Advanced Studies in Pure Mathematics 40, 2004
Representation Theory of Algebraic Groups and Quantum Groups
pp. 219–252

Quantum affine algebras and crystal bases

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§1. Introduction

The notion of a *quantum group* was introduced by Drinfel'd and Jimbo, independently, in their study of the quantum Yang-Baxter equation arising from 2-dimensional solvable lattice models ([3, 9]). Quantum groups are certain families of Hopf algebras that are deformations of universal enveloping algebras of Kac-Moody algebras. For the past 20 years, the quantum groups turned out to be the fundamental algebraic structure behind many branches of mathematics and mathematical physics.

In [18, 19, 25], Kashiwara and Lusztig independently developed the theory of *crystal bases* (or *canonical bases*) for quantum groups which provides a powerful combinatorial and geometric tool to study the representations of quantum groups. A *crystal basis* can be understood as a basis at $q = 0$ and is given a structure of colored oriented graph, called the *crystal graph*, with arrows defined by the *Kashiwara operators.* The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable modules over quantum groups. In particular, they have extremely simple behavior with respect to taking the tensor product.

In this paper, we will discuss some of the recent developments in crystal basis theory for quantum affine algebras in connection with combinatorics of *Young walls* ([12, 16, 17]). In Section 2, 3 and 4, we briefly review the basic properties of Kac-Moody algebras, quantum groups and

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Received February 25, 2002.
*This research was supported by KOSEF Grant # 98-0701-01-5-L and the Young Scientist Award, Korean Academy of Science and Technology
AMS Classification 2000: 17B37, 17B65, 17B67.

crystal bases ([8, 11, 19]). In Section 5, we fix some notations for quantum affine algebras, and in Section 6, we recall the *path realization* of crystal graphs for quantum affine algebras using the notion of *perfect crystals* ([13, 14]). In Section 7, we discuss the surprising connection between the crystal basis theory for the quantum affine algebra $U_q(\widehat{sl_n})$ and the modular representation theory of Hecke algebras ([1, 6, 26]). In Section 8, we explain the combinatorics of Young walls using the example of the quantum affine algebra $U_q(B_n^{(1)})$ and give a realization of the crystal graph for the basic representations in terms of *reduced proper Young walls.* In Section 9, we give a construction of the *Fock space representation* of quantum affine algebras using combinatorics of Young walls ([16, 17]).

This paper is based on my talks given at the International Conference on Representation Theory of Algebraic Groups and Quantum Groups which was held at Sophia University, Tokyo, Japan, in August 2001. I would like to express my sincere gratitude to the organizing committee of the conference and the staff members for their hospitality and cooperation.

§2. Kac-Moody algebras

Let I be a finite index set. A square matrix $A = (a_{ij})_{i,j \in I}$ is called a *generalized Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ for all $i \in I$, (ii) $a_{ij} \in \mathbf{Z}_{<0}$ for all $i, j \in I$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. In this paper, we assume that A is *symmetrizable*; i.e., there is a diagonal matrix $D = \mathrm{diag}(s_i \in \mathbf{Z}_{>0} | \, i \in I)$ with positive integral entries such that DA is symmteric.

Consider a free abelian group

$$P^\vee = \left(\bigoplus_{i \in I} \mathbf{Z} h_i \right) \oplus \left(\bigoplus_{j=1}^{\mathrm{corank}\, A} \mathbf{Z} d_j \right),$$

and let $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$. The free abelian group $P^\vee$ is called the *dual weight lattice* and the $\mathbf{Q}$-vector space $\mathfrak{h}$ is called the *Cartan subalgebra.* The *weight lattice* and the set of *simple coroots* are defined to be

$$P = \{\lambda \in \mathfrak{h}^* | \, \lambda(P^\vee) \subset \mathbf{Z}\}, \qquad \Pi^\vee = \{h_i | \, i \in I\}.$$

We denote by $\Pi = \{\alpha_i | \, i \in I\}$ the set of *simple roots*, which is a linearly independent subset of $\mathfrak{h}^*$ satisfying

$$\alpha_i(h_j) = a_{ji} \quad \text{for all} \quad i, j \in I.$$

Definition 2.1. The quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ defined above is called a *Cartan datum* associated with A.

We denote by $P^+ = \{\lambda \in P | \ \lambda(h_i) \geq 0 \ \text{ for all } i \in I\}$ the set of *dominant integral weights.* The free abelian group $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ is called the *root lattice.* We set $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. There is a partial ordering on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. Since the generalized Cartan matrix A is symmetrizable, there is a nondegenerate symmetric bilinear form $(\ | \ )$ on $\mathfrak{h}^*$ satisfying

$$s_i = \frac{(\alpha_i|\alpha_i)}{2} \quad \text{and} \quad \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = a_{ij} \quad \text{for all } \ i,j \in I.$$

Definition 2.2. The *Kac-Moody algebra* $\mathfrak{g}$ associated with a Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the Lie algebra over $\mathbf{Q}$ generated by the elements e_i, f_i $(i \in I)$ and $h \in \mathfrak{h}$ subject to the defining relations:

$$\begin{gathered}
[h, h'] = 0, \quad [e_i, f_j] = \delta_{ij} h_i \quad \text{for all } \ h, h' \in \mathfrak{h}, \ i, j \in I, \\
[h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i \quad \text{for all } \ h \in \mathfrak{h}, \ i \in I, \\
(\mathrm{ad} e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } \ i \neq j.
\end{gathered}$$

Let $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) be the subalgebra of $\mathfrak{g}$ generated by e_i (resp. f_i) for $i \in I$. Then we have the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition* $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V | \ hv = \mu(h)v \ \text{ for all } h \in \mathfrak{h}\}$. If $\dim_{\mathbf{Q}} V_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$, we define the *character* of V by

$$\mathrm{ch} V = \sum_{\mu\in\mathfrak{h}^*} (\dim_{\mathbf{Q}} V_\mu) e^\mu,$$

where e^μ are basis elements of the group algebra $\mathbf{Q}[\mathfrak{h}^*]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in \mathfrak{h}^*$.

A weight module V over $\mathfrak{g}$ is called a *highest weight module* with highest weight λ if there exists a non-zero vector $v_\lambda \in V$ such that (i) $V = U(\mathfrak{g})v_\lambda$, (ii) $h v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$, (iii) $e_i v_\lambda = 0$ for all $i \in I$. For example, let $J(\lambda)$ denote the left ideal of $U(\mathfrak{g})$ generated by e_i, $h - \lambda(h)1$ for $i \in I$, $h \in \mathfrak{h}$, and set $M(\lambda) = U(\mathfrak{g})/J(\lambda)$. Then, via left multiplication, $M(\lambda)$ becomes a highest weight $\mathfrak{g}$-module with highest weight λ. The $\mathfrak{g}$-module $M(\lambda)$ is called the *Verma module*, and it satisfies the following properties:

Proposition 2.3 ([11]).

(a) *$M(\lambda)$ is a free $U(\mathfrak{g}^-)$-module of rank 1.*

(b) *Evey highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.*

(c) *$M(\lambda)$ contais a unique maximal submodule $R(\lambda)$.*

The irreducible quotient $V(\lambda) = M(\lambda)/R(\lambda)$ is called the *irreducible highest weight $\mathfrak{g}$-module with highest weight λ.*

Definition 2.4. The *category $\mathcal{O}_{int}$* consists of $\mathfrak{g}$-modules M satisfying the following properties:

(i) $M = \bigoplus_{\mu \in P} M_\mu$, where $M_\mu = \{v \in M |\ hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$,

(ii) there exist finitely many $\lambda_1, \cdots, \lambda_s \in P$ such that

$$\mathrm{wt}(M) := \{\mu \in P|\ M_\mu \neq 0\} \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+),$$

(iii) e_i and f_i $(i \in I)$ are locally nilpotent on M.

The basic properties of the category $\mathcal{O}_{int}$ are given in the following proposition.

Proposition 2.5.

(a) *For each $i \in I$, let $\mathfrak{g}_{(i)}$ be the subalgebra of $\mathfrak{g}$ generated by e_i, f_i, h_i, which is isomorphic to the 3-dimensional simple Lie algebra sl_2. Then every $\mathfrak{g}$-module M in the category $\mathcal{O}_{int}$ is a direct sum of finite dimensional irreducible $\mathfrak{g}_{(i)}$-submodules.*

(b) *The category $\mathcal{O}_{int}$ is semisimple. Moreover, every irreducible object in the category $\mathcal{O}_{int}$ has the form $V(\lambda)$ with $\lambda \in P^+$.*

§3. Quantum groups

Let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be a Cartan datum associated with a symmetrizable generalized Cartan matrix A. For an indeterminate q, set $q_i = q^{s_i}$ and define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

We will also use the notation $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$.

Definition 3.1. The *quantum group* $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ subject to the following defining relations:

$$
\begin{aligned}
& q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad (h, h' \in P^\vee), \\
& q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee,\ i \in I), \\
& e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where} \quad K_i = q^{s_i h_i}, \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0 \qquad (i \neq j), \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0 \qquad (i \neq j).
\end{aligned}
$$

The quantum group $U_q(\mathfrak{g})$ has a *Hopf algebra* structure with the comultiplication Δ, counit ε, and antipode S defined by

$$
\begin{gathered}
\Delta(q^h) = q^h \otimes q^h, \\
\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \\
\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
S(q^h) = q^{-h}, \quad S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i
\end{gathered}
$$

for $h \in P^\vee$ and $i \in I$.

Let U^+ (resp. U^-) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (resp. f_i) for $i \in I$, and let U^0 be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h $(h \in P^\vee)$. Then we have the *triangular decomposition*:

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+.$$

A $U_q(\mathfrak{g})$-module V^q is called a *weight module* if it admits a *weight space decomposition* $V^q = \bigoplus_{\mu \in P} V_\mu$, where $V_\mu^q = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$. If $\dim_{\mathbf{Q}(q)} V_\mu^q < \infty$ for all $\mu \in P$, we define the *character* of V^q by

$$\mathrm{ch} V = \sum_{\mu \in P} \left(\dim_{\mathbf{Q}(q)} V_\mu\right) e^\mu,$$

where e^μ are basis elements of the group algebra $\mathbf{Q}(q)[P]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in P$.

A weight module V^q over $U_q(\mathfrak{g})$ is called a *highest weight module with highest weight* λ if there exists a non-zero vector $v_\lambda \in V^q$ (called the *highest weight vector*) such that (i) $V^q = U_q(\mathfrak{g})v_\lambda$, (ii) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$, (iii) $e_i v_\lambda = 0$ for all $i \in I$.

We construct the *Verma module* over $U_q(\mathfrak{g})$ as in the Kac-Moody algebra case. Let $J^q(\lambda)$ denote the left ideal of $U_q(\mathfrak{g})$ generated by e_i, $q^h - q^{\lambda(h)}1$ for $i \in I$, $h \in P^\vee$, and set $M^q(\lambda) = U_q(\mathfrak{g})/J^q(\lambda)$. Then, via left multiplication, $M^q(\lambda)$ becomes a highest weight $U_q(\mathfrak{g})$-module with highest weight λ called the Verma module.

Also, as in the Kac-Moody algebra case, we have:

Proposition 3.2 (cf. [8]).

(a) *$M^q(\lambda)$ is a free U^--module of rank 1.*

(b) *Evey highest weight $U_q(\mathfrak{g})$-module with highest weight λ is a homomorphic image of $M^q(\lambda)$.*

(c) *$M^q(\lambda)$ contains a unique maximal submodule $R^q(\lambda)$.*

The irreducible quotient $V^q(\lambda) = M^q(\lambda)/R^q(\lambda)$ is called the *irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ.*

Definition 3.3. The *category $\mathcal{O}^q_{int}$* consists of $U_q(\mathfrak{g})$-modules M^q satisfying the following properties:

(i) $M^q = \bigoplus_{\mu \in P} M^q_\mu$, where $M^q_\mu = \{v \in M^q \mid q^h v = q^{\mu(h)} v$ for all $h \in P^\vee\}$,

(ii) there exist finitely many $\lambda_1, \cdots, \lambda_s \in P$ such that

$$\mathrm{wt}(M^q) := \{\mu \in P \mid M^q_\mu \neq 0\} \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+),$$

(iii) e_i and f_i $(i \in I)$ are locally nilpotent on M^q.

Proposition 3.4.

(a) *For each $i \in I$, let $U_{(i)}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$, which is isomorphic to the quantum group $U_{q_i}(sl_2)$. Then every $U_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{int}$ is a direct sum of finite dimensional irreducible $U_{(i)}$-submodules.*

(b) *The category $\mathcal{O}^q_{int}$ is semisimple. Moreover, every irreducible object in the category $\mathcal{O}^q_{int}$ has the form $V^q(\lambda)$ with $\lambda \in P^+$.*

Let $\mathbf{A}_1 = \{f/g \in \mathbf{Q}(q) \mid f, g \in \mathbf{Q}[q],\ g(1) \neq 0\}$ be the subring $\mathbf{Q}(q)$ consisting of the rational functions in q that are regular at $q = 1$. We define the $\mathbf{A}_1$-*form* of the quantum group $U_q(\mathfrak{g})$ to be the $\mathbf{A}_1$-subalgebra $U_{\mathbf{A}_1}$ generated by the elements e_i, f_i, q^h, $\dfrac{q^h - 1}{q - 1}$ for $h \in P^\vee$, $i \in I$.

Similarly, for the irreducible highest weight module $V^q(\lambda) = U_q(\mathfrak{g})v_\lambda$ and the Verma module $M^q(\lambda) = U_q(\mathfrak{g})u_\lambda$ over $U_q(\mathfrak{g})$, we define their $\mathbf{A}_1$-*forms* by

$$V^q(\lambda)_{\mathbf{A}_1} = U_{\mathbf{A}_1}v_\lambda, \qquad M^q(\lambda)_{\mathbf{A}_1} = U_{\mathbf{A}_1}u_\lambda.$$

Let $\mathbf{J}_1$ be the unique maximal ideal of $\mathbf{A}_1$ generated by $q-1$ and consider the isomorphism of fields

$$\mathbf{A}_1/\mathbf{J}_1 \xrightarrow{\sim} \mathbf{Q} \qquad \text{given by} \qquad f(q) + \mathbf{J}_1 \longmapsto f(1).$$

(In particular, q is mapped onto 1.) Define the $\mathbf{Q}$-linear vector spaces

$$\begin{aligned} U_1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} U_{\mathbf{A}_1}, \\ V^1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} V^q(\lambda)_{\mathbf{A}_1}, \\ M^1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} M^q(\lambda)_{\mathbf{A}_1}. \end{aligned}$$

They are called the *classical limits* of $U_q(\mathfrak{g})$, $V^q(\lambda)$ and $M^q(\lambda)$, respectively.

As we can see in the following proposition, the structure of the quantum group $U_q(\mathfrak{g})$ *tends to* that of $U(\mathfrak{g})$ as $q \to 1$. Similarly, as $q \to 1$, the structure of the irreducible highest weight module $V^q(\lambda)$ (resp. the Verma module $M^q(\lambda)$) over $U_q(\mathfrak{g})$ with highest weight $\lambda \in P^+$ *tends to* that of $V(\lambda)$ (resp. $M(\lambda)$) over $U(\mathfrak{g})$. Moreover, the dimensions of the weight spaces are invariant under the deformation.

Proposition 3.5 ([8, 24]).
(a)

$$U_1 \cong U(\mathfrak{g}), \qquad V^1 \cong V(\lambda), \qquad M^1 \cong M(\lambda).$$

(b)

$$\begin{aligned} \dim_{\mathbf{Q}(q)} V^q(\lambda)_\mu &= \dim_{\mathbf{Q}} V(\lambda)_\mu, \\ \dim_{\mathbf{Q}(q)} M^q(\lambda)_\mu &= \dim_{\mathbf{Q}} M(\lambda)_\mu \quad \textit{for all} \quad \mu \in P. \end{aligned}$$

§4. Crystal bases

In this section, We briefly review the *crystal basis theory* for quantum groups developed by Kashiwara ([18, 19]). For simplicity, we will drop the superscript q from $U_q(\mathfrak{g})$-modules. Fix an index $i \in I$ and let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$. By the representation theory of $U_q(sl_2)$, every element $v \in M_\lambda$ can be written uniquely as

$$v = \sum_{k\geq 0} f_i^{(k)} v_k,$$

where $k \geq -\lambda(h_i)$ and $v_k \in \ker e_i \cap M_{\lambda+k\alpha_i}$. We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on M, called the *Kashiwara operators*, by

$$\tilde{e}_i v = \sum_{k\geq 1} f_i^{(k-1)} v_k, \qquad \tilde{f}_i v = \sum_{k\geq 0} f_i^{(k+1)} v_k.$$

Let $\mathbf{A}_0 = \{f/g \in \mathbf{Q}(q) \mid f, g \in \mathbf{Q}[q],\ g(0) \neq 0\}$ be the subring of $\mathbf{Q}(q)$ consisting of the rational functions in q that are regular at $q = 0$.

Definition 4.1. A *crystal basis* of M is a pair (L, B), where
(i) L is a free $\mathbf{A}_0$-submodule M such that $M \cong \mathbf{Q}(q) \otimes_{\mathbf{A}_0} L$,
(ii) B is a basis of the $\mathbf{Q}$-vector space L/qL,
(iii) $L = \bigoplus_{\lambda\in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$,
(iv) $B = \bigsqcup_{\lambda\in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
(v) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all $i \in I$,
(vi) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$,
(vii) for $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

The set B is given a colored oriented graph structure with the arrows defined by

$$b \xrightarrow{\ i\ } b' \quad \text{if and only if} \quad \tilde{f}_i b = b'.$$

The graph B is called the *crystal graph* of M and it reflects the combinatorial structure of M. For instance, we have

$$\dim_{\mathbf{Q}(q)} M_\lambda = \# B_\lambda \quad \text{for all} \quad \lambda \in P.$$

Let B be a crystal graph for a $U_q(\mathfrak{g})$-module M in the category $\mathcal{O}_{int}^q$. For each $b \in B$ and $i \in I$, we define

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in B\}, \qquad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\}.$$

Then the crystal graph B satisfies the following properties.

Proposition 4.2 ([19, 20, 21]).
(a) *For all $i \in I$ and $b \in B$, we have*

$$\begin{aligned}\varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle,\\ \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i,\\ \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i.\end{aligned}$$

(b) *If* $\tilde{e}_i b \in B$, *then*

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.$$

(c) *If* $\tilde{f}_i b \in B$, *then*

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1.$$

Moreover, the crystal bases have extremely simple behavior with respect to taking the tensor product.

Proposition 4.3 ([18, 19]).
Let M_j $(j = 1, 2)$ *be a* $U_q(\mathfrak{g})$*-module in the category* $\mathcal{O}^q_{int}$ *and let* (L_j, B_j) *be its crystal basis. Set*

$$L = L_1 \otimes_{\mathbf{A}_0} L_2, \quad B = B_1 \times B_2.$$

Then (L, B) *is a crystal basis of* $M_1 \otimes_{\mathbf{Q}(q)} M_2$ *with the Kashiwara operators on* B *given by*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

The *tensor product rule* in Proposition 4.3 gives a very convenient combinatorial description of the action of Kashiwara operators on the multi-fold tensor product of crystal graphs. Let M_j be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ with a crystal basis (L_j, B_j) $(j = 1, \cdots, N)$. Fix an index $i \in I$ and consider a vector $b = b_1 \otimes \cdots \otimes b_N \in B_1 \otimes \cdots \otimes B_N$. To each $b_j \in B_j$ $(j = 1, \cdots, N)$, we assign a sequence of $-$'s and $+$'s with as many $-$'s as $\varepsilon_i(b_j)$ followed by as many $+$'s as $\varphi_i(b_j)$:

$$\begin{aligned} b &= b_1 \otimes b_2 \otimes \cdots \otimes b_N \\ &\longmapsto (\underbrace{-, \cdots, -}_{\varepsilon_i(b_1)}, \underbrace{+, \cdots, +}_{\varphi_i(b_1)}, \cdots\cdots, \underbrace{-, \cdots, -}_{\varepsilon_i(b_N)}, \underbrace{+, \cdots, +}_{\varphi_i(b_N)}). \end{aligned}$$

In this sequence, we cancel out all the $(+, -)$-pairs to obtain a sequence of $-$'s followed by $+$'s:

$$i\text{-sgn}(b) = (-, -, \cdots, -, +, +, \cdots, +).$$

The sequence i-sgn(b) is called the i*-signature* of b.

Now the tensor product rule tells that $\tilde{e}_i$ acts on b_j corresponding to the rightmost $-$ in the i-signature of b and $\tilde{f}_i$ acts on b_k corresponding to the leftmost $+$ in the i-signature of b:

$$\tilde{e}_i b = b_1 \otimes \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_N,$$
$$\tilde{f}_i b = b_1 \otimes \cdots \otimes \tilde{f}_i b_k \otimes \cdots \otimes b_N.$$

We define $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) if there is no $-$ (resp. $+$) in the i-signature of b.

We close this section with the existence and uniquenes theorem for crystal bases.

Theorem 4.4 ([19]).

(a) *Let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$ and highest weight vector v_λ. Let $L(\lambda)$ be the free $\mathbf{A}_0$-submodule of $V(\lambda)$ spanned by the vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda$ $(i_k \in I, r \in \mathbf{Z}_{\geq 0})$ and set*

$$B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda)\} \setminus \{0\}.$$

Then $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$ and every crystal basis of $V(\lambda)$ is isomorphic to $(L(\lambda), B(\lambda))$.

(b) *Define a $\mathbf{Q}$-algebra automorphism of $U_q(\mathfrak{g})$ by*

$$\overline{q} = q^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h} \quad \text{for } i \in I, h \in P^\vee.$$

Let $\mathbf{A} = \mathbf{Q}[q, q^{-1}]$ and define $V(\lambda)_\mathbf{A} = U_\mathbf{A}^- v_\lambda$, where $U_\mathbf{A}^-$ is the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)}$ $(i \in I, n \in \mathbf{Z}_{\geq 0})$. Then there exists a unique $\mathbf{A}$-basis $G(\lambda) = \{G(b) \mid b \in B(\lambda)\}$ of $V(\lambda)_\mathbf{A}$ such that

$$\overline{G(b)} = G(b), \quad G(b) \equiv b \mod qL(\lambda) \quad \text{for all } b \in B(\lambda).$$

The basis $G(\lambda)$ of $V(\lambda)$ given in Theorem 4.4 is called the *global basis* or the *canonical basis* of $V(\lambda)$ associated with the crystal graph $B(\lambda)$ ([19, 25]).

§5. Quantum affine algebras

Let $I = \{0, 1, \cdots, n\}$ be an index set and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of affine type. We denote by

$$P^\vee = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_n \oplus \mathbf{Z}d$$

the dual weight lattice and $\Pi^\vee = \{h_i \mid i \in I\}$ the simple coroots. The simple roots α_i and the fundamental weights Λ_i are given by

$$\begin{aligned} &\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0,i}, \\ &\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \qquad (i, j \in I). \end{aligned}$$

We define the *affine weight lattice* to be

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbf{Z}\}.$$

The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called an *affine Cartan datum.* To each affine Cartan datum, we can associate an infinite dimensional Lie algebra $\mathfrak{g}$ called the *affine Kac-Moody algebra* ([11]). The center of the affine Kac-Moody algebra $\mathfrak{g}$ is 1-dimensional and is generated by the *canonical central element*

$$c = c_0 h_0 + c_1 h_1 + \cdots + c_n h_n.$$

Moreover, the imaginary roots of $\mathfrak{g}$ are nonzero integral multiples of the *null root*

$$\delta = d_0 \alpha_0 + d_1 \alpha_1 + \cdots + d_n \alpha_n.$$

Here, c_i and d_i $(i \in I)$ are the non-negative integers given in [11].

Using the fundamental weights and the null root, the affine weight lattice can be written as

$$P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_n \oplus \mathbf{Z}\delta.$$

Set

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbf{Z}_{\geq 0} \quad \text{for all} \quad i \in I\}.$$

The elements of P (resp. P^+) are called the *affine weights* (resp. *affine dominant integral weights*). The *level* of an affine dominant integral weight $\lambda \in P^+$ is defined to be the nonnegative integer $\lambda(c)$.

Definition 5.1. The *quantum affine algebra* $U_q(\mathfrak{g})$ is the quantum group associated with the affine Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$.

The subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$ $(i \in I)$ is denoted by $U_q'(\mathfrak{g})$, and is also called the *quantum affine algebra.*

Let

$$\overline{P}^\vee = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_n \quad \text{and} \quad \overline{\mathfrak{h}} = \mathbf{Q} \otimes_{\mathbf{Z}} \overline{P}^\vee.$$

Consider α_i and Λ_i $(i \in I)$ as linear functionals on $\overline{\mathfrak{h}}$, and set

$$\overline{P} = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_n.$$

The elements of $\overline{P}$ are called the *classical weights.* The algebra $U_q'(\mathfrak{g})$ can be regarded as the quantum affine algebra associated with the *classical Cartan datum* $(A, \Pi, \Pi^\vee, \overline{P}, \overline{P}^\vee)$.

The projection $\mathrm{cl} : P \longrightarrow \overline{P}$ will be denoted by $\lambda \longmapsto \overline{\lambda}$ and we will fix an embedding $\mathrm{aff} : \overline{P} \longrightarrow P$ such that

$$\mathrm{cl} \circ \mathrm{aff} = \mathrm{id}, \qquad \mathrm{aff} \circ \mathrm{cl}(\alpha_i) = \alpha_i \quad \text{for} \quad i \neq 0.$$

We define

$$\overline{P}^+ = \mathrm{cl}(P^+) = \{\lambda \in \overline{P} |\ \lambda(h_i) \geq 0 \ \text{ for all } \ i \in I\}.$$

The elements of $\overline{P}^+$ are called the *classical dominant integral weights.* A classical dominant integral weight $\lambda \in \overline{P}^+$ is said to have *level* $l \in \mathbf{Z}_{\geq 0}$ if $\lambda(c) = l$. Note that it has the same level as its affine counterpart.

§6. Perfect crystals and paths

By extracting properties of the crystal graphs, we define the notion of abstract *crystals* as follows ([20, 21]).

Definition 6.1. An *affine crystal* (resp. *classical crystal*) is a set B together with the maps $\mathrm{wt} : B \to P$ (resp. $\mathrm{wt} : B \to \overline{P}$), $\varepsilon_i : B \to \mathbf{Z} \cup \{-\infty\}$, $\varphi_i : B \to \mathbf{Z} \cup \{-\infty\}$, $\tilde{e}_i : B \to B \cup \{0\}$, and $\tilde{f}_i : B \to B \cup \{0\}$ satisfying the following conditions:

(i) for all $i \in I$, $b \in B$, we have

$$\begin{aligned} \varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle, \\ \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i, \\ \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i, \end{aligned}$$

(ii) if $\tilde{e}_i b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1,$$

(iii) if $\tilde{f}_i b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1,$$

(iv) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for all $i \in I$, $b, b' \in B$,

(v) if $\varepsilon_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

For instance, the crystal graphs for $U_q(\mathfrak{g})$-modules (resp. $U'_q(\mathfrak{g})$-modules) in the category $\mathcal{O}^q_{int}$ are affine crystals (resp. classical crystals).

Definition 6.2. Let B_1 and B_2 be (affine or classical) crystals. A *morphism* $\psi : B_1 \to B_2$ of crystals is a map $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ satisfying the conditions:

(i) $\psi(0) = 0$,

(ii) if $b \in B_1$ and $\psi(b) \in B_2$, then

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b),$$

(iii) if $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \psi(b) = \psi(b')$.

A morphism of crystals is said to be *strict* if it commutes with the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$.

Definition 6.3. The *tensor product* $B_1 \otimes B_2$ of the crystals B_1 and B_2 is defined to be the set $B_1 \times B_2$ whose crystal structure is given by

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle), \\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varepsilon_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle), \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}
\end{aligned}$$

Here, we denote $b_1 \otimes b_2 = (b_1, b_2)$ and use the convention that $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

We now define the notion of *perfect crystals*. Let B be a classical crystal. For $b \in B$, we define

$$\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i.$$

Note that

$$\mathrm{wt}(b) = \varphi(b) - \varepsilon(b).$$

For a positive integer $l > 0$, set

$$\overline{P}_l^+ = \{\lambda \in \overline{P}^+ \mid \langle c, \lambda\rangle = l\}. \tag{6.1}$$

Definition 6.4. For $l \in \mathbf{Z}_{>0}$, we say that a finite classical crystal $\mathbf{B}$ is a *perfect crystal of level* l if

(i) there is a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal basis whose crystal graph is isomorphic to $\mathbf{B}$,

(ii) $\mathbf{B} \otimes \mathbf{B}$ is connected,

(iii) there exists some $\lambda_0 \in \overline{P}$ such that

$$\mathrm{wt}(\mathbf{B}) \subset \lambda_0 + \frac{1}{d_0} \sum_{i \neq 0} \mathbf{Z}_{\leq 0} \alpha_i, \qquad \#(\mathbf{B}_{\lambda_0}) = 1,$$

(iv) for any $b \in \mathbf{B}$, we have $\langle c, \varepsilon(b) \rangle \geq l$,

(v) for each $\lambda \in \overline{P}_l^+$, there exist unique $b^\lambda \in \mathbf{B}$ and $b_\lambda \in \mathbf{B}$ such that

$$\varepsilon(b^\lambda) = \lambda, \qquad \varphi(b_\lambda) = \lambda.$$

A finite dimensional $U_q'(\mathfrak{g})$-module $\mathbf{V}$ is called a *perfect representation* of *level* l if it has a crystal basis (L, B) such that B is isomorphic to a perfect crystal of level l.

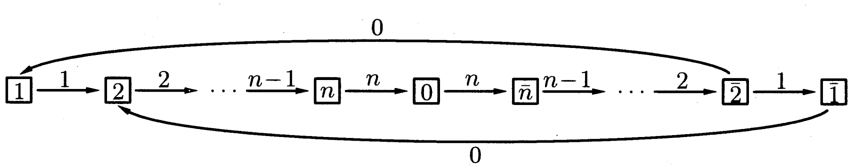
Remark. For a perfect crystal $\mathbf{B}$ of level l, define

$$\mathbf{B}^{\min} = \{ b \in \mathbf{B} \mid \langle c, \varepsilon(b) \rangle = l \}.$$

Then the maps $\varepsilon, \varphi : \mathbf{B}^{\min} = \{ b \in B \mid \langle c, \varepsilon(b) \rangle = l \} \longrightarrow \overline{P}_l^+$ are bijective.

In the following, we give an example of perfect crystals of level 1 for the quantum affine algebra $U_q(B_n^{(1)})$ $(n \geq 3)$.

Example 6.5.



Here, we have

$$b^{\Lambda_0} = \boxed{1}, \quad b_{\Lambda_0} = \boxed{\bar{1}}; \quad b^{\Lambda_1} = \boxed{\bar{1}}, \quad b_{\Lambda_1} = \boxed{1};$$
$$b^{\Lambda_n} = \boxed{0}, \quad b_{\Lambda_n} = \boxed{0}.$$

Fix a positive integer $l > 0$ and let $\mathbf{B}$ be a perfect crystal of level l. By definition, for any classical dominant integral weight $\lambda \in \overline{P}_l^+$, there exists a unique element $b_\lambda \in \mathbf{B}$ such that $\varphi(b_\lambda) = \lambda$. Set

$$\mu = \lambda - \mathrm{wt}(b_\lambda) = \varepsilon(b_\lambda),$$

and denote by u_μ the highest weight vector of the crystal graph $B(\mu)$. Then, using the fact that $\mathbf{B}$ is perfect, one can show that the vector $u_\mu \otimes b_\lambda$ is the unique maximal vector in $B(\mu) \otimes \mathbf{B}$. Moreover, we have:

Theorem 6.6 ([13]). *Let* $\mathbf{B}$ *be a perfect crystal of level* $l > 0$. *Then for any dominant integral weight* $\lambda \in \overline{P}_l^+$, *there exists a crystal isomorphism*

$$\Psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes \mathbf{B} \quad \textit{given by} \quad u_\lambda \longmapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where b_λ *is the unique element in* $\mathbf{B}$ *such that* $\varphi(b_\lambda) = \lambda$.

Set

$$\lambda_0 = \lambda, \quad \lambda_{k+1} = \varepsilon(b_{\lambda_k}),$$

and

$$b_0 = b_\lambda, \quad b_{k+1} = b_{\lambda_{k+1}}.$$

By taking the composition of crystal isomorphism given in Theorem 6.6 repeatedly, we obtain a sequence of crystal isomorphisms

$$B(\lambda) \xrightarrow{\sim} B(\lambda_1) \otimes \mathbf{B} \xrightarrow{\sim} B(\lambda_2) \otimes \mathbf{B} \otimes \mathbf{B} \xrightarrow{\sim} \cdots\cdots$$

given by

$$u_\lambda \longmapsto u_{\lambda_1} \otimes b_0 \longmapsto u_{\lambda_2} \otimes b_1 \otimes b_0 \longmapsto \cdots\cdots,$$

which yields the infinite sequences

$$\mathbf{w}_\lambda = (\lambda_k)_{k=0}^\infty = (\cdots, \lambda_{k+1}, \lambda_k, \cdots, \lambda_1, \lambda_0) \quad \text{in} \quad (\overline{P}_l^+)^\infty$$

and

$$\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0 \quad \text{in} \quad \mathbf{B}^{\otimes\infty}.$$

Thus for each $k \geq 1$, we get a crystal isomorphism

$$\Psi_k : B(\lambda) \xrightarrow{\sim} B(\lambda_k) \otimes \mathbf{B}^{\otimes k}$$

given by

$$u_\lambda \longmapsto u_{\lambda_k} \otimes b_{k-1} \otimes \cdots \otimes b_1 \otimes b_0.$$

Since $\mathbf{B}$ is perfect, we have $\varphi(b_j) = \lambda_j$ and $\varepsilon(b_j) = \lambda_{j+1}$. It follows that the sequences

$$\mathbf{w}_\lambda = (\lambda_k)_{k=0}^{\infty} = (\cdots, \lambda_{k+1}, \lambda_k, \cdots, \lambda_1, \lambda_0)$$

and

$$\mathbf{p}_\lambda = (b_k)_{k=0}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$$

are periodic with the same period. That is, there is a positive integer $N > 0$ such that $\lambda_{j+N} = \lambda_j$, $b_{j+N} = b_j$ for all $j = 0, 1, \cdots, N-1$.

Definition 6.7.
(a) The sequence

$$\mathbf{p}_\lambda = (b_k)_{k=0}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$$

is called the *ground-state path* of weight λ.
(b) A λ-*path* in $\mathbf{B}$ is a sequence

$$\mathbf{p} = (\mathbf{p}(k))_{k=0}^{\infty} = \cdots \otimes \mathbf{p}(k) \otimes \cdots \otimes \mathbf{p}(1) \otimes \mathbf{p}(0)$$

such that $\mathbf{p}(k) = b_k$ for all $k \gg 0$.

Let $\mathbf{P}(\lambda) = \mathbf{P}(\lambda, \mathbf{B})$ be the set of all λ-paths in $\mathbf{B}$. We define the $U_q'(\mathfrak{g})$-crystal structure on $\mathbf{P}(\lambda)$ as follows. Let $\mathbf{p} = (\mathbf{p}(k))_{k=0}^{\infty}$ be a λ-path in $\mathbf{P}(\lambda)$ and let $N > 0$ be a positive integer such that $\mathbf{p}(k) = b_k$ for all $k \geq N$. For each $i \in I$, we define

$$\begin{aligned}
\overline{\mathrm{wt}}(\mathbf{p}) &= \lambda_N + \textstyle\sum_{k=0}^{N-1} \overline{\mathrm{wt}}\mathbf{p}(k), \\
\tilde{e}_i\mathbf{p} &= \cdots \otimes \mathbf{p}(N+1) \otimes \tilde{e}_i(\mathbf{p}(N) \otimes \cdots \otimes \mathbf{p}(0)), \\
\tilde{f}_i\mathbf{p} &= \cdots \otimes \mathbf{p}(N+1) \otimes \tilde{f}_i(\mathbf{p}(N) \otimes \cdots \otimes \mathbf{p}(0)), \\
\varepsilon_i(\mathbf{p}) &= \max(\varepsilon_i(\mathbf{p}') - \varphi_i(b_N), 0), \\
\varphi_i(\mathbf{p}) &= \varphi_i(\mathbf{p}') + \max(\varphi_i(b_N) - \varepsilon_i(\mathbf{p}'), 0),
\end{aligned}$$

where $\mathbf{p}' = \mathbf{p}(N-1) \otimes \cdots \otimes \mathbf{p}(1) \otimes \mathbf{p}(0)$.
Then we have the *path realization* of the classical crystal $B(\lambda)$:

Theorem 6.8 ([13]).
(a) *The maps* $\overline{wt} : \mathbf{P}(\lambda) \to \overline{P}$, $\tilde{e}_i, \tilde{f}_i : \mathbf{P}(\lambda) \to \mathbf{P}(\lambda) \cup \{0\}$, $\varepsilon_i, \varphi_i : \mathbf{P}(\lambda) \to \mathbf{Z}$ *define a classical crystal structure on* $\mathbf{P}(\lambda)$.
(b) *There exists an isomorphism of classical crystals*

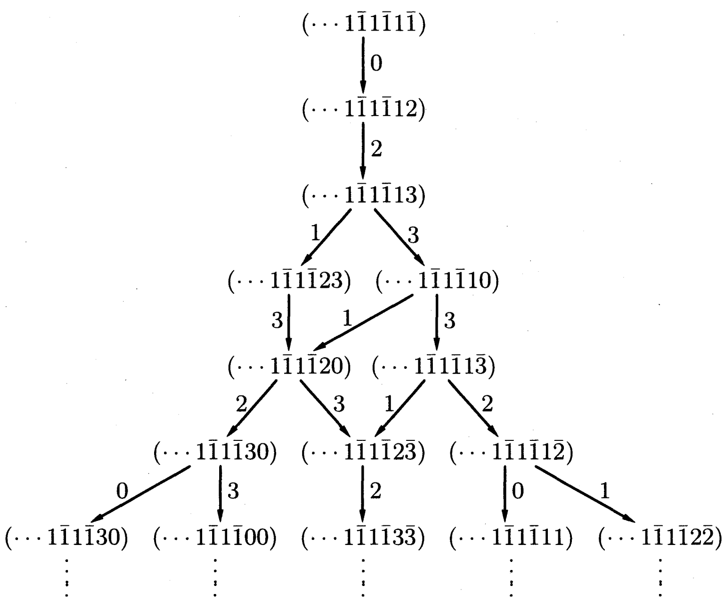
$$\Psi : B(\lambda) \xrightarrow{\sim} \mathbf{P}(\lambda) \quad \textit{given by} \quad u_\lambda \longmapsto \mathbf{p}_\lambda.$$

In the following example, we give the ground-state paths for the basic representations of the quantum affine algebra $U_q(B_3^{(1)})$ and illustrate the top part of their crystal graphs.

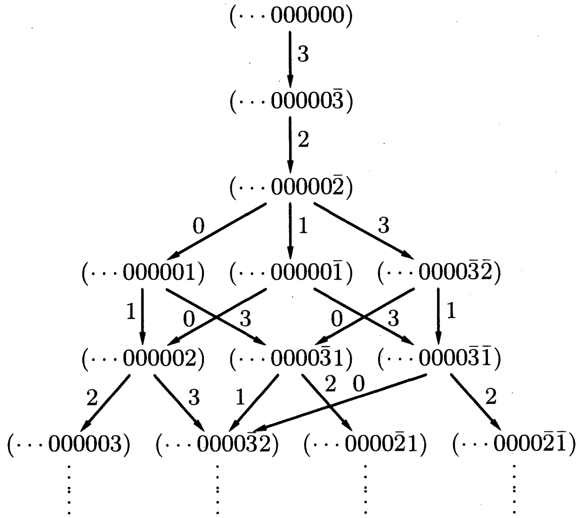
Example 6.9.
(a) Ground-state paths:

$$
\begin{aligned}
\mathbf{p}_{\Lambda_0} &= (\mathbf{p}_{\Lambda_0}(k))_{k=0}^{\infty} = (\cdots, 1, \bar{1}, 1, \bar{1}, 1, \bar{1}),\\
\mathbf{p}_{\Lambda_1} &= (\mathbf{p}_{\Lambda_1}(k))_{k=0}^{\infty} = (\cdots, \bar{1}, 1, \bar{1}, 1, \bar{1}, 1),\\
\mathbf{p}_{\Lambda_n} &= (\mathbf{p}_{\Lambda_n}(k))_{k=0}^{\infty} = (\cdots, 0, 0, 0, 0, 0, 0)
\end{aligned}
$$

(b) Crystal graph $\mathbf{P}(\Lambda_0)$



(c) Crystal graph $\mathbf{P}(\Lambda_3)$



§7. Hecke algebras and crystal bases

Let $U_q(\widehat{sl_n})$ be the quantum affine algebra associated with the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$, where $A = (a_{ij})_{i,j=1}^{n-1}$ is the generalized Cartan matrix of affine type $A_{n-1}^{(1)}$,

$$P^\vee = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_{n-1} \oplus \mathbf{Z}d,$$
$$P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1} \oplus \mathbf{Z}\delta,$$
$$\Pi^\vee = \{h_0, h_1, \cdots, h_{n-1}\},$$
$$\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}.$$

Here, Λ_i are the *fundamental weights* defined by

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad \text{for} \quad i = 0, 1, \cdots, n-1,$$

and $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$ is the *null root.*

In [26], Misra and Miwa gave a realization of the crystal graph $B(\Lambda_0)$ for the basic representation $V(\Lambda_0)$ of the quantum affine algebra $U_q(\widehat{sl_n})$

in terms of *n-reduced colored Young diagrams.* More precisely, we will build a colored Young diagram $Y = (y_k)_{k=0}^{\infty}$ with each box colored by $0, 1, \cdots, n-1$ following the pattern given below:

					0	1
					$n-1$	0
					$n-2$	$n-1$
					$\vdots$	$\vdots$
0	1	2	3	$\cdots$	0	1
$n-1$	0	1	2	$\cdots$	$n-1$	0

Here, y_k $(k = 0, 1, 2, \cdots)$ denotes the k-th column of Y reading from right to left. The heights of the columns of Y are weakly decreasing as we proceed from right to left and we have $y_k = 0$ for $k \gg 0$.

A colored Young diagram $Y = (y_k)_{k=0}^{\infty}$ is called *n-reduced* if $y_k - y_{k+1} < n$ for all $k \geq 0$. Let $\mathcal{Y}(\Lambda_0)$ be the set of all n-reduced colored Young diagrams. We will define the Kashiwara operators on $\mathcal{Y}(\Lambda_0)$ as follows. Fix an index $i \in I = \{0, 1, \cdots, n-1\}$. To each column y_k of Y, we assign its *i-signature* by

$$i\text{-signature of } y_k = \begin{cases} + & \text{if the top of } y_k \text{ is } i-1, \\ - & \text{if the top of } y_k \text{ is } i, \\ \cdot & \text{otherwise.} \end{cases}$$

Then we get an infinite sequence of $+$'s and $-$'s. From this infinite sequence, we cancel all the $(+, -)$-pairs to obatin a finite sequence of $-$'s followed by $+$'s:

$$i\text{-signature of } Y = (-, \cdots, -, +, \cdots, +),$$

which is called the *i-signature* of Y. Now we define $\tilde{e}_i Y$ (resp. $\tilde{f}_i Y$) to be the colored Young diagram obtained from Y by removing the i-box from the column (resp. by adding an i-box to the column) corresponding to the rightmost $-$ (resp. leftmost $+$) in the i-signature of Y.

Proposition 7.1 ([26]). *With the Kashiwara operators defined in this way, the set $\mathcal{Y}(\Lambda_0)$ becomes a $U_q(\widehat{sl_n})$-crystal. Moreover, there exists a $U_q(\widehat{sl_n})$-crystal isomorphism*

$$\mathcal{Y}(\Lambda_0) \xrightarrow{\sim} B(\Lambda_0).$$

For a nonzero complex number ζ, let $H_N(\zeta)$ denote the Hecke algebra of type A_{N-1}. Recall that when ζ is a primitive n-th root of unity, every finite dimensional irreducible $H_N(\zeta)$-module appears as the unique irreducible quotient $D(Y)$ of the Specht module $S(Y)$ corresponding to an n-reduced Young diagram Y with N boxes (see, for example, [4, 5]). Since Y can be viewed as a crystal basis element of $B(\Lambda_0)$ for $U_q(\widehat{sl_n})$, using the $U_q(\widehat{sl_n})$-module action on the space of all colored Young diagrams (see, for example, [7, 10, 26]), one should be able to write the global basis element $G(Y)$ as a linear combination of colored Young diagrams:

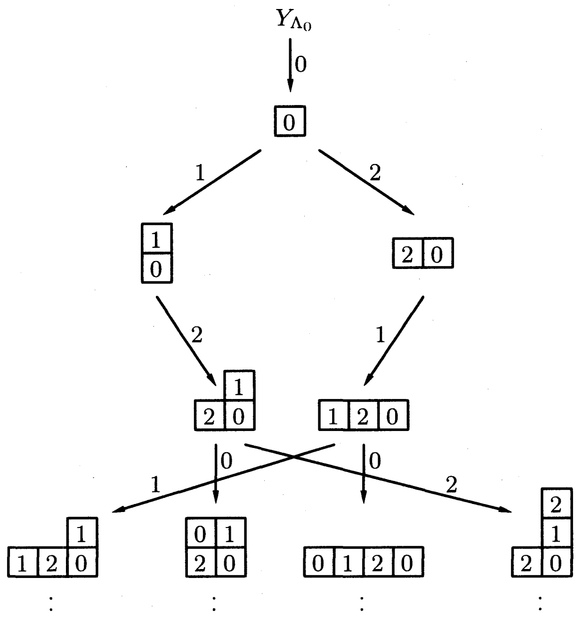
$$G(Y) = \sum_{Y'} d_{Y',Y}(q) Y' \quad \text{for} \quad d_{Y',Y}(q) \in \mathbf{Z}[q].$$

In [23], Lascoux, Leclerc and Thibon gave a recursive algorithm of computing the polynomials $d_{Y',Y}(q)$ and conjectured that

$$d_{Y',Y}(1) = [S(Y') : D(Y)],$$

where $[S(Y') : D(Y)]$ denotes the multiplicity of $D(Y)$ occurring in a composition series of $S(Y')$. This conjecture was proved by Ariki ([1]) and Grojnowski ([6]). In fact, the polynomials $d_{Y',Y}(q)$ coincide with the affine Kazhdan-Lusztig polynomials.

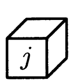
Example 7.2. By Proposition 7.1, the crystal graph $B(\Lambda_0)$ for the quantum affine algebra $U_q(\widehat{sl_3})$ is realized as the set of 3-reduced colored Young diagrams.



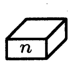
§8. Combinatorics of Young walls

In [12], we generalized the idea of [26] to the other classical quantum affine algebras (except $U_q(C_n^{(1)})$) and gave a realization of the crystal graphs $B(\Lambda)$ for the basic representations in terms of new combinatorial objects called the *Young walls*, which can be viewed as generalizations of colored Young diagrams. We will explain the main idea of [12] with the example of the quantum affine algebra $U_q(B_n^{(1)})$ because this case contains all the characteristics of combinatorics of Young walls.

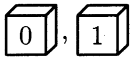
The Young walls are built of colored blocks of three different shapes (called the blocks of type I, type II, and type III, respectively) :



$(2 \leq j \leq n-1)$: unit width, unit height, unit thickness,

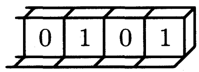


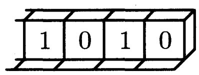
: unit width, half-unit height, unit thickness,

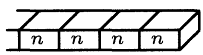


: unit width, unit height, half-unit thickness.

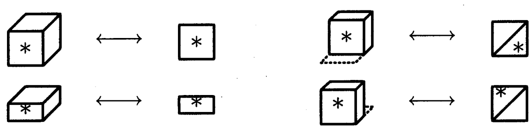
For each fundamental weight Λ of level 1; i.e., for $\Lambda = \Lambda_0$, Λ_1 or Λ_n, we fix a frame Y_Λ called the *ground-state wall of weight* Λ:

$Y_{\Lambda_0} =$  ,

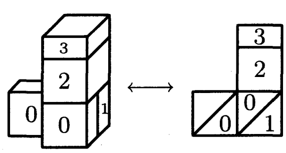
$Y_{\Lambda_1} =$  ,

$Y_{\Lambda_n} =$  .

On these frames, we build the walls of thickness less than or equal to one unit extending to the left. For convenience, we will use the following notations:



For example, when $n = 3$, we have



The rules for building the walls are given as follows:

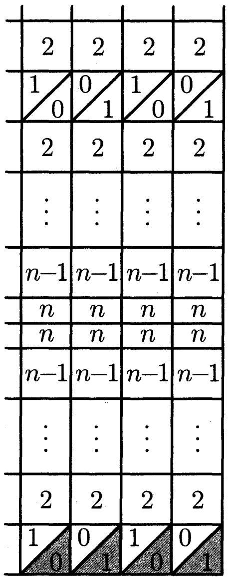
(1) The walls are built on top of the ground-state walls.

(2) The colored blocks should be stacked in columns. No block can be placed on top of a column of half-unit thickness.

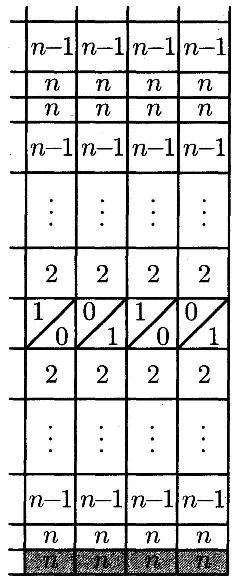
(3) Except for the right-most column, there should be no free space to the right of any block.

(4) The colored blocks should be stacked in the pattern given below:

On Y_{Λ_0} :



On Y_{Λ_n} :



A wall Y built on the ground-state wall Y_Λ following the rules given above is called a *Young wall* on Y_Λ, for the heights of its columns are weakly decreasing as we proceed from right to left. We often write $Y = (y_k)_{k=0}^\infty = (\cdots\cdots, y_2, y_1, y_0)$ as an infinite sequence of its colums.

Definition 8.1.
(a) A column of a Young wall is called a *full column* if its height is a multiple of the unit length and its top is of unit thickness.
(b) A Young wall is said to be *proper* if none of the full columns have the same height.

We denote by $\mathcal{P}(\Lambda)$ the set of all proper Young walls on Y_Λ. We will define the action of Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ on $\mathcal{P}(\Lambda)$.

Definition 8.2. Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall on Y_Λ.
(a) A block of color i in Y is called a *removable i-block* if Y remains a proper Young wall after removing the block. A column in Y is said to be *i-removable* if the top of that column is a removable i-block.
(b) A place in Y is called an *i-admissible slot* if one may add an i-block to obtain another proper Young wall. A column in Y is said to be *i-admissible* if the top of that column is an i-admissible slot.

Fix an index $i \in I$ and let $Y = (y_k)_{k=0}^{\infty} \in \mathcal{P}(\Lambda)$ be a proper Young wall. To each column y_k of Y, we assign its *i-signature* as follows:

(1) we assign $--$ if the column y_k is twice i-removable (the i-block will be of half-unit height in this case);
(2) we assign $-$ if the column is once i-removable, but not i-admissible (the i-block may be of unit height or of half-unit height);
(3) we assign $-+$ if the column is once i-removable and once i-admissible (the i-block will be of half-unit height in this case);
(4) we assign $+$ if the column is once i-admissible, but not i-removable (the i-block may be of unit height or of half-unit height);
(5) we assign $++$ if the column is twice i-admissible (the i-block will be of half-unit height in this case).

Then we get an infinite sequence of $+$'s and $-$'s. From this infinite sequence, we cancel out every $(+,-)$-pair to obtain a finite sequence of $-$'s followed by $+$'s, reading from left to right. This sequence is called the *i-signature* of Y. Now, we define the crystal structure on $\mathcal{P}(\Lambda)$ as follows.

(1) We define $\tilde{e}_i Y$ to be the proper Young wall obtained from Y by removing the i-block corresponding to the rightmost $-$ in the i-signature of Y. We define $\tilde{e}_i Y = 0$ if there exists no $-$ in the i-signature of Y.
(2) We define $\tilde{f}_i Y$ to be the proper Young wall obtained from Y by adding an i-block to the column corresponding to the leftmost $+$ in the i-signature of Y. We define $\tilde{f}_i Y = 0$ if there exists no $+$ in the i-signature of Y.

We also define the maps

$$\mathrm{wt} : \mathcal{P}(\Lambda) \longrightarrow P, \quad \varepsilon_i : \mathcal{P}(\Lambda) \longrightarrow \mathbf{Z}, \quad \varphi_i : \mathcal{P}(\Lambda) \longrightarrow \mathbf{Z}$$

by

$$\mathrm{wt}(Y) = \Lambda - \sum_{i \in I} k_i \alpha_i,$$

$$\varepsilon_i(Y) = \text{the number of } -\text{'s in the } i\text{-signature of } Y,$$

$$\varphi_i(Y) = \text{the number of } +\text{'s in the } i\text{-signature of } Y,$$

where k_i is the number of i-blocks in Y that have been added to the ground-state wall Y_Λ.

Proposition 8.3 ([12]).
The set $\mathcal{P}(\Lambda)$ together with the maps $\mathrm{wt} : \mathcal{P}(\Lambda) \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda) \cup \{0\}$ and $\varepsilon_i, \varphi_i : \mathcal{P}(\Lambda) \to \mathbf{Z}$ becomes a $U_q(B_n^{(1)})$-crystal.

Recall that $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + 2\alpha_n$ is the null root of the quantum affine algebra $U_q(B_n^{(1)})$.

Definition 8.4.

(a) The part of a column in a proper Young wall is called a *δ-column* if it has the same number of colored blocks as the null root δ in some cyclic order.

(b) A δ-column in a proper Young wall is called *removable* if it can be removed to yield another proper Young wall.

(c) A proper Young wall is said to be *reduced* if none of its columns contain a removable δ-column.

Let $\mathcal{Y}(\Lambda) \subset \mathcal{P}(\Lambda)$ be the set of all reduced proper Young walls on Y_Λ.

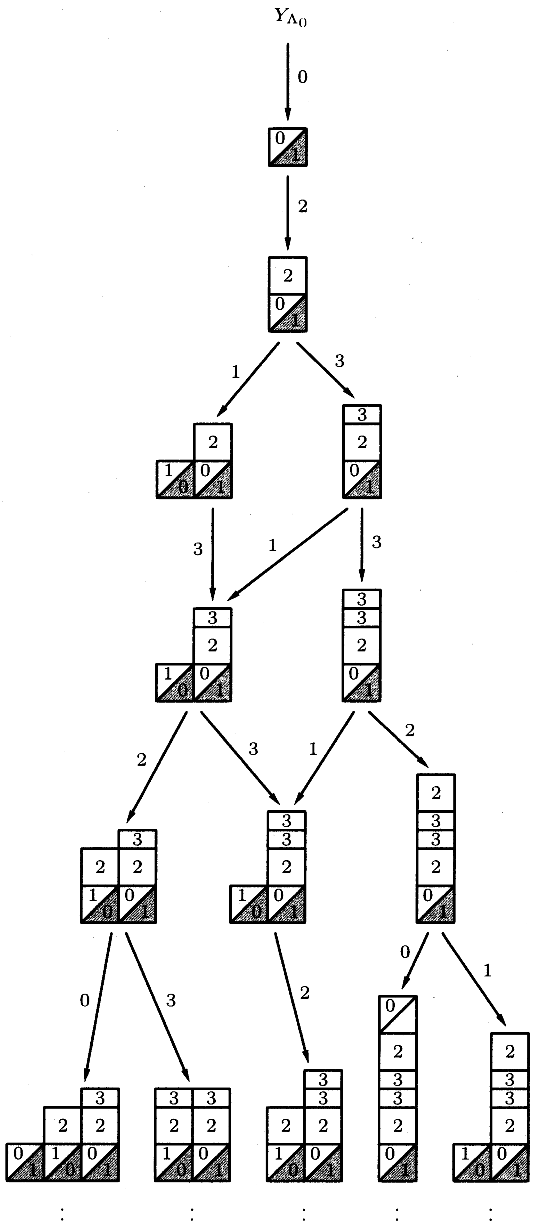
Theorem 8.5 ([12]).

The set $\mathcal{Y}(\Lambda)$ is a connected $U_q(B_n^{(1)})$-crystal. Moreover, there is a $U_q(B_n^{(1)})$-crystal isomorphism

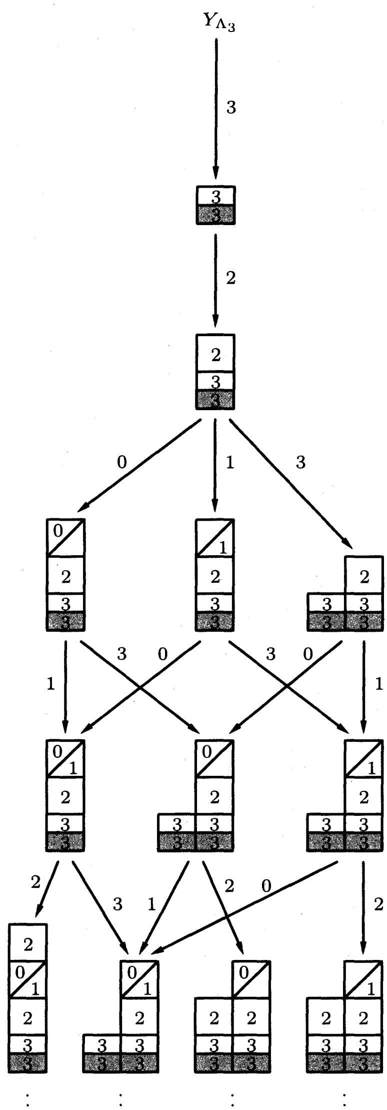
$$\mathcal{Y}(\Lambda) \xrightarrow{\sim} B(\Lambda),$$

where $B(\Lambda)$ is the crystal graph of the basic representation $V(\Lambda)$ of the quantum affine algebra $U_q(B_n^{(1)})$.

Example 8.6. (a) The crystal $\mathbf{Y}(\Lambda_0)$ for $U_q(B_3^{(1)})$



(b) The crystal $\mathbf{Y}(\Lambda_3)$ for $U_q(B_3^{(1)})$



Remark. (a) For the other classical quantum affine algebras (except $U_q(C_n^{(1)})$), one can prove that the crystal graph of a basic representation can be realized as the affine crystal consisting of reduced proper Young walls ([12]).

(b) As an application, we obtain a realization of crystal graphs for finite dimensional irreducible modules over quantum classical algebras (see [15]).

(c) The colored Young diagrams introduced in [26] can be regarded as the Young walls consisting of regular cubes only. In [10], Jimbo, Misra, Miwa and Okado extended the idea of [26] to higher level irreducible representations of $U_q(\widehat{sl_n})$. The crystal graph of a level l irreucible highest weight representation was characterized as the set of l-tuples of n-reduced colored Young diagrams satisfying certain addtional conditions. From our point of view, they can be viewed as reduced proper Young walls with l layers, which provides us with a clue to the construction of crystal bases for the higer level irreducible highest weight representations of other classical quantum affine algebras.

(d) As we have seen in Section 7, when ζ is a primitive n-th root of unity, the finite dimensional irreducible $H_N(\zeta)$-modules can be parametrized by n-reduced colored Young diagrams with N boxes. We expect that there exist some interesting algebraic structures whose irreducible representations (at some specialization) are parametrized by reduced proper Young walls. In [2], Brundan and Kleshchev verified this idea by showing that the irreducible representations of the Hecke-Clifford superalgebra $\mathcal{H}_N(\zeta)$ with ζ a primitive $(2n+1)$-th root of unity are parametrized by reduced proper Young walls of type $A_{2n}^{(2)}$ with N blocks.

§9. Fock space representation

Let $\mathcal{F}(\Lambda) = \bigoplus_{Y \in \mathcal{P}(\Lambda)} \mathbf{Q}(q)Y$ be the $\mathbf{Q}(q)$-vector space with a basis $\mathcal{P}(\Lambda)$. In [16, 17], Kang and Kwon defined a $U_q(\mathfrak{g})$-module structure on $\mathcal{F}(\Lambda)$, the *Fock space representation*, and showed that the crystal $\mathcal{P}(\Lambda)$ is exactly the crystal graph of $\mathcal{F}(\Lambda)$. The Fock space $\mathcal{F}(\Lambda)$ can be regarded as the q-deformed wedge space arising from a level 1 perfect representation ([22]).

We recall how to define the action of e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ on proper Young walls in $\mathcal{P}(\Lambda)$. Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall on Y_Λ. We denote by $|y_k|$ the number of blocks in y_k added to Y_Λ. Then the *associated partition* is defined to be $|Y| = (\cdots, |y_k|, \cdots, |y_1|, |y_0|)$. For $Y = (y_k)_{k=0}^\infty$, $Z = (z_k)_{k=0}^\infty$ in $\mathcal{P}(\Lambda)$, we define $|Y| \trianglerighteq |Z|$ if and only if $\sum_{k=l}^\infty |y_k| \geq \sum_{k=l}^\infty |z_k|$ for all $l \geq 0$.

Since the action of q^h is easily defined:

$$q^h Y = q^{\langle h, \mathrm{wt}(Y) \rangle} \quad \text{for} \quad h \in P^\vee, \ Y \in \mathcal{P}(\Lambda),$$

we will focus on the action of e_i and f_i on Y $(i \in I)$.

Case 1. Suppose that the i-blocks are of type I.

If b is a removable i-block in y_k of Y, then let $Y_R(b) = (y_{k-1}, \cdots, y_1, y_0)$ be the wall consisting of the columns lying at the right of b, and set $R_i(b;Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b))$. (The wall $Y_R(b)$ should be regarded as a $U_{(i)}$-crystal, where no block can be added on y_l for $l \geq k$.) We denote by $Y \nearrow b$ the Young wall obtained by removing b from Y. Then we define

$$e_i Y = \sum_b q_i^{-R_i(b;Y)} (Y \nearrow b),$$

where b runs over all removable i-blocks in Y.

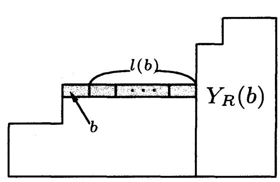
On the other hand, if b is an admissible i-slot in y_k of Y, then let $Y_L(b) = (\cdots, y_{k+2}, y_{k+1})$ be the Young wall consisting of the columns in Y lying at the left of b, and set $L_i(b;Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b))$. (The wall $Y_L(b)$ may be a proper Young wall on another ground-state wall $Y_{\Lambda'}$.) We denote by $Y \swarrow b$ the Young wall obtained by adding an i-block at b. Then we define

$$f_i Y = \sum_b q_i^{L_i(b;Y)} (Y \swarrow b),$$

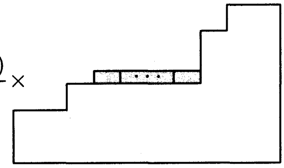
where b runs over all admissible i-slots in Y.

Case 2. Suppose that the i-blocks are of type II.

Let b be a removable i-block in y_k of Y. If the i-signature of y_k is $--$, or if the i-signature of y_k is $-$ and there is another i-block below b, define $Y \nearrow b$ to be the Young wall obtained by removing b from Y. If the i-signature of y_k is $-+$, or if the i-signature of y_k is $-$ and there is no i-block below b, define $Y \nearrow b = q^{-1}(1 - (-q^2)^{l(b)+1})Z$, where Z is the Young wall obtained by removing b from Y and $l(b)$ is the number of y_l's with $l < k$ such that $|y_l| = |y_k|$.

$Y =$ 

$$Y \nearrow b = \frac{(1-(-q^2)^{l(b)+1})}{q} \times$$



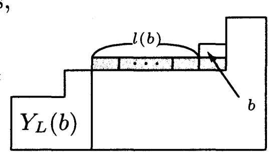
In either case, set $Y_R(b) = (y_{l-1}, \cdots, y_0)$, where l is the integer such that $|y_k| = |y_{k-1}| = \cdots = |y_{l+1}| < |y_l|$, and let $R_i(b;Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b))$. Then we define

$$e_i Y = \sum_b q_i^{-R_i(b;Y)}(Y \nearrow b),$$

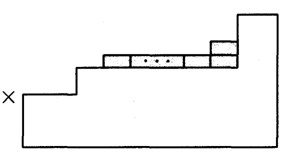
where b runs over all removable i-blocks in Y.

On the other hand, suppose that b is an admissible i-slot in y_k of Y. If the i-signature of y_k is $++$, or if the i-signature of y_k is $+$ and there is no i-block below b, then we define $Y \swarrow b$ to be the Young wall obtained by adding an i-block at b. If the i-signature of y_k is $-+$, or if the i-signature of y_k is $+$ and there is another i-block below b, then we define $Y \swarrow b = q^{-1}(1-(-q^2)^{l(b)+1})Z$, where Z is the Young wall obtained by adding an i-block at b and $l(b)$ is the number of y_l's with $l > k$ such that $|y_l| = |y_k|$. That is,

$$Y =$$



$$Y \swarrow b = \frac{(1-(-q^2)^{l(b)+1})}{q} \times$$



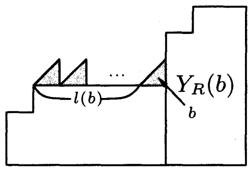
In either case, set $Y_L(b) = (\cdots, y_{l+2}, y_{l+1})$, where l is the integer such that $|y_{l+1}| < |y_l| = |y_{l-1}| = \cdots = |y_k|$, and let $L_i(b;Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b))$. Then we define

$$f_i Y = \sum_b q_i^{L_i(b;Y)}(Y \swarrow b),$$

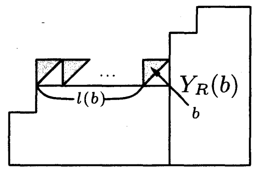
where b runs over all admissible i-slots in Y.

Case 3. Suppose that the i-blocks are of type III.

If b is a removable i-block in y_k of Y, then we define $Y \nearrow b$ to be the Young wall obtained by removing b from Y. We also consider the following i-block b in y_k of Y, which we call a *virtually removable i-block*:

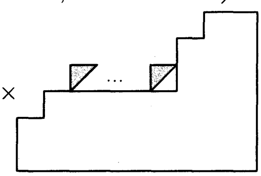


or

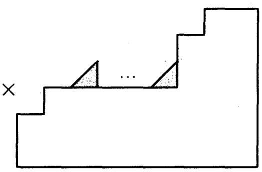


In this case, we define $Y \nearrow b$ to be

$(-q_i)^{l(b)} \times$



or $(-q_i)^{l(b)} \times$



,

where $l(b)$ is given in the above figure. In either case, set

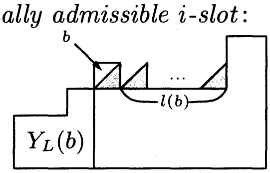
$$Y_R(b) = (y_{k-1}, \cdots, y_0), \quad R_i(b;Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b)),$$

and define

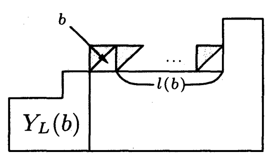
$$e_i Y = \sum_b q_i^{-R_i(b;Y)} (Y \nearrow b),$$

where b runs over all removable and virtually removable i-blocks in Y.

On the other hand, if b is an admissible i-slot in y_k of Y, then we define $Y \swarrow b$ to be the Young wall obtained by adding an i-block at b. We also consider the following i-slot b in y_k of Y, which we call a *virtually admissible i-slot*:

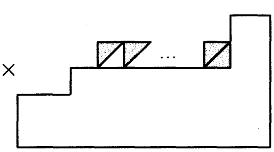


or

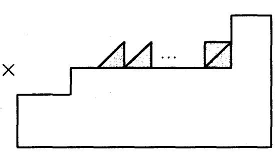


In this case, we define $Y \swarrow b$ to be

$(-q_i)^{l(b)} \times$



or $(-q_i)^{l(b)} \times$



,

where $l(b)$ is given in the above figure. In either case, set

$$Y_L(b) = (\cdots, y_{k+2}, y_{k+1}), \quad L_i(b;Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b)),$$

and define

$$f_i Y = \sum_b q_i^{L_i(b;Y)} (Y \swarrow b),$$

where b runs over all admissible and virtually admissible i-slots in Y.

Theorem 9.1 ([16, 17]).

(a) *The Fock space $\mathcal{F}(\Lambda)$ is an integrable $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$.*

(b) *Let $\mathcal{L}(\Lambda) = \bigoplus_{Y \in \mathcal{P}(\Lambda)} \mathbf{A}_0 Y$. Then the pair $(\mathcal{L}(\Lambda), \mathcal{P}(\Lambda))$ is a crystal basis of the Fock space $\mathcal{F}(\Lambda)$.*

(c)

$$\mathcal{F}(\Lambda) = \begin{cases} \bigoplus_{m=0}^{\infty} V(\Lambda - m\delta)^{\oplus p(m)} & \text{if } \mathfrak{g} \neq D^2_{n+1}, \\ \bigoplus_{m=0}^{\infty} V(\Lambda - 2m\delta)^{\oplus p(m)} & \text{if } \mathfrak{g} = D^2_{n+1}. \end{cases}$$

Remark. In [16, 17], Kang and Kwon generalized the Lascoux-Leclerc-Thibon algorithm to obtain an effective algorithm for constructing the global basis $G(\Lambda)$ of the basic representation $V(\Lambda)$ for all classical quantum affine algebras except $U_q(C_n^{(1)})$. More precisely, for each reduced proper Young wall $Y \in \mathcal{Y}(\Lambda)$, the generalized Lascoux-Leclerc-Thibon algorithm yields the global basis element

$$G(Y) = \sum_{Y' \in \mathcal{P}(\Lambda)} K_{Y,Y'}(q) Y',$$

where the coefficients $K_{Y,Y'}(q)$ satisfy the following conditions:

(i) $K_{Y,Y'}(q) \in \mathbf{Z}[q]$,

(ii) $K_{Y,Y'}(q) = 0$ unless $|Y| \trianglerighteq |Y'|$,

(iii) $K_{Y,Y'}(q) = 1$ and $K_{Y,Y'}(0) = 0$ if $Y \neq Y'$.

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