) *$\mathscr{S}(\Lambda_{r,n})$ has a triangular decomposition*

$$\mathscr{S}(\Lambda_{r,n}) = \mathscr{S}^-(\Lambda_{r,n}) \cdot \mathscr{S}^+(\Lambda_{r,n}) = \mathscr{S}^-(\Lambda_{r,n}) \cdot \Big(\sum_{\nu \in \Lambda_{r,n}} R\varphi_\nu\Big) \cdot \mathscr{S}^+(\Lambda_{r,n}).$$

Thus, $\{\, \varphi^d_{\lambda\mu}\varphi^e_{\mu\nu} \mid \lambda, \mu, \nu \in \Lambda_{r,n}, d \in \mathcal{D}_{\lambda\mu}$ and $e^{-1} \in \mathcal{D}_{\mu\nu} \,\}$ is a basis of $\mathscr{S}(\Lambda_{r,n})$.

Du and Rui call $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$ the Borel subalgebras of $\mathscr{S}(\Lambda_{r,n})$. Surprisingly, the Borel subalgebras of the cyclotomic Schur algebras are isomorphic to the Borel subalgebras of the q-Schur algebras; hence, they are really type A algebras.

The right hand side of part (iii) is written so as to suggest the triangular decomposition of quantum groups; however, this is slightly misleading because $\varphi^d_{\lambda\mu}\big(\sum_\nu r_\nu\varphi_\nu\big)\varphi^e_{\sigma\tau} = \delta_{\mu\sigma}r_\mu\varphi^d_{\lambda\mu}\varphi^e_{\sigma\tau}$, for $r_\nu \in R$.

Du and Rui are able to say quite a lot about the representation theory of these subalgebras. Because $\mathscr{S}^\pm(\Lambda_{r,n})$ are quasi-hereditary, they have standard modules and costandard modules; denote these by $\Delta^\pm(\mu)$ and $\nabla^\pm(\mu)$ respectively, for $\mu \in \Lambda_{r,n}$. Also, if $\mu \in \Lambda^+_{r,n}$ then the Weyl module $W^\mu = \Delta(\mu)$ is a standard module of $\mathscr{S}(\Lambda_{r,n})$ and its contragredient dual $(W^\mu)^* = \nabla(\mu)$ is a costandard module (duality with respect to $*$).

Theorem 5.12 (Du and Rui). *Suppose that R is a field.*

(i) *The Borel subalgebras $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$ are quasi-hereditary, with respect to the poset $\Lambda_{r,n}$. Moreover, $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$ are Ringel dual to each other.*

(ii) (a) *Each costandard module of $\mathscr{S}^-(\Lambda)$ is one dimensional and, hence, simple; moreover, every simple module appears this way.*

(b) *Dually, each standard module of $\mathscr{S}^-(\Lambda_{r,n})$ is a projective indecomposable $\mathscr{S}^-(\Lambda_{r,n})$-module.*

(c) *Explicitly, if $\mu \in \Lambda^+_{r,n}$ then $\Delta^-(\mu) = \mathscr{S}^-(\Lambda_{r,n})\varphi_\mu$ and $\nabla^-(\mu) = \Delta^-(\mu)/\operatorname{Rad}\Delta^-(\mu)$; moreover, $\{\,\varphi_\mu \mid \mu \in \Lambda_{r,n}\,\}$ is a complete set of primitive idempotents in $\mathscr{S}^-(\Lambda_{r,n})$.*

(iii) *Suppose that $\mu \in \Lambda_{r,n}$. Then*

$$\mathscr{S}(\Lambda_{r,n}) \otimes_{\mathscr{S}^+(\Lambda_{r,n})} \Delta^+(\mu) \cong \begin{cases} \Delta(\mu), & \text{if } \mu \in \Lambda^+_{r,n}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\operatorname{Hom}_{\mathscr{S}^-(\Lambda_{r,n})}\big(\mathscr{S}(\Lambda_{r,n}), \nabla^-(\mu)\big) \cong \begin{cases} \nabla(\mu), & \text{if } \mu \in \Lambda^+_{r,n}, \\ 0, & \text{otherwise,} \end{cases}$$

Ringel duality interchanges the standard and costandard modules of $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$, so part (ii) also describes the simple and projective $\mathscr{S}^+(\Lambda_{r,n})$-modules.

Du and Rui also give the dimensions of the standard and costandard modules for the Borel subalgebras and show that the Borel subalgebras are Ringel dual to each other.

5.6. Tilting modules

Let A be a quasi-hereditary algebra (see [**32, 46**]), and let Λ^+ be its poset of weights. Then for each $\lambda \in \Lambda^+$ we have a standard module $\Delta(\lambda)$, a costandard module $\nabla(\Lambda)$ and a simple module $L(\lambda)$. The simple module $L(\lambda)$ is the head of $\Delta(\lambda)$ and the simple socle of $\nabla(\lambda)$; further, $\nabla(\lambda)$ is the contragredient dual of $\Delta(\lambda)$ if A possesses a suitable involution.

Let $\mathcal{F}(\Delta)$ be the full subcategory of A-**mod** consisting of those modules which have a Δ-filtration; thus $X \in \mathcal{F}(\Delta)$ if X has a filtration $X = X_1 \supset X_2 \supset \cdots \supset X_m \supset 0$ with $X_i/X_{i+1} \cong \Delta(\lambda_i)$ for $1 \le i \le m$. If $X \in \mathcal{F}(\Delta)$ and $\lambda \in \Lambda^+$ let $[X{:}\Delta(\lambda)] = \#\{\, 1 \le i \le m \mid X_i/X_{i+1} \cong \Delta(\lambda)\,\}$; this is independent of the choice of filtration because the equivalence classes of standard modules are a basis of the Grothendieck group of A. Similarly, let $\mathcal{F}(\nabla)$ be the full subcategory of A-modules which have a ∇-filtration.

Ringel [**110**] has proved that for each $\lambda \in \Lambda^+$ there is a unique indecomposable A-module $T(\lambda) \in \mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)$ such that $[T(\lambda){:}\Delta(\lambda)] = 1$ and $[T(\lambda){:}\Delta(\mu)] \ne 0$ only if $\mu \ge \lambda$; we call $T(\lambda)$ a (partial) tilting module for A. Moreover, every module in $\mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)$ is isomorphic to a direct sum of tilting modules.

If Λ is saturated then the cyclotomic Schur algebra $\mathscr{S}(\Lambda)$ is quasi-hereditary by Theorem 4.14, so we may ask for a description of the tilting modules of $\mathscr{S}(\Lambda)$. When $r = 1$ Donkin [**45, 46**] determined the tilting modules of the q-Schur algebras. To describe this, recall from the previous section that $\mathscr{S}_q(d;n) = \operatorname{End}_{\mathscr{H}}(V^{\otimes n})$. Donkin showed that the tilting modules of $\mathscr{S}_q(d;n)$ are precisely the indecomposable direct summands of the exterior powers $\wedge^\lambda V = \wedge^{\lambda_1}V\otimes\cdots\otimes\wedge^{\lambda_d}V$. For another approach to the tilting modules of the q-Schur algebras see [**50**].

Even though we do not know how to describe $\oplus_\mu M^\mu$ as a tensor product the exterior powers of $\mathscr{S}(\Lambda)$ still admit a similar description. In introducing M^λ we said that it should be thought of as an induced trivial module; the analogue of an induced sign representation for $\mathscr{H}$ is the module $N^\lambda = n_\lambda\mathscr{H}$, where $n_\lambda = y_\lambda u_\lambda^- = u_\lambda^- y_\lambda$ and

$$y_\lambda = \sum_{w\in\mathfrak{S}_\lambda} (-q)^{-\ell(w)}T_w \quad\text{and}\quad u_\lambda^- = \prod_{s=1}^{r-1}\ \prod_{k=1}^{|\lambda^{(s+1)}|+\cdots+|\lambda^{(r)}|} (L_k - Q_s).$$

For each multipartition λ let $E^\lambda = \theta_\lambda\mathscr{S}(\Lambda)$, where $\theta_\lambda \in \operatorname{Hom}_{\mathscr{H}}(\mathscr{H}, N^\lambda)$ is the map $\theta_\lambda(h) = n_\lambda h$ for $h \in \mathscr{H}$. Then E^λ is a right $\mathscr{S}(\Lambda)$-module and we have the following.

Theorem 5.13 (Mathas [**105**]). *Suppose that R is a field, and that Λ is a saturated set of multicompositions containing ω. Then the tilting modules of $\mathscr{S}(\Lambda_{r,n})$ are the indecomposable summands of the modules $\{\, E^\lambda \mid \lambda \in \Lambda^+ \,\}$.*

The key tool in the proof of Theorem 5.13 is the use of Specht filtrations and dual Specht filtrations of $\mathscr{H}$–modules; this is a bit surprising because Specht filtrations are generally not as good as Weyl filtrations (since it can happen that $S^\lambda \cong S^\mu$ when $\lambda \neq \mu$).

The tilting modules of $\mathscr{S}(\Lambda)$ have all of the expected properties. For example, $[T(\lambda){:}\nabla(\mu)] = [\Delta(\mu'){:}L(\lambda')]$ for all $\lambda, \mu \in \Lambda^+$. (Here μ' is the multipartition conjugate to μ.) Furthermore, the Ringel dual of $\mathscr{S}(\Lambda)$ is the algebra $\mathscr{S}'(\Lambda) = \operatorname{End}_{\mathscr{H}}\big(\bigoplus_{\mu\in\Lambda} N^\mu\big)$ and $\mathscr{S}'(\Lambda) \cong \mathscr{S}(\Lambda)$.

The theory of Young modules for $\mathscr{H}$ (cf. [**78**]), is also developed in [**105**]. The Young modules (and twisted Young modules) are the indecomposable direct summands of the modules M^λ and N^λ, for λ a multicomposition of n; they are indexed by the multipartitions of n. The Young modules are the image under the Schur functor of the corresponding indecomposable projective, injective or tilting modules for the algebras $\mathscr{S}(\Lambda)$ or $\mathscr{S}'(\Lambda)$.

§6. Some open problems

In this final chapter we discuss some open problems for the Ariki-Koike algebras and the cyclotomic Schur algebras. We are mostly interested in the connections between the representation theory of the Ariki-Koike algebras and cyclotomic Schur algebras with the representation theory of the finite groups of Lie type.

Problem 6.1. *Prove the conjectures of Broué, Malle and Michel stated in Conjecture 2.5 and* [**19**].

We also pose the more general (and more vague) problem.

Problem 6.2. *Find a link between the representation theory of the cyclotomic Schur algebras and the modular representation theory of the finite groups of Lie type.*

At best, there is only circumstantial evidence for such a connection when $r > 2$. If we believe in the conjectures of the Broué school then there are strong ties between the representation theory of cuspidal representations of $\mathrm{GL}_d(q)$ in characteristic zero, so it is not unreasonable to expect that the modular theory of the cyclotomic Schur algebras and Ariki-Koike algebras also carry information about the modular representations of $\mathrm{GL}_d(q)$.

6.1. Quantum groups and geometry

The results of Ariki and Sakamoto and Shoji from §5.4 show that in some circumstances the module categories of the Ariki-Koike algebras and the cyclotomic Schur algebras are connected with the module categories of Levi subalgebras of $U_v(\mathfrak{gl}_d)$. Unfortunately, these results apply only in cases where the Ariki-Koike algebras are Morita equivalent to direct sums of tensor products of Iwahori-Hecke algebras of type A and when the cyclotomic q-Schur algebras were Morita equivalent to direct sums of tensor products of q-Schur algebras.

Problem 6.3. *Realize the cyclotomic Schur algebras as a quotient of a quantum group $U_{\mathcal{A}}(\mathfrak{g})$ over an arbitrary integral domain.*

We could ask for a generalization of the results of Sakamoto and Shoji (Theorem 5.9) and Ariki (Theorem 5.10); however, as the conjectures of Broué's school only ask for a derived equivalence it seems to me that we cannot expect something so simple here.

That the cyclotomic Schur algebras might be realizable as a quotient of a quantum group is suggested by the cyclotomic Jantzen sum formula (Theorem 5.6) and by the existence of the Borel subalgebras and the triangular decomposition of $\mathscr{S}(\Lambda)$ (Theorem 5.11). Both of these results hint at connections with quantum groups and at some undiscovered geometry.

Note also that the existence of the Borel subalgebras allows us to consider the dual Weyl modules of the cyclotomic Schur algebras as induced modules and so gives us cohomological techniques to play with.

6.2. Tensor products

First consider the case $r = 1$. If λ is a partition of n and μ is a partition of m then $S^\lambda \boxtimes S^\mu$ is a module for the Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)$. We can identify $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)$ with the subalgebra $\mathscr{H}_q(\mathfrak{S}_{(n,m)})$ of $\mathscr{H}_q(\mathfrak{S}_{n+m})$ where $\mathfrak{S}_{(n,m)} = \mathfrak{S}_n \times \mathfrak{S}_m$. Thus, $\mathscr{H}_q(\mathfrak{S}_{n+m})$ is a free $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)$-module and we can define the $\mathscr{H}_q(\mathfrak{S}_{n+m})$-module

$$S^\lambda \otimes S^\mu = (S^\lambda \boxtimes S^\mu) \otimes_{\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)} \mathscr{H}_q(\mathfrak{S}_{n+m}).$$

When $\mathscr{H}_q(\mathfrak{S}_n)$ is semisimple, this decomposes as a direct sum of Specht modules according to the Littlewood-Richardson rule.

In the case of the q-Schur algebras it is even easier. If λ and μ are both partitions of length at most d then the Weyl modules W^λ and W^μ are homogeneous polynomial representations for $U_v(\mathfrak{gl}_d)$ of degree n and m respectively; therefore, $W^\lambda \otimes W^\mu$ is a polynomial representation of

$U_v(\mathfrak{gl}_d)$ of degree $n+m$ — since $U_v(\mathfrak{gl}_d)$ is a Hopf algebra. Hence, $W^\lambda \otimes W^\mu$ is an $\mathscr{S}_q(d; n+m)$-module since $\mathscr{S}(d; N)$-**mod** is the category of polynomial representations of $U_v(\mathfrak{gl}_d)$ of homogeneous degree N. Again, in the semisimple case the decomposition of $W^\lambda \otimes W^\mu$ into irreducibles is given by the Littlewood-Richardson rule.

When we try and extend either of these constructions to the cyclotomic case we run into problems. First, for the Ariki-Koike algebras there is no obvious way to consider $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n}) \otimes \mathscr{H}_{q,\mathbf{Q}'}(W_{s,m})$ as a free submodule of $\mathscr{H}_{q,\mathbf{Q}\cup\mathbf{Q}'}(W_{t,n+m})$ for any t, unless $rs = 0$. Secondly, we do not have an interpretation of the module category of a cyclotomic Schur algebra in terms of homogeneous representations of a quantum group.

Problem 6.4. *Find a good tensor product operation for the categories $\mathscr{H}$-**mod** and $\mathscr{S}(\Lambda)$-**mod**.*

Of course, a strong enough link with quantum groups would give us this for free. The correct approach is probably via the affine Hecke algebra (or possibly the work of Shoji [**117**]).

If we knew how to take tensor products of modules for the cyclotomic Schur algebras then we could try and solve the following problem.

Problem 6.5. *Find an analogue of the Steinberg tensor product theorem for the cyclotomic Schur algebras.*

Evidence for the existence of such a result, as well as an indication of what it might look like, are given by Uglov's [**120**] action of the Heisenberg algebra upon the generalized Fock spaces.

6.3. Decomposition numbers at roots of unity

The decomposition numbers of the Ariki-Koike algebras are known in characteristic zero, thanks to Ariki's theorem and the work of Uglov (assuming that $Q_s \neq 0$ for any s); see Corollary 3.20.

Problem 6.6. *Compute the decomposition numbers of the cyclotomic q-Schur algebras in characteristic zero.*

By Theorem 3.19 the decomposition matrix of $\mathscr{H}_q(\mathfrak{S}_n)$ can be computed from the canonical basis of $L_v(\Lambda_0)$. The easiest way to compute the canonical basis of $L_v(\Lambda_0)$ is to work in the Fock space $\mathcal{F}$, an infinite rank free $\mathbb{C}[v, v^{-1}]$-module with a basis given by the set of all partitions of all integers. Leclerc and Thibon's idea [**95**] was to define a canonical basis on the whole of the Fock space; they did this using the action of a Heisenberg algebra on $\mathcal{F}$. Leclerc and Thibon conjectured that the decomposition matrices of the q-Schur algebra were given by computing the canonical basis of $\mathcal{F}$ and then specializing $v = 1$; this was proved by Varagnolo and Vasserot [**121**].

Hence, this problem has been solved when $r = 1$. Furthermore, as remarked in §5.4, when $P_n(q, \mathbf{Q}) \neq 0$ we also know the answer because by Theorem 4.15 $\mathscr{S}(\Lambda)$ is Morita equivalent to a direct sum of tensor products of q-Schur algebras.

Now, the decomposition matrices of the Ariki-Koike algebras in characteristic zero are obtained by computing the canonical basis of highest weight modules $L_v(\Lambda)$, for the various dominant weights Λ. This time, $L_v(\Lambda)$ embeds in a generalized Fock space $\mathcal{F}_\Lambda$ and Uglov has shown how to compute a canonical basis for the whole of this space; this gives a canonical basis element for each multipartition. For $n \geq 0$ the canonical basis of $\mathcal{F}_\Lambda$ at $v = 1$ gives a square unitriangular matrix, indexed by the multipartitions of n, which contains the decomposition matrix of the Ariki-Koike algebra $\mathscr{H}_n$ as a submatrix (delete those columns corresponding to the multipartitions λ with $D^\lambda = 0$); compare with Corollary 5.2. The indexing of the rows and columns is wrong; however, once this difference in labeling is taken into account, I expect that this will give the decomposition matrix of $\mathscr{S}(\Lambda_{r,n})$.

6.4. Dipper-James theory

Let q be a prime power and let $\mathrm{GL}_n(q)$ be the general linear group over a field with q elements. Dipper and James [**39**] proved that the decomposition matrix of $\mathrm{GL}_n(q)$ in non-defining characteristic is completely determined by the decomposition matrix of the q^d–Schur algebras, for $d \geq 1$. Recently Brundan, Dipper, and Kleshchev [**27**] have rewritten this theory using cuspidal algebras. They also make the Dipper-James result on decomposition matrices much more explicit; see [**27**, Theorem 4.4d].

To date, no one has succeeded in generalizing this theory to the cyclotomic q-Schur algebras. The best results in this direction were obtained by Gruber and Hiss [**74**] who, for linear primes, worked with a Morita equivalent version of the cyclotomic Schur algebras when $r = 2$ (type B), to give similar results for other finite reductive groups $G_n(q)$. See the survey article of Dipper, Geck, Hiss and Malle [**36**] for the current status of this theory.

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