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Cellular algebras and diagram algebras in representation theory

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Abstract.

We discuss a circle of ideas for addressing problems in representation theory using the philosophy of cellular algebras, applied to algebras described in terms of diagrams. Cellular algebras are often generically semisimple, and have non-semisimple specialisations whose representation theory may be discussed by solving problems in linear algebra, which are formulated in the semisimple context, and are therefore tractable in some significant cases. This applies in particular to certain "Temperley-Lieb" quotients of Hecke algebras, both finite dimensional and affine, which may be described in terms of bases consisting of diagrams. This leads to the application of cellular algebra theory to an analysis of their representation theory, with corresponding consequences for the relevant Hecke algebras. A particular case is the determination of the decomposition numbers of some standard modules for the affine Hecke algebra of GL_n. These decomposition numbers are known (by Kazhdan-Lusztig) to be expressible in terms of the dimensions of the stalks of certain intersection cohomology sheaves, and we discuss how our results imply the rational smoothness of some varieties associated with quiver representations.

§1. Introduction

We discuss a circle of ideas for addressing problems in representation theory using the philosophy of cellular algebras, applied to algebras described in terms of diagrams. Cellular algebras are often generically semisimple, and have non-semisimple specialisations whose representation theory may be discussed by solving problems in linear algebra, which are formulated in the semisimple context, and are therefore tractable in

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This paper is organised as follows. In §2 we recall the basic notion of a cellular algebra, and how the cellular structure may be used to analyse the representations of its non-semisimple specialisations. The finite dimensional Temperley-Lieb algebras of type A_n and B_n are introduced with their cellular structure and cell modules. In §3 we discuss the infinite dimensional affine Temperley-Lieb algebra $TL_n^a(q)$ from this point of view, recalling the results of [17]. In §4 we describe the interrelationships among these algebras. This is done by relating them all to the affine Hecke algebra of GL_n. An understanding of the interconnections among the underlying generalised Artin braid groups is useful here, to make the connection. In §5 we discuss the lifting of cell modules from $TL_n^a(q)$ to modules for the affine Hecke algebra, and explain the results of [18] which identify these liftings in the Grothendieck ring as "standard modules". In §6 implications for the decomposition numbers of certain standard modules are treated, and finally, in §7, we explain their interpretation in terms of the intersection cohomology of certain quiver varieties.

The approach outlined here could also be used to study the "modular representation theory" of $\widehat{H_n^a}(q)$, but we do not do this here. The results may also be interpreted in terms of a generalisation to the non-generic case of the "multisegments" of Zelevinsky and Bernstein. Since the "annular algebras" of V. Jones ([20]) are quotients of the algebras $T_n^a(q)$, their representation theory may be thought of as a subset of the story below. Hence our work throws light on the connection between the work of Jones on link invariants (cf. [21]) and affine Hecke algebras. There are also close connections between this work and subfactors of C^*-algebras (see [22, 23]).

§2. Cellular algebras and Temperley-Lieb algebras

Cellular algebras were first defined in [15] in terms of the existence of a basis with specified multiplicative properties. Subsequently, there has been a significant literature (cf. [25], [26], [27], [28] and [13]) in which structural definitions have been given, in terms of the existence of certain filtrations, and in which cellular algebras have been compared with quasi-hereditary algebras; in particular much is now known about their characterisation in terms of global dimension, Cartan matrix and representation type. The main point of a cellular structure is that it is preserved under specialisation, so that cellularity is a particularly effective way of studying algebras which are not semisimple, but are deformations of semisimple algebras.

In this work, we shall use the original definition. Let R be a commutative ring with 1, and let A be an R-algebra which is free over R, and which we assume for the moment to be of finite rank (although many of the concepts will be applied when A has infinite rank). To specify a cellular structure for A, we require a "cell datum", which consists of: (i) a poset $\mathcal{T}$, (ii) for each $t \in \mathcal{T}$, a set $M(t)$, and (iii) an injection $\amalg_{t\in\mathcal{T}} M(t) \times M(t) \xrightarrow{C} A$, whose image is an R-basis $\{C^t_{S,T}\}$ of A which satisfies

$$\begin{aligned} aC^t_{S,T} &= \sum_{S'\in M(t)} r_a(S',S)C^t_{S',T} + \text{ lower terms} \\ &(a \in A, r_a(S',S) \in R), \end{aligned} \tag{2.1}$$

where "lower terms" indicates a linear combination of basis elements $C^{t'}_{S'',T''}$ with $t' < t$. In addition, one requires that the map $C^t_{S,T} \mapsto C^t_{T,S}$, extended R-linearly, be an anti-automorphism $a \mapsto a^*$ of A.

We shall review briefly the representation theory of cellular algebras. To start with, for each $t \in \mathcal{T}$, one has the *cell module* $W(t)$; this has R-basis $\{C_S \mid S \in M(t)\}$, and A acts on the left by the rule

$$a.C_S = \sum r_a(S',S)C_{S'} \quad \text{for} \quad a \in A, \ S \in M(t). \tag{2.2}$$

It follows from 2.1 that 2.2 defines an action.

The cell module $W(t)$ has a natural R-bilinear form $\phi_t : W(t) \times W(t) \longrightarrow R$, defined by

$$(C^t_{S,T})^2 = \phi_t(C_S, C_T)C^t_{S,T} + \text{ lower terms.} \tag{2.3}$$

It is easily shown ([15]) that the form ϕ_t is symmetric and invariant under the action of A, i.e. that for any two elements $x, y \in W(t)$ and

$a \in A$, we have $\phi_t(x,y) = \phi_t(y,x)$ and $\phi_t(a.x,y) = \phi_t(x,a^*.y)$. An immediate consequence is that the radical $\mathrm{Rad}(t) = \mathrm{Rad}\,\phi_t = \{x \in W(t) \mid \phi_t(x,y) = 0 \text{ for all } y \in W(t)\}$ of ϕ_t is an A-submodule of $W(t)$. Write $L(t) := W(t)/\mathrm{Rad}(t)$; clearly $L(t)$ is zero if and only if the form $\phi_t = 0$.

The next statement summarises the representation theory of A when R is a field.

Proposition 2.4 ([15]). *Maintain the above notation, and in addition assume that R is a field. Let $\mathcal{T}^0$ be the set of $t \in \mathcal{T}$ such that $L(t) \neq 0$ (or equivalently $\phi_t \neq 0$). Then* (i) *The modules $L(t)$ $(t \in \mathcal{T}^0)$ are absolutely irreducible and form a complete set of representatives of the distinct isomorphism classes of irreducible A-modules.*

(ii) *For $s \in \mathcal{T}, t \in \mathcal{T}^0$ let d_{st} be the multiplicity of $L(t)$ in $W(s)$. Then the decomposition matrix $D = (d_{st})$ is upper unitriangular. In particular* $\mathrm{Hom}(W(t_1), W(t_2)) = 0$ *unless $t_1 \geq t_2$.*

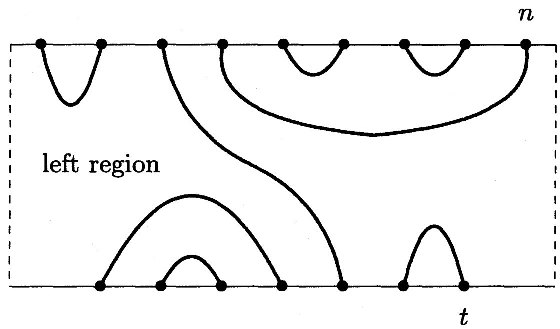
(iii) *For $s_1, s_2 \in \mathcal{T}^0$, let c_{s_1,s_2} be the multiplicity if the irreducible module $L(s_2)$ in the projective cover $P(s_1)$ of $W(s_1)$. If C is the (Cartan) matrix (c_{s_1,s_2}) then $C = D^tD$, where D^t denotes the transpose of D.*

(iv) *The cell module $W(t)$ is irreducible if and only if, for all $t' \in \mathcal{T}, t' \neq t$,* $\mathrm{Hom}(W(t'), W(t)) = 0$. *Equivalently, the form ϕ_t is non-degenerate on $W(t)$.*

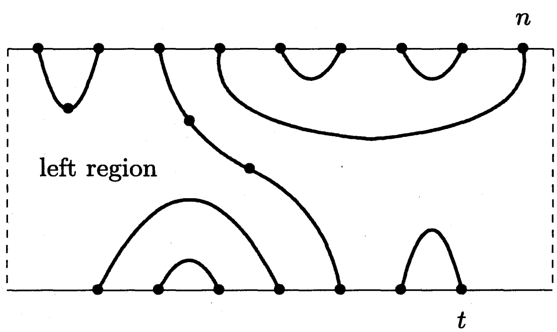
(v) *The following assertions are equivalent. The algebra A is semisimple. Each cell module is absolutely irreducible. The forms ϕ_t are all non-degenerate. There are no non-trivial homomorphisms between distinct cell modules.*

We shall now see how these concepts apply to the Temperley-Lieb algebras of type A and B. Let R be a commutative ring with 1; write $R^\times$ for the group of invertible elements of R and let $q, Q \in R^\times$. For any invertible element $x \in R$, write $\delta_x = -(x + x^{-1})$. Our notation for parameters, which may appear arbitrary, is chosen with a view to the links with Hecke algebras described below. The usual Temperley-Lieb algebra $TL_n(q)$ may be described in terms of planar diagrams (see [20], [10] or [15]) in a way which is well known. The Temperley-Lieb algebra $TLB_n(q,Q)$ of type B_n has a corresponding description in terms of "marked diagrams" (cf. [42], [36], [37]). When the base ring is an algebraically closed field, $TLB_n(q,Q)$ is sometimes known as the "blob algebra" (see [6]). The algebra $TL_n(q)$ is realised as the subalgebra of $TLB_n(q,Q)$ which is spanned by unmarked diagrams. We describe now $TLB_n(q,Q)$ as a cellular algebra, using the language of [16] for diagrams.

If t, n are positive integers of the same parity, a finite (planar) diagram $\mu : t \longrightarrow n$ is represented by a set of non-intersecting arcs which are contained in the "fundamental rectangle" (see below). These arcs divide the fundamental rectangle into regions, among which there is a unique "left region" as shown below.



A *marked diagram* is a (finite planar) diagram, where the interior of the boundary arcs of the leftmost region may be marked with one or more • symbols ("marks") (see below).



The R-linear combinations of unmarked diagrams from t to n constitute the morphisms in the Temperley-Lieb category $\mathbf{T}$, where the objects are the non negative integers $\mathbb{Z}_{\geq 0}$. If t, n respectively denote the number of bottom and top nodes in a diagram D, then D is a morphism from t to n in $\mathbf{T}$. Composition of morphisms corresponds to concatenation of diagrams, with closed loops being deleted and replaced by the scalar δ_q. In the composition $D_1 D_2$ of diagrams D_1 and D_2, D_1 is placed above

D_2. This is consistent with morphisms being composed from right to left. The endomorphisms of $n \in \mathbf{T}$ form an algebra $\mathrm{Hom}_{\mathbb{T}}(n, n) \cong TL_n(q)$. Marked diagrams may be similarly concatenated according to rules we shall now state; this produces a new category, $\mathbb{TB}$, the Temperley-Lieb category of type B. The composition rules are as follows.

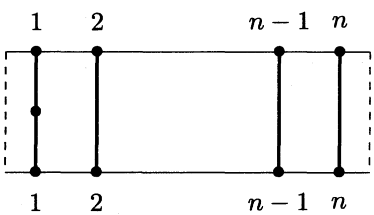
A marked diagram is *proper* if it has no loops and each arc has at most one mark. The following rules reduce the concatenation of any two diagrams to an R linear combination of proper diagrams:

2.5. (i) *If μ is a diagram and L is a loop with no marks, $\mu \amalg L = \delta_q \mu$.*

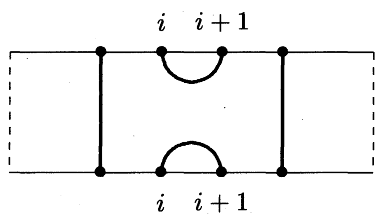
(ii) *If, in (i), L has one mark, $\mu \amalg L = \kappa\mu$, where $\kappa = \frac{q}{Q} + \frac{Q}{q}$.*

(iii) *If some arc of μ has more than one mark and μ' is the diagram obtained by removing a mark from the arc concerned, then $\mu = \delta_Q \mu'$.*

Now consider the following marked diagrams from n to n.



$c_0 : n \to n$



$c_i : n \to n,\ 1 \le i \le n-1$

The next result describes $TLB_n(q, Q)$ algebraically.

Proposition 2.6 (cf. [18, 5.8]). (i) *The Temperley-Lieb algebra $TLB_n(q, Q)$ is generated as R-algebra by $\{c_0, c_1, \ldots, c_{n-1}\}$ subject to the relations*

$$\begin{aligned} c_0^2 &= \delta_Q c_0 \\ c_i^2 &= \delta_q c_i \ \textit{for}\ 1 \le i \le n-1 \\ c_i c_{i+1} c_i &= c_i \ \textit{for}\ 1 \le i \le n-2 \\ c_i c_{i-1} c_i &= c_i \ \textit{for}\ 2 \le i \le n-1 \\ c_i c_j &= c_j c_i \ \textit{if}\ |i-j| > 1,\ 0 \le i, j \le n-1 \\ c_1 c_0 c_1 &= \kappa c_1, \end{aligned} \tag{2.7}$$

where $\kappa = \frac{q}{Q} + \frac{Q}{q}$ and $\delta_x = -(x + x^{-1})$ for $x \in R^\times$.

(ii) *The elements $c_1, \ldots, c_{n-1}$ generate a subalgebra of $TLB_n(q, Q)$ isomorphic to the usual Temperley-Lieb algebra $TL_n(q)$ of type A_{n-1}.*

The stated relations are easily seen to hold by 2.5. The fact that all relations are consequences of these depends on the connection between the Hecke algebra description of $TLB_n(q,Q)$ (§4 below) and the diagrammatic description here. The proof may be found in [42, Satz (4.5) p. 77].

We now specify the various elements of a cellular structure (see above) for $TLB_n(q,Q)$. Take $\mathcal{T} = \{t \in \mathbb{Z} \mid |t| \leq n,\ t \equiv n \pmod 2\}$, partially ordered as follows: $t \leq s$ if $|t| < |s|$ or $|t| = |s|$ and $t \leq s$.

To define the sets $M(t)$, first take $t \in \mathcal{T}, t \geq 0$. Then $M(t)$ is the set of monic diagrams $D : t \to n$ with no marked through strings, where "monic" means that there are t through strings, as in [GL2], where it is shown that this is equivalent to D being a monic morphism in the category-theoretic sense. In general, let $M(t) = M(|t|)$. Then $C : M(t) \times M(t) \to TLB_n(q,Q)$ is defined as follows. Let $S, T \in M(t)$. For $t \geq 0$, define $C^t_{S,T} = S \circ T^*$, where * denotes reflection in a horizontal axis. For $t < 0$, define $C^t_{S,T} = S \circ c_o \circ T^*$, where $c_0 = c_0(t) : t \to t$ is the generator shown below 2.5. This is the diagram $S \circ T^*$, with the leftmost through string marked. The cellular axioms above, in particular 2.1 are easily checked.

The cell modules $W_t(n)$ are now defined in complete analogy with the $W_{t,z}(n)$ of [GL2]. For any $t \in \mathcal{T}$, $W_t(n)$ has basis $M(t)$. If $t \geq 0$, $TLB_n(q,Q)$ acts via composition in the category $\mathbb{TB}$; explicitly, if $D \in M(t)$ and $\omega \in TLB_n(q,Q)$, then $\omega.D = \omega D$ (composition in $\mathbb{TB}$) if $\omega \circ D \in M(t)$, and $\omega.D = 0$ otherwise.

For $t < 0$, one may think of $W_t(n)$ as having basis the set $\{D \circ c_0(t) \mid D \in M(|t|)\}$ of monic diagrams $: t \to n$ in $\mathbb{TB}$ with the leftmost through string marked. Then the action of $TLB_n(q,Q)$ is essentially multiplication in $\mathbb{TB}$, as in the case $t \geq 0$. Thus if $t < 0$, then $\omega.(D \circ c_0(t)) = 0$ if $\omega \circ D$ is not monic, while $\omega.(D \circ c_0) = \omega D c_0$ (composition in $\mathbb{TB}$) if $\omega \circ D$ is monic.

It is easily seen (cf. [42]) that the dimension (i.e. rank over R) of $W_t(n)$ is $\binom{n}{\frac{n-|t|}{2}}$.

For each integer i with $1 \leq i \leq n-1$, the diagrams $\{c_0, c_1, \ldots, c_{i-1}\}$ generate an algebra isomorphic to $TLB_i(q,Q)$, and we shall require information concerning the restriction of the cell modules $W_t(n)$ to the subalgebras $TLB_i(q,Q)$. Let us first consider the restriction of $W_t(n)$ from $TLB_n(q,Q)$ to $TLB_{n-1}(q,Q)$ when $t \in \mathcal{T}, t \geq 0$. Now $TLB_{n-1}(q,Q)$ is spanned by those diagrams in $TLB_n(q,Q)$ with a string joining the rightmost dots in the top and bottom rows. The basis elements of $W_t(n)$ divide naturally into those which have a through string to the rightmost top dot, and those which do not. The former evidently span a

$TLB_{n-1}(q,Q)$- submodule isomorphic to $W_{t-1}(n-1)$. A diagram in the latter set may be identified with a monic diagram $t+1 \to n-1$ by "pulling down" the rightmost dot from the top to the bottom row. In this way, the quotient of the $TLB_{n-1}(q,Q)$-module $W_t(n)$ by $W_{t-1}(n-1)$ is easily identified as $W_{t+1}(n-1)$. Using a similar argument for the case $t \leq 0$ we obtain

Proposition 2.8. *Let $t \in \mathbb{Z}$, $0 \leq |t| \leq n$, $n+t \in 2\mathbb{Z}$. Assume $\delta_q \neq 0$.*

(i) *If $t \geq 0$, we have a short exact sequence*

$$(2.9)\quad 0 \to W_{t-1}(n-1) \to Res^{TLB_n(q,Q)}_{TLB_{n-1}(q,Q)} W_t(n) \to W_{t+1}(n-1) \to 0.$$

(ii) *If $t < 0$, we have a short exact sequence*

$$(2.10)\quad 0 \to W_{t+1}(n-1) \to Res^{TLB_n(q,Q)}_{TLB_{n-1}(q,Q)} W_t(n) \to W_{t-1}(n-1) \to 0.$$

Here we adopt the convention that $W_j(k) = 0$ if $|j| > k$.

Repeated application of 2.8 yields

Corollary 2.11. *Let $t \in \mathbb{Z}$, $0 \leq |t| \leq n$, $n+t \in 2\mathbb{Z}$ and suppose $t \geq 0$. There is a filtration of $W_t(n)$ by R-submodules $W^{(i)}, i = n, n-1, n-2, \ldots, 1$ as in 2.12 below.*

(2.12)

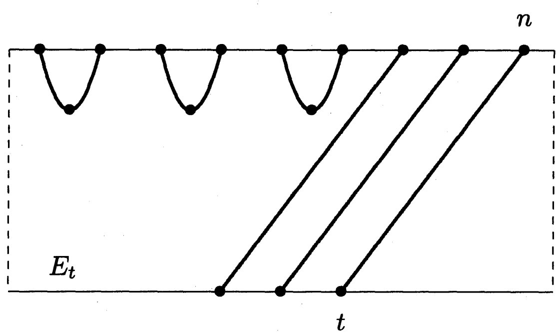
$$W_t(n) \supset W_{t-1}(n-1) \supset \cdots \supset W_0(n-t) \supset W_{-1}(n-t-1) \supset$$
$$W_0(n-t-2) \supset W_{-1}(n-t-3) \supset \cdots \supset W_0(2) \supset W_{-1}(1).$$

Thus

$$(2.13)\qquad W^{(i)} \cong \begin{cases} W_{t-n+i}(i) \text{ if } n-t \leq i \leq n \\ W_0(i) \text{ if } i \text{ is even and } 0 \leq i \leq n-t \\ W_{-1}(i) \text{ if } i \text{ is odd and } 0 \leq i \leq n-t. \end{cases}$$

For each $i = 1, 2, 3, \ldots, n$, $W^{(i)}$ is a $TLB_i(q,Q)$-submodule of $W_t(n)$.

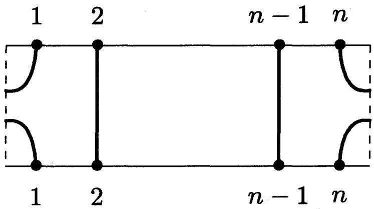
To illustrate the construction in 2.11, we note that the diagram $E_t = E_{t,n}$ below lies in each of the submodules $W^{(i)}$.



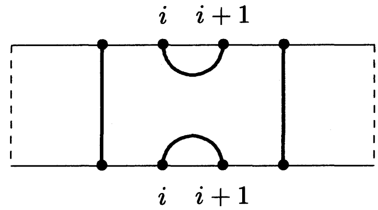
Given the details of the cellular structure of $TLB_n(q,Q)$, all the results of Proposition 2.4 are available. However we turn next to the affine Temperley-Lieb algebra.

§3. The diagram version of the affine Temperley-Lieb algebra

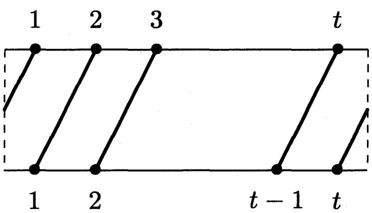
There are several algebras which could be (and have been) called the "affine Temperley-Lieb algebra". In this section we discuss the diagrammatic version $T_n^a(q)$, which was introduced in [16, 2.7] as the algebra of endomorphisms of an object of the affine Temperley-Lieb category $\mathbb{T}^a$ (see [GL2, (2.5)]), and whose representation theory was completely analysed in *loc. cit.* using cellular theory, although the algebras are infinite dimensional. It also occurs independently in the work of Green [12] and Fan–Green [8]. We maintain the notation of the last section for R, q, etc. Details may be found in [GL2], but a good approximation to the picture is obtained if one thinks of affine diagrams as arcs drawn on the surface of a cylinder joining $2n$ marked points, n on each circle component of the boundary, in pairs. The arcs must not intersect, and diagrams are multiplied by concatenation in the usual way. These diagrams are represented by periodic diagrams drawn between two horizontal lines, each diagram being determined by the "fundamental rectangle", from which the cylinder is obtained by identifying vertical edges. In this interpretation, the generators $\{f_1, \ldots, f_n, \tau_n\}$ of $T_n^a(q)$ are represented by the diagrams shown below.



$f_n : n \to n$



$f_i : n \to n,\ 1 \le i \le n-1$



$\tau_t : t \to t$

$\tau_0 : 0 \to 0$

The elements $\{f_1, \dots, f_n\}$ of the algebra $T_n^a(q)$ satisfy the relations 3.1 below.

$$
(3.1)\qquad
\begin{aligned}
f_i^2 &= \delta_q f_i \\
f_i f_{i\pm 1} f_i &= f_i \\
f_i f_j &= f_j f_i \text{ if } |i-j| \ge 2 \text{ and } \{i,j\} \ne \{1,n\},
\end{aligned}
$$

Further, $\tau_n f_i \tau_n^{-1} = f_{i+1}$, where the index is taken mod n. We shall sometimes omit the subscript from the "twist" τ_n when there is no ambiguity.

The reduction of a concatenation of affine diagrams to a linear combination of diagrams is simpler that in the case of $TLB_n(q,Q)$: circles which circumnavigate the cylinder (these are called infinite in [16]) remain; they are simply powers of τ_0. If D is an affine diagram and C is a finite or contractible closed loop, then in the category $\mathbb{T}^a$, $D \amalg C = \delta_q D$.

In [16], we defined cell modules $W_{t,z}(n)$ for the algebra $T_n^a(q)$, (where $t \in \mathbb{Z}, 0 \le t \le n, t+n \in 2\mathbb{Z}$, and $z \in R^\times$) and when R is an algebraically closed field of characteristic zero, completely determined their composition factors and multiplicities. We briefly review this material now.

Recall [16] that an affine diagram from t to n is monic if it has t through strings. It follows that $t \le n$ and $t \equiv n (\text{mod } 2)$. For such a positive integer t, let X_t be the $T_n^a(q)$-module with basis *all* monic affine diagrams:$t \longrightarrow n$, with $T_n^a(q)$ action given by composition in the category $\mathbb{T}^a$, modulo diagrams with fewer than t through strings. Thus X_t may be thought of as a quotient of the left $T_n^a(q)$-module $\mathrm{Hom}_{\mathbb{T}^a}(t,n)$ by the submodule spanned by diagrams with fewer than t through strings. Let $z \in R^\times$. The $T_n^a(q)$ module $W_{t,z}(n)$ is defined as the quotient of X_t by the ideal generated by the set

$$I_z := \{\gamma\tau_t - \chi_z(t)\gamma\} \tag{3.2}$$

over all affine diagrams $\gamma \in X_t$, where $\chi_z(t) = z$ if $t \neq 0$ and $\chi_z(0) = z + z^{-1}$.

While the module X_t is evidently of infinite rank over R, $W_{t,z}(n)$ is shown in [16, §2] to be free of rank $\binom{n}{\frac{n-t}{2}}$ over R. A basis of $W_{t,z}(n)$ is provided by the *standard* diagrams (see [16, 1.7, 2.7]) from t to n in $\mathbb{T}^a$.

The R-linear map $w \mapsto w^*$, where d^* denotes the reflection of a diagram $d \in \mathbb{T}^a$ in a horizontal line defines an involution on the category $\mathbb{T}^a$ (i.e. an involutory functor from $\mathbb{T}^a$ to $(\mathbb{T}^a)^{\mathrm{opp}}$), which induces an anti-automorphism of $T_n^a(q)$ for each n. Suppose μ, ν are standard diagrams from t to n. Let $\phi_{t,z} : W_{t,z}(n) \times W_{t,z^{-1}}(n) \longrightarrow R$ be the R-linear extension of the map $(\mu,\nu) \mapsto \phi_{t,z}(\mu,\nu) = \chi_z(\nu^*\mu)$, where $\chi_z : T_t^a(q) \to R$ is the R-linear map which annihilates non-monic diagrams and takes τ_t^i to χ_z^i (see 3.2). Then $\phi_{t,z}$ is bilinear and ([16, (2.7)]) invariant in the sense that $\phi_{t,z}(w\mu,\nu) = \phi_{t,z}(\mu, w^*\nu)$, $(\mu,\nu \in W_{t,z}(n), w \in T_n^a(q))$. We now have all the ingredients of a cellular structure and can state the analogue of 2.4 for $T_n^a(q)$.

Definition 3.3. *Let $\Lambda^a(n)^+$ be the set*

$$\Lambda^a(n)^+ = \{(t,z) \mid t \in \mathbb{Z}_{\geq 0},\ 0 \leq t \leq n,\ n-t \in 2\mathbb{Z};\ z \in R^\times\}. \tag{3.4}$$

Define $\Lambda^a(n)$ by

$$\Lambda^a(n) = \begin{cases} \Lambda^a(n)^+ & \text{if } q^2 \neq -1 \\ \Lambda^a(n)^+ \setminus \{(0,\pm q)\} & \text{if } q^2 = -1. \end{cases} \tag{3.5}$$

Define the equivalence relation $\approx$ on $\Lambda^a(n)^+$ as that which identifies $(0,z)$ and $(0,z^{-1})$ for all $z \in R^\times$, and write

$$\begin{aligned} \Lambda^a(n)^0 &= \Lambda^a(n)/\approx \\ \Lambda^a(n)^{0+} &= \Lambda^a(n)^+/\approx \end{aligned} \tag{3.6}$$

If $(t_1,z_1),(t_2,z_2) \in \Lambda^a(n)^+$, then $W_{t_1,z_1}(n) \cong W_{t_2,z_2}(n)$ if $(t_1,z_1) \approx (t_2,z_2)$. Thus $\Lambda^a(n)^{0+}$ plays the role of $\mathcal{T}$ in §2. We write $\mathrm{Rad}\,\phi_{t,z}$ for the submodule $\{x \in W_{t,z}(n) \mid \phi_{t,z}(x,y) = 0 \text{ for all } y \in W_{t,z^{-1}}(n)\}$ of $W_{t,z}(n)$.

Theorem 3.7 ([16, (2.8)]). *Let R be an algebraically closed field and maintain the above notation.*

(i) *For $(t,z) \in \Lambda^a(n)^+$, $L_{t,z}(n) := W_{t,z}(n)/\operatorname{Rad}\phi_{t,z}$ is either an (absolutely) irreducible $T_n^a(q)$ module or zero, and $L_{t,z}(n) \neq 0$ if and only if $(t,z) \in \Lambda^a(n)$.*

(ii) *All irreducible $T_n^a(q)$ modules are realised thus, and if $(t_1,z_1) \not\sim (t_2,z_2)$, then $L_{t_1,z_1}(n) \not\cong L_{t_2,z_2}(n)$.*

(iii) *The cell module $W_{t,z}(n)$ is irreducible if and only if for any $(t',z') \in \Lambda^a(n)^0$, $(t',z') \neq (t,z)$ we have*

$$\operatorname{Hom}_{T_n^a(q)}(W_{t',z'}(n), W_{t,z}(n)) = 0.$$

Equivalently, the form $\phi_{t,z}$ is a perfect pairing.

It follows that the distinct irreducible $T_n^a(q)$-modules are parametrised by $\Lambda^a(n)^0$, while the distinct cell modules are parametrised by $\Lambda^a(n)^{0+}$. These two sets therefore play the roles of $\mathcal{T}^0$ and $\mathcal{T}$ respectively in §2. Where there is little danger of confusion, we abuse notation by denoting the elements of $\Lambda^a(n)^{0+}$ as pairs (t,z), rather than equivalence classes of pairs. Thus we speak of $W_{t,z}(n)$ and $L_{t,z}(n)$ for $(t,z) \in \Lambda^a(n)^{0+}$. It follows from (3.7) that to understand the composition factors of the $T_n^a(q)$-module $W_{t,z}(n)$, it suffices to understand the spaces $\operatorname{Hom}_{T_n^a(q)}(W_{s,y}(n), W_{t,z}(n))$ for all pairs $(t,z),(s,y)$. This analysis is caried out in [16], and leads to the following results.

Let $\preceq$ be the partial order on $\Lambda^a(n)^+$ which is generated by the preorder $\overset{\circ}{\prec}$ which stipulates that $(t,z) \overset{\circ}{\prec} (s,y)$ if

$$\begin{aligned} &0 \le t \le s \le n, \quad s = t + 2\ell \quad (\ell \in \mathbb{Z},\ \ell > 0) \quad \text{and} \\ &z^2 = q^{\epsilon(s,z)s} \text{ and } y = zq^{-\epsilon(s,z)\ell} \text{ for } \epsilon(s,z) = \pm 1. \end{aligned} \tag{3.8}$$

Note that 3.8 implies that

$$y^2 = q^{\epsilon(s,z)t} \text{ and } z^t = y^s \tag{3.9}$$

and

$$(t,z) \preceq (t',z') \implies z^t = (z')^{t'}. \tag{3.10}$$

It suffices to verify 3.10 when $(t,z) \overset{\circ}{\prec} (t',z')$, in which case it follows easily from 3.8.

It is easily verified that [17, 4.1] the partial order $\preceq$ on $\Lambda^a(n)^+$ induces a partial order, also denoted $\preceq$, on the set $\Lambda^a(n)^{0+} = \Lambda^a(n)^+/\approx$.

The following result is proved in [16, Theorem 5.1].

Theorem 3.11. *Let R be a field of characteristic 0 or $p > 0$, where $pe > n$ and e is the multiplicative order of q^2. Then* (i) *We have*

$$\dim \operatorname{Hom}_{T_n^a(q)}(W_{s,y}(n), W_{t,z}(n)) = \begin{cases} 1 & \text{if } (t,z) \preceq (s,y) \\ 0 & \text{otherwise} \end{cases} . \tag{3.12}$$

(ii) *In the Grothendieck group $\Gamma(T_n^a(q))$, we have for any $(t,z) \in \Lambda^a(n)^{0+}$,*

$$W_{t,z}(n) = \sum_{\substack{(s,y)\in\Lambda^a(n)^0 \\ (t,z)\preceq(s,y)}} L_{s,y}(n) \tag{3.13}$$

Thus the matrix expressing the cell modules in terms of the irreducibles in $\Gamma(T_n^a(q))$ is upper unitriangular, and has entries 0 or 1. Now if (t,z) is confined to $\Lambda^a(n)^0$, the relation 3.13 can clearly be inverted.

The result is (cf. [17, Theorem (4.5)])

Theorem 3.14. *In the notation above, if $(t,z) \in \Lambda^a(n)^0$,*

$$L_{t,z}(n) = \sum_{\substack{(s,y)\in\Lambda^a(n)^0 \\ (t,z)\preceq(s,y)}} n_{t,z}^{s,y} W_{s,y}(n) \tag{3.15}$$

where $n_{t,z}^{s,y} = 0$ or ± 1.

§4. Interrelationships among various Hecke and Temperley-Lieb algebras

Our purpose in this section is to give a concise statement of the various connections among several Hecke algebras and Temperley-Lieb algebras. Behind these connections are facts concerning generalised Artin groups and their interpretation in terms of braids and ribbons. The topological background may be found in [7] and [39], and a more leisurely exposition may be found in [18, §2].

4.1. Some generalised Artin braid groups

The generalised Artin group Γ_n of type B_n is well known to be the fundamental group of the space of unordered n-tuples of distinct points in $\mathbb{C}$. It may therefore be described in terms of "cylindrical braids", and the following facts are well known (cf. *loc. cit.*). The group Γ_n has generating set $\{\xi_1, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ with relations

$$
\begin{aligned}
&\sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ if } \ |i-j| \neq 1\\
&\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } \ i = 1, 2, \ldots, n-1\\
&\xi_1\sigma_1\xi_1\sigma_1 = \sigma_1\xi_1\sigma_1\xi_1\\
&\xi_1\sigma_i = \sigma_i\xi_1 \ \text{ if } \ i \neq 1.
\end{aligned}
\tag{4.1}
$$

Define elements $\tau = \xi_1\sigma_1\sigma_2\ldots\sigma_{n-1}$ and $\sigma_n = \tau\sigma_{n-1}\tau^{-1}$ in Γ_n. Then τ is represented as a "twist", and the following facts are well known (cf [18, §2]).

Proposition 4.2. (i) *We have $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ for each i, where the subscripts are taken mod n.*

(ii) *The element τ^n is in the centre of Γ_n (this follows from (i)).*

(iii) *The elements $\{\sigma_1, \ldots, \sigma_{n-1}\}$ are Coxeter generators of the Artin group $\mathcal{B}_n$ of type A_{n-1}, i.e. of the classical braid group on n strings.*

(iv) *The elements $\{\sigma_1, \ldots, \sigma_n\}$ are Coxeter generators of the Artin group Δ_n of affine type $\widetilde{A}_{n-1}$.*

(v) *There is an exact sequence $1 \longrightarrow \Delta_n \longrightarrow \Gamma_n \longrightarrow \mathbb{Z} \longrightarrow 1$, where $\Delta_n \longrightarrow \Gamma_n$ is inclusion and $\Gamma_n \longrightarrow \mathbb{Z}$ is the map taking $\tau^r\sigma_{i_1}^{n_1}\ldots\sigma_{i_l}^{n_l} \in \Gamma_n$ to $r \in \mathbb{Z}$.*

(vi) *If $\xi_{i+1} = \sigma_i\xi_i\sigma_i$ $(i = 1, 2, \ldots, n-1)$, then $\xi_1, \ldots, \xi_n$ generate a free abelian subgroup of rank n of Γ_n and $\tau^n = \xi_1\ldots\xi_n$ is in the centre of Γ_n.*

4.2. Some Hecke algebras

Let R be a commutative ring as in §2, and fix elements $q, Q \in R^\times$. The various Hecke algebras with which we are concerned all emanate from the group ring $R\Gamma_n$, where Γ_n is the Artin braid group of type B_n as above.

Definition 4.3. *Let S_i be the element $S_i = (\sigma_i - q)(\sigma_i + q^{-1})$ of $R\Gamma_n$ $(i = 1, 2, \ldots, n)$. The affine Hecke algebra $\widehat{H_n^a}(q)$ of GL_n over R is defined by*

$$\widehat{H_n^a}(q) = R\Gamma_n/\langle S_1\rangle.$$

Note that since $S_1, \ldots, S_n$ are all conjugate in $R\Gamma_n$, the ideal $\langle S_1\rangle$ is equal to $\langle S_1, \ldots, S_n\rangle$. Let $\eta : R\Gamma_n \longrightarrow \widehat{H_n^a}(q)$ be the natural map. We then write

$$
\begin{aligned}
\eta(\sigma_i) &= T_i \ \text{ for } \ i = 1, \ldots, n\\
\eta(\xi_i) &= X_i \ \text{ for } \ i = 1, \ldots, n\\
\eta(\tau) &= V.
\end{aligned}
\tag{4.4}
$$

Note that since $\tau = \xi_1\sigma_1\sigma_2\dots\sigma_{n-1}$, we have

$$V = X_1T_1\dots T_{n-1}. \tag{4.5}$$

We shall need to identify several sub- and quotient algebras of $\widehat{H_n^a}(q)$, and for this purpose we introduce the symmetric group $W \cong \mathrm{Sym}_n$, the corresponding affine Weyl group $W^a \cong \mathrm{Sym}_n \ltimes \mathbb{Z}^{n-1}$, which is a Coxeter group of rank n, and the Weyl group WB_n of type B_n.

The next proposition collects some well known facts concerning $\widehat{H_n^a}(q)$, many of which may be found in §3 of [33].

Proposition 4.6. (i) *The elements $T_1, \dots, T_n$ generate a subalgebra $H_n^a(q)$ of $\widehat{H_n^a}(q)$, which has R-basis $\{T_w \mid w \in W^a \cong Sym_n \ltimes \mathbb{Z}^{n-1}\}$, where, if $w = s_{i_1}\dots s_{i_\ell}$ is a reduced expression for $w \in W^a$, $T_w = T_{i_1}\dots T_{i_\ell}$. We refer to this as the "unextended" Hecke algebra of type $\widetilde{A_{n-1}}$.*

(ii) *The elements $T_1, \dots, T_{n-1}$ generate a subalgebra $H_n(q)$ of $\widehat{H_n^a}(q)$ which has (finite) R-basis $\{T_w \mid w \in W \cong Sym_n\}$.*

(iii) *We have $\widehat{H_n^a}(q) \cong R\mathbb{Z}\otimes H_n^a(q) \cong R\langle V\rangle \otimes H_n^a(q)$, where the tensor product is twisted, using the action of V on $H_n^a(q)$: $VT_iV^{-1} = T_{i+1}$, where the subscript is taken mod n.*

(iv) *(Bernstein) We have $\widehat{H_n^a}(q) \cong H_n(q) \otimes R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ as R-module, and the multiplication is given by the "Bernstein relations": for $i \in \{1, \dots, n-1\}$, write s_i for the corresponding simple reflection in W and ${}^{s_i}f$ for the image of $f \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ under the natural action of $W \cong Sym_n$. Then*

$$T_if - ({}^{s_i}f)T_i = (q - q^{-1})\frac{f - ({}^{s_i}f)}{1 - X_iX_{i+1}^{-1}}.$$

The (finite rank) Hecke algebra of type B_n, arises as follows. Let $WB_n := \mathrm{Sym}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the hyperoctahedral group. This is the quotient of Γ_n obtained by stipulating that each generator is an involution. Thus if $\{s_0, s_1, \dots, s_{n-1}\}$ are the images under the natural map of the generators $\{\xi_1, \sigma_1, \dots, \sigma_{n-1}\}$, WB_n is generated by the s_i, which are involutions and satisfy the relations analogous to 4.1 above. Let $Q \in R^\times$. The *Hecke algebra* $HB_n(q, Q)$ of type B_n with parameters (q, Q) is defined as

$$\begin{aligned} HB_n(q,Q) =& \widehat{H_n^a}(q)/\langle (X_1 - Q)(X_1 + Q^{-1})\rangle \\ =& R\Gamma_n/\langle (\xi_1 - Q)(\xi_1 + Q^{-1}), (\sigma_1 - q)(\sigma_1 + q^{-1})\rangle. \end{aligned}$$

Proposition 4.7. *Let $\eta_Q : \widehat{H_n^a}(q) \longrightarrow HB_n(q,Q)$ be the natural map. Write $T_i \in HB_n(q,Q)$ for the image of $T_i \in \widehat{H_n^a}(q)$ under η_Q ($i = 1,\dots,n-1$) (relying on the context to distinguish between them), and write $T_0 = \eta_Q(X_1)$. Then $HB_n(q,Q)$ has R-basis $\{T_w \mid w \in WB_n\}$, where, if $w = s_{i_1}\dots s_{i_\ell}$ is a reduced expression for $w \in WB_n$, $T_w = T_{i_1}\dots T_{i_\ell}$.*

As a special case of 4.7, if w is the Coxeter element $s_0s_1\dots s_{n-1}$ of WB_n, the corresponding basis element T_w of $HB_n(q,Q)$ is written $V(= T_0T_1\dots T_{n-1})$.

Another algebra which is often considered in this context is the affine Hecke algebra of SL_n, which is denoted $\widetilde{H_n^a(q)}$ in [17], and whose representation theory is discussed there. This is defined as the quotient of $\widehat{H_n^a}(q)$ by the ideal generated by $V^n - 1$. It is clear that in the notation of Proposition (4.6), $\widetilde{H_n^a(q)} \cong R(\mathbb{Z}/n\mathbb{Z}) \otimes H_n^a(q)$.

As a summary of the Hecke algebras which play a role in our story, we have the sequence

$$H_n(q) \xrightarrow{\text{incl}} H_n^a(q) \xrightarrow{\text{incl}} \widehat{H_n^a}(q) \xrightarrow{\eta_Q} HB_n(q,Q), \tag{4.8}$$

where the composition of the three maps is the obvious inclusion of $H_n(q)$ in $HB_n(q,Q)$ as the subalgebra generated by $T_1,\dots,T_{n-1}$.

4.3. Temperley-Lieb algebras of various types

We shall next explain how to realise the various algebras which have been introduced above by means of diagrams as quotients of Hecke algebras. Recall that $W = \langle s_1,\dots,s_{n-1}\rangle \cong \mathrm{Sym}_n$ above. Write $W_i = \langle s_i, s_{i+1}\rangle \cong \mathrm{Sym}_3$ for $i = 1,2,\dots,n-2$. Define the element $E_i \in H_n(q) \subset H_n^a(q) \subset \widehat{H_n^a}(q)$ by

$$E_i = \sum_{w\in W_i} q^{\ell(w)}T_w \tag{4.9}$$

where $\ell(w)$ denotes the usual length function. Let I (resp. $\widehat{I}$) denote the ideal of $H_n^a(q)$ (resp. $\widehat{H_n^a}(q)$) generated by E_1. Note that since the E_i are all conjugate (even in $H_n(q)$), this is the same as the ideal generated by all the E_i.

Definition 4.10. *The affine Temperley-Lieb algebras $TL_n^a(q)$ and $\widehat{TL_n^a}(q)$ are defined by*

$$\begin{aligned} TL_n^a(q) &= H_n^a(q)/I \\ \widehat{TL_n^a}(q) &= \widehat{H_n^a}(q)/\widehat{I} \end{aligned} \tag{4.11}$$

It is known (cf. [16], [17]) that if $C_i = -(T_i + q^{-1}) \in H_n^a(q)$ $(i = 1, \dots, n)$, then in $H_n^a(q)$, $C_iC_{i+1}C_i - C_i = C_{i+1}C_iC_{i+1} - C_{i+1} = -q^3E_i$, where the indices are taken mod n. If we abuse notation by writing $C_i \in TL_n^a(q)$ for the image of $C_i \in H_n^a(q)$ under the natural map, it follows easily that $TL_n^a(q)$ is generated by $\{C_1, \dots, C_n\}$ subject to the relations

$$\begin{aligned} C_i^2 &= \delta_q C_i \\ C_iC_{i\pm 1}C_i &= C_i \qquad (4.12) \\ C_iC_j &= C_jC_i \text{ if } |i-j| \geq 2 \text{ and } \{i,j\} \neq \{1,n\}, \end{aligned}$$

Just as in the case of Hecke algebras, we also have the affine Temperley-Lieb algebra $\widetilde{TL_n^a(q)}$ of SL_n (cf. [17]). In analogy with the (4.6), we have

$$\widetilde{TL_n^a(q)} \cong \widehat{TL_n^a}(q)/\langle V^n - 1\rangle \cong R(\mathbb{Z}/n\mathbb{Z}) \otimes TL_n^a(q). \qquad (4.13)$$

Proposition 4.14. (i) *We have*

$$\widehat{TL_n^a}(q) \cong R\langle V\rangle \otimes TL_n^a(q), \qquad (4.15)$$

where $V \in \widehat{TL_n^a}(q)$ (identified with the image of $V \in \widehat{H_n^a}(q)$) permutes the C_i cyclically.

(ii) *The elements $\{C_1, \dots, C_{n-1}\}$ generate a subalgebra of $\widehat{TL_n^a}(q)$ which is isomorphic to $TL_n(q)$. It may be mapped monomorphically into $T_n^a(q)$ via $C_i \mapsto f_i$ $(i = 1, \dots, n-1)$.*

(iii) *There is a family of surjections $\phi_\alpha : \widehat{TL_n^a}(q) \longrightarrow T_n^a(q)$ $(\alpha \in R^\times)$, defined by $\phi_\alpha(C_i) = f_i$ and $\phi_\alpha(V) = \alpha\tau_n$. Each ϕ_α restricts to the same monomorphism $TL_n^a(q) \to T_n^a(q)$.*

(iv) *The kernel of ϕ_α is generated by the element $\nu_\alpha = \alpha^{-2}V^2C_{n-1} - C_1C_2\dots C_{n-1}$ $(= \alpha^{-2}C_1V^2 - C_1C_2\dots C_{n-1})$ of $\widehat{TL_n^a}(q)$.*

(v) *If R is an algebraically closed field of characteristic prime to n, any irreducible finite dimensional representation of $\widehat{TL_n^a}(q)$ is the pullback via ϕ_α (for some $\alpha \in R^\times$) of an irreducible representation of $T_n^a(q)$.*

Sketch of proof. The proofs of (i) and (ii) are easy. The first part of (iii) follows immediately from the relations above, while the second follows from the fact (cf. [16, (2.9)]) that 4.12 gives a presentation of $TL_n^a(q)$. Next, one verifies easily that $\tau_n^2 f_{n-1} = f_1f_2\dots f_{n-1}$ in $T_n^a(q)$ (see [17, 1.11]), which shows that $\nu_\alpha \in \mathrm{Ker}\phi_\alpha$. The fact that ν_α generates the kernel may be found in [12] or [8]. This relation also appears in [20]. We now indicate how to prove (v). Let ρ be an irreducible representation of $\widehat{TL^a_n}(q)$. Since V^n is in the centre of $\widehat{TL^a_n}(q)$ (cf. 4.2(vi)), it follows that V^n acts as a scalar, say $\lambda \in R^\times$ in ρ. Hence ρ factors through $\widehat{TL^a_n}(q)/\langle V^n - \lambda.1\rangle$, which is isomorphic (cf. 4.13) via a map which is the identity on $TL^a_n(q)$, to $\widetilde{TL^a_n(q)}$. But by [17, Theorem (2.8)], every irreducible representation of $\widetilde{TL^a_n}(q)$ is the inflation via a homomorphism like ϕ_α of an irreducible representation of $T^a_n(q)$. □

We shall need an algebraic description of the Temperley-Lieb algebra of type B_n which we have met in §2 as an algebra with a basis of marked planar diagrams. It is constructed as a quotient of $HB_n(q,Q)$ as follows. Recall that since $H_n(q) \subset HB_n(q,Q)$ (4.8) the elements T_i, C_i of 4.14, as well as E_i defined in 4.9 all lie in $HB_n(q,Q)$. In addition, we have (cf. 4.7) $C_0 := -(T_0 + Q^{-1}) \in HB_n(q,Q)$. Clearly $HB_n(q,Q)$ is generated as algebra by $C_0, C_1, \dots, C_{n-1}$.

Proposition 4.16 ([18], §5). *There is a surjective homomorphism of associative R-algebras $HB_n(q,Q) \xrightarrow{\eta_4} TLB_n(q,Q)$, defined on the generators C_i by $\eta_4(C_i) = c_i$ $(i = 0,1,\dots,n-1)$. The kernel of η_4 is generated by E_1 and $C_1C_0C_1 - \kappa C_1 \in HB_n(q,Q)$, where $\kappa = \frac{q}{Q} + \frac{Q}{q}$ as in 2.7.*

Define $t_i \in TLB_n(q,Q)$ by $\eta_4(T_i) = t_i$ for $i = 0,1,\dots,n-1$. Then we have $t_0 = -(c_0 + Q^{-1})$ and $t_i = -(c_i + q^{-1})$ for $i = 1,2,\dots,n-1$. The t_i are all invertible. Because of its special role, we denote by v the element $\eta_4(V)$ of $TLB_n(q,Q)$ (see immediately following 4.7). Then $TLB_n(q,Q)$ is clearly generated by $\{v, c_1, \dots, c_{n-1}\}$, and $vc_iv^{-1} = c_{i+1}$ for $1 \le i \le n-2$.

The diagram below summarises the relationships among the algebras we have now introduced.

$$
\begin{tikzcd}
H_n(q) \arrow[r, "\eta_1"] \arrow[d, "\text{incl}"] & TL_n(q) \arrow[d, "\text{incl}"] & \\
H^a_n(q) \arrow[r, "\eta_2"] \arrow[d, "\text{incl}"] & TL^a_n(q) \arrow[d, "\text{incl}"] & \\
\widehat{H^a_n}(q) \arrow[r, "\eta_3"] \arrow[d, "\eta_Q"] & \widehat{TL^a_n}(q) \arrow[r, "\phi_\alpha"] \arrow[d, "\gamma_Q"] & T^a_n(q) \\
HB_n(q,Q) \arrow[r, "\eta_4"] & TLB_n(q,Q) &
\end{tikzcd}
\tag{4.17}
$$

We complete this section with a result which is instrumental in applying our results on $T_n^a(q)$ to the representation theory of $\widehat{H_n^a}(q)$.

Theorem 4.18 ([18], (5.11)). *Let $\beta \in R$ satisfy $\beta^2 = -q^{n-2}$. Then there is a unique surjective homomorphism $g_\beta : T_n^a(q) \longrightarrow TLB_n(q,Q)$ such that $g_\beta(f_i) = c_i$ for $i = 1, 2, \ldots, n-1$, and $g_\beta(\tau) = \beta v$. If α, μ satisfy $\alpha^{-1}\mu = \beta$, then the following diagram commutes,*

$$\begin{array}{ccccc} \widehat{H_n^a}(q) & \xrightarrow{\eta_3} & \widehat{TL_n^a}(q) & \xrightarrow{\phi_\alpha} & T_n^a(q) \\ \Big\downarrow{\scriptstyle \eta_{Q,\mu}} & & \Big\downarrow{\scriptstyle \gamma_{Q,\mu}} & & \Big\downarrow{\scriptstyle g_\beta} \\ HB_n(q,Q) & \xrightarrow{\eta_4} & TLB_n(q,Q) & \xrightarrow{id} & TLB_n(q,Q) \end{array} \tag{4.19}$$

where $\eta_{Q,\mu}(T_i) = \eta_Q(T_i)$ for $1 \le i \le n$, and $\eta_{Q,\mu}(X_1) = \mu\eta_Q(X_1) = \mu T_0$, and $\gamma_{Q,\mu}(T_i) = t_i$ (see immediately following 4.16) and $\gamma_{Q,\mu}(V) = \mu\gamma_Q(V) = \mu v$.

The proof of Theorem (4.18) involves proving the relation $c_1 v^2 = -q^{-(n-2)}c_1c_2\ldots c_{n-1}$ in $TLB_n(q,Q)$ (cf. the sentence following 4.16), and analysing the diagram 4.17.

§5. Lifting representations from finite to infinite dimensional algebras

5.1. Commutative diagrams and lifting representations

Let A and B be R-algebras and $\psi : A \to B$ be an R-algebra homomorphism. If $\rho : B \to \mathrm{End}_R(M)$ is a representation of B (where M is a free R-module) one may lift ρ to a representation $\psi^*\rho$ of A by composing with ψ. This is called the inflation of ρ via ψ We shall discuss in this section the lifting of representations in the diagram 4.19. Note that by 4.14(v), if R is an algebraically closed field of characteristic not dividing n, then every finite dimensional irreducible representation of $\widehat{TL_n^a}(q)$ is the lift via ϕ_α of an irreducible represetation of $T_n^a(q)$, for some $\alpha \in R^\times$.

It follows that the inflations to $\widehat{H_n^a}(q)$ of irreducible representations of $T_n^a(q)$ are precisely those irreducible representations of $\widehat{H_n^a}(q)$ which factor through $\widehat{TL_n^a}(q)$. Our next result describes the lifting of cell modules from $TLB_n(q,Q)$ to $T_n^a(q)$.

Theorem 5.1 ([18], Corollary (6.15)). *Suppose R is any commutative ring and suppose that q, Q are elements of $R^\times$. Let $\beta \in R^\times$ satisfy $\beta^2 = -q^{n-2}$ and let $g_\beta : T_n^a(q) \to TLB_n(q,Q)$ be the surjection defined in 4.18. For $t \in \mathbb{Z}$ such that $|t| \le n$ and $t \equiv n \pmod 2$, define $\epsilon_t := \frac{t}{|t|}$ for $t \neq 0$, and $\epsilon_t = 1$ if $t = 0$.*

Then the inflation $g_\beta^* W_t(n)$ of the cell module $W_t(n)$ of $TLB_n(q, Q)$ is isomorphic to the cell module $W_{|t|, z_t^{\epsilon_t}}(n)$ of $T_n^a(q)$, where

$$z_t = (-1)^t \beta Q^{-1} q^{-\frac{1}{2}(n+t-2)}. \tag{5.2}$$

It follows from 5.1 that the inflation $g_\beta^* L_t(n)$ of the irreducible quotient $L_t(n)$ (cf. 2.4(i)) of $W_t(n)$ is the irreducible $T_n^a(q)$-module $L_{|t|, z_t^{\epsilon_t}}$. This is because of the implied description of the irreducible quotients of cell modules in terms of homomorphisms between cell modules (see 2.4(ii)). It therefore follows (see 3.7) that

Corollary 5.3. *Fix $q \in R^\times$ and assume R is a field. Then every irreducible $T_n^a(q)$-module is the inflation via g_β of an irreducible $TLB_n(q, Q)$-module for some $Q \in R^\times$.*

To see this, one need only observe that given t, β, q and $z \in R^\times$, the equation $z_t = z$ (see 5.2) has the solution $Q = (-1)^t \beta z^{-1} q^{-\frac{1}{2}(n+t-2)}$.

Our objective is to study the inflations of $T_n^a(q)$-modules to $\widehat{H_n^a}(q)$, and to apply our knowledge of decomposition numbers for $T_n^a(q)$ (see 3.11) to the representations of $\widehat{H_n^a}(q)$. For this purpose we focus on the part of the diagram 4.19 below.

$$\begin{array}{ccc} \widehat{H_n^a}(q) & \xrightarrow{\psi_\alpha} & T_n^a(q) \\ \Big\downarrow{\scriptstyle \xi_{Q,\mu}} & & \Big\downarrow{\scriptstyle g_\beta} \\ TLB_n(q, Q) & \xrightarrow{\text{id}} & TLB_n(q, Q) \end{array} \tag{5.4}$$

where $\psi_\alpha = \phi_\alpha \circ \eta_3$ and $\xi_{Q,\mu} = \eta_4 \circ \eta_{Q,\mu}$.

By the commutativity of the diagram 5.4, we have

$$\psi_\alpha^* W_{t,z}(n) \cong \psi_\alpha^* g_\beta^* W_t(n) \cong \xi_{Q,\mu}^* W_t(n), \tag{5.5}$$

where $z = z_t$ as above, and α, β and μ are related by $\beta^2 = -q^{n-2}$ and $\mu\alpha^{-1} = \beta$. We wish to interpret the inflations $\psi_\alpha^* W_{t,z}(n)$ of the cell modules of $T_n^a(q)$ in terms of the *"standard modules"* of $\widehat{H_n^a}(q)$, and we therefore give a brief description of these.

5.2. Standard modules for $\widehat{H_n^a}(q)$

We assume henceforth that the ground ring R is $\mathbb{C}$, although many of our results apply more generally (see [18]). The standard modules for $\widehat{H_n^a}(q)$ may be defined cohomologically ([5, 24, 43]) or as induced modules ([41, 4, 44, 45]). The various definitions agree only up to equivalence in the Grothendieck group $\Gamma(\widehat{H_n^a}(q))$ of finite dimensional $\widehat{H_n^a}(q)$-modules, which suffices for our purpose, since we shall be discussing composition factors. Let $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{G} = \mathrm{Lie}(G)$. The standard modules are parametrised by the set $\widehat{\mathcal{P}}$ of equivalence classes of pairs (s, N) modulo G, where $s \in G$ is semisimple, $N \in \mathfrak{G}$ is nilpotent and $s.N = q^2 N$ (where G acts on $\mathfrak{G}$ through the adjoint representation).

Let us explain a key property of this correspondence. The algebra $\widehat{H_n^a}(q)$ has a subalgebra (4.6(iv)) $U(n) = \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, which may be thought of as the group algebra $\mathbb{C}(\mathbb{Z}^n)$, where $\mathbb{Z}^n = \langle X_1, \dots, X_n \rangle$. Any one-dimensional representation χ of $U(n)$ therefore corresponds to a character of $\mathbb{Z}^n$ and hence to the sequence $(\chi(X_1), \dots, \chi(X_n))$. Let $(s, N) \in \widehat{\mathcal{P}}$, so that N corresponds to a partition $\lambda = (\lambda_1 \le \lambda_2 \le \dots \le \lambda_\ell)$, where $\sum \lambda_i = n$ and $s = \mathrm{diag}(s_1, \dots, s_n)$ where $s_{i+1} = q^{-2} s_i$ whenever $i, i+1$ lie in the same "block" of N. Thus the sequence of eigenvalues of s consists of ℓ subsequences of size $\lambda_1, \lambda_2, \dots, \lambda_\ell$, which we call blocks, and each block is of the form $a, aq^{-2}, aq^{-4}, \dots$ for some $a \in \mathbb{C}^\times$. If $M_{s,N}$ is the corresponding standard module, its dimension is $K = \frac{n!}{\lambda_1! \dots \lambda_\ell!}$ and there is a filtration

$$0 \subset M_1 \subset M_2 \subset \dots \subset M_K = M_{s,N} \tag{5.6}$$

where $\dim M_i = i$, each term M_i is a $U(n)$-submodule of $M_{s,N}$, and the characters of $U(n)$ on the K quotients M_i / M_{i-1} $(i = 1, 2, \dots, K)$ are given by the K sequences $(\chi_1, \dots, \chi_n)$ obtained by permuting the sequence $(s_1, \dots, s_n)$ while keeping each block in its original order.

5.3. Lifting cell modules to standard modules

To state our main result, it is convenient to note that that there is an involution $\iota : \widehat{H_n^a}(q) \to \widehat{H_n^a}(q)$ which takes T_i to $-T_i^{-1}$ $(i = 1, \dots, n-1)$ and X_j to X_j^{-1} $(j = 1, \dots, n)$; this follows by noting that the images of the T_i and X_j under ι satisfy the the relations 4.6(iv). The surjection $\theta_\alpha : \widehat{H_n^a}(q) \to T_n^a(q)$ is defined by (cf. 5.4)

$$\theta_\alpha = \psi_\alpha \circ \iota. \tag{5.7}$$

Theorem 5.8 ([18], (9.9)). *Let $\widehat{H_n^a}(q)$ be the affine Hecke algebra of $\mathrm{GL}_n(\mathbb{C})$ and let $\Gamma(\widehat{H_n^a}(q))$ be the Grothendieck group of finite dimensional $\widehat{H_n^a}(q)$-modules. Let $W_{t,z}(n)$ be a cell module for the diagram algebra $T_n^a(q)$. Then, with θ_α as in 5.7, we have*

$$[\theta_\alpha^* W_{t,z}(n)] = [M_{s,N}] \tag{5.9}$$

where $[V]$ denotes the class in $\Gamma(\widehat{H_n^a}(q))$ of an $\widehat{H_n^a}(q)$-module V and $M_{s,N}$ is the Kazhdan-Lusztig standard module of $\widehat{H_n^a}(q)$ corresponding *to the pair* $(s, N) \in \widehat{\mathcal{P}}$ *where* $N = N_k$ *is the nilpotent Jordan matrix corresponding to the 2-step partition* $(n-k, k)$ *and* s *is the diagonal matrix* $\operatorname{diag}(a, aq^{-2}, \ldots, aq^{-(n+t-2)}, b, bq^{-2}, \ldots, bq^{-(n-t-2)})$, *where the parameters* a, b *and* k *are given by*

$$
\begin{aligned}
k =& \frac{n-t}{2} \\
a =& (-1)^{n+1}\alpha z q^{\frac{1}{2}(n+t-2)} \qquad (5.10)\\
b =& (-1)^{n+1}\alpha z^{-1} q^{\frac{1}{2}(n-t-2)}
\end{aligned}
$$

Some remarks concerning the proof. We begin by noting (see [1, Theorem 3.2, p.798]) that $M_{s,N}$ is equivalent in $\Gamma(\widehat{H_n^a}(q))$ to an induced module, which has an easy characterisation [18, (9.3)] in terms of a generating vector. One then shows that when the pair (q, z) is generic (see [18, (8.1)] for a definition; it suffices that there be no non-trivial polynomial F such that $F(q, z) = 0$) then $\theta_\alpha^* W_{t,z}(n)$ is actually isomorphic to the induced module. Finally a number-theoretic specialisation argument is used to prove the result in general.

The key to the proof is the treatment of the generic case. The heart of the proof in this case is the determination of the characters of the algebra $U(n)$, i.e. the simultaneous eigenvalues of the elements X_i, on $\theta_\alpha^* W_{t,z}(n)$, or equivalently, on $\psi_\alpha^* W_{t,z}(n)$. For this, we use the identification $\psi_\alpha^* W_{t,z}(n) \cong \xi_{Q,\mu}^* W_t(n)$ as in (5.5), where the parameters are as stated there. Now to analyse the action of the X_i on $\xi_{Q,\mu}^* W_t(n)$, we start by observing that $X_1 X_2 \ldots X_n = V^n$ is in the centre of $\widehat{H_n^a}(q)$ (4.2(v)), and acts on $\psi_\alpha^* W_{t,z}(n)$ as a scalar (since τ_n^n acts on $W_{t,z}(n)$ as a scalar), which may be easily computed. We now use the filtration 2.12 of $W_t(n)$ to obtain one for $\psi_\alpha^* W_{t,z}(n) \cong \xi_{Q,\mu}^* W_t(n)$, noting that by the above argument, $X_1 X_2 \ldots X_i$ acts via a (known) scalar on the i^{th} term of the filtration. By this means one finds a simultaneous eigenvector for the X_i, and using the fact that $X_i = (X_1 X_2 \ldots X_i)(X_1 X_2 \ldots X_{i-1})^{-1}$ one knows the corresponding character of $U(n)$. Proceeding by induction, one obtains a filtration of $\psi_\alpha^* W_{t,z}(n)$ like that described in 5.6.

Assuming genericity, the characters of $U(n)$ arising in this filtration are all distinct, so that $\psi_\alpha^* W_{t,z}(n)$ is semisimple as $U(n)$-module. This permits the determination of the action of the T_j on a certain summand, completing the proof for the generic case. The passage from the generic case to the general case involves a number-theoretic argument (cf. [3]). We refer the reader to [18, §§6,7,8 and 9] for all details. □

We denote the pair s, N in the statement 5.8 above by $s(a, b), N_k$.

§6. On the decomposition numbers of standard modules

It is clear from Theorem 5.8 and the remark preceding 5.1 that the standard modules of $\widehat{H^a_n}(q)$ which factor through representations of $\widehat{TL^a_n}(q)$ are precisely the $M_{s,N}$ where $N = N_k$ is a two-step nilpotent element of $\mathfrak{G}$ as above. Since the relations 5.10 may be inverted to give α, z and t in terms of s, k, these modules are all realised, up to Grothendieck equivalence, as inflations of cell modules of $T^a_n(q)$. Hence the results of Theorems 3.11 and 3.14 may be applied to analyse their composition factors.

After working through the details of the correspondence provided by Theorem 5.8 and the properties of the cell modules $W_{t,z}(n)$, the following precise statement results.

Proposition 6.1 ([18], 9.12). *Let $\mathcal{P}$ be the set equivalence classes of pairs $(s, N) \in \widehat{\mathcal{P}}$ where $N \in \mathfrak{G}$ is two-step nilpotent, i.e. $N \sim N_k$ for some k with $0 \le 2k \le n$. Let $\widetilde{\Omega}$ be the set of triples (t, α, z) $(t \in \mathbb{Z}, 0 \le t \le n, n-t \in 2\mathbb{Z}; \alpha, z \in \mathbb{C}^\times)$ and let Ω be the set of equivalence classes of triples in $\widetilde{\Omega}$ under the equivalence generated by the relations $(t, \alpha, z) \sim (t, -\alpha, -z)$, $(n, \alpha, z) \sim (n, y^{-1}z\alpha, y)$ and $(0, \alpha, z) \sim (0, \alpha, z^{-1})$. Then (with the obvious abuse of notation) we have well defined $\widehat{H^a_n}(q)$-modules $M_{s,N}$, $(s, N) \in \mathcal{P}$ and $\theta^*_\alpha W_{t,z}(n)$, $(t, \alpha, z) \in \Omega$, and there is a bijection $f : \mathcal{P} \to \Omega$ such that if $(s, N) \in \mathcal{P}$ corresponds to $(t, \alpha, z) \in \Omega$, $[M_{s,N}] = [\theta^*_\alpha W_{t,z}(n)]$.*

This correspondence permits us to define the irreducible "top quotients" $L_{s,N}$ for $(s, N) \in \mathcal{P}$ as the pullback of the corresponding top quotient of $W_{t,z}(n)$, which is either zero or irreducible. Translating the statement 3.7(i) into the language of pairs, we obtain

Proposition 6.2. *The module $L_{s(a,b),N_k}$ is zero if and only if*

$$(6.3) \quad q^2 = -1, \ n \text{ is even}, \ k = \frac{n}{2}, \ a = \alpha, \ b = -\alpha, \ \text{for some } \alpha \in \mathbb{C}^\times.$$

We therefore denote by $\mathcal{P}_0$ the set of (equivalence classes of) pairs in $\mathcal{P}$ which do not satisfy the condition 6.3. By Theorem 3.7(i), $\mathcal{P}_0$ parametrises the irreducible $\widehat{H^a_n}(q)$-modules which factor through $\widehat{TL^a_n}(q)$.

The partial order $\preceq$ on $\Lambda^a(n)^{0^+}$ induces one on $\mathcal{P}$ using the correspondence 6.1. Specifically we have

Proposition 6.4. *Suppose that under the correspondence 6.1 above, the triples (t_1, α, z_1), (t_2, α, z_2) correspond to the pairs $(s(a_1, b_1), N_{k_1})$, $(s(a_2, b_2), N_{k_2})$ respectively. Then $(t_1, z_1) \overset{\circ}{\prec} (t_2, z_2)$ if and only if there* *exists $\ell > 0$ and $\epsilon = \pm 1$ such that if we write $2k_i = n - t_i$ for $i = 1, 2$, then*

$$
\begin{aligned}
k_2 &= k_1 - \ell \geq 0 \\
a_1 b_1^{-1} &= q^{t_1 + \epsilon t_2} \\
(a_2, b_2) &= (a_1, b_1) \text{ if } \epsilon = 1 \\
&\quad (b_1, a_1) \text{ if } \epsilon = -1
\end{aligned}
\tag{6.5}
$$

Now Theorems 3.11 and 3.14 yield immediately

Theorem 6.6. *Let $\mathcal{P}$ and $\mathcal{P}_0$ be the sets of semisimple-nilpotent pairs defined above, and let $\preceq$ be the partial order on $\mathcal{P}$ generated by the relation 6.5. Then in the Grothendieck group of finite-dimensional $\widehat{H_n^a}(q)$-modules,*

$$[M_{s,N}] = \sum_{\substack{(s',N') \in \mathcal{P}_0 \\ (s,N) \preceq (s',N')}} [L_{s',N'}] \quad \text{for any pair } (s, N) \in \mathcal{P}, \tag{6.7}$$

and for $(s, N) \in \mathcal{P}_0$,

$$[L_{s,N}] = \sum_{\substack{(s',N') \in \mathcal{P} \\ (s,N) \preceq (s',N')}} n_{s,N}^{s',N'} [M_{s',N'}] \tag{6.8}$$

where $n_{s,N}^{s',N'} = 0$ or ± 1.

We conclude this section by giving some applications to the structure of the standard modules which are easy consequences of Theorem 6.6

Corollary 6.9. (i) *The standard modules $M_{s,N}$ ($(s, N) \in \mathcal{P}$) are multiplicity free.*

(ii) *If q is not a root of unity, the standard modules have at most 2 composition factors.*

(iii) *In all cases, $M_{s,N}$ has composition length bounded by $\left[\frac{n}{2}\right]$.*

(iv) *The maximum composition length of $M_{s,N}$ is $\left[\frac{n}{2}\right]$, and therefore is unbounded as $n \to \infty$.*

§7. Connection with representation varieties of quivers

In this section we continue to consider the representations of $\widehat{H_n^a}(q)$, where the underlying ring is $\mathbb{C}$. In this case, we explain an interpretation of 6.7,6.9 in terms of the intersection cohomology of certain varieties.

Specifically, we show how (6.9)(i) is equivalent to the rational smoothness of certain varieties which we define below. This interpretation of decomposition numbers in terms of intersection cohomology is due to Chriss-Ginzburg [5], Kazhdan-Lusztig [24] and Lusztig [33, 34, 35], (see also [11]) and the exposition here owes much to A. Henderson, who has given an alternative direct geometric proof of our (6.9)(i) starting from the geometric point of view (see [19]). Write $G = GL_n(\mathbb{C})$ and $\mathfrak{G} = \mathrm{Lie}(G)$. The multiplicities of the irreducible $\widehat{H_n^a}(q)$-modules in the standard modules $M_{s,N}$ are given in terms of the intersection cohomology of certain complexes on the closures of the orbits of the group $Z_G(s)$ acting on the variety

$$\mathcal{N}_{s,q} = \{N \in \mathfrak{G}_{\mathrm{nil}} \mid s \cdot N = q^2 N\}. \tag{7.1}$$

Specifically, for $N \in \mathcal{N}_{s,q}$, let $\mathcal{O}_N$ be its orbit under $Z_G(s)$. Let $\leq$ be the partial order on the (finite) set of orbits of $Z_G(s)$ on $\mathcal{N}_{s,q}$ which is defined by orbit closure. Let $\mathscr{H}^k(IC(\overline{\mathcal{O}_N}))$ denote the k^{th} cohomology sheaf of the perverse extension of the constant sheaf on $\mathcal{O}_N$ to the closure $\overline{\mathcal{O}_N}$. It is known (see below) that $\mathscr{H}^k(IC(\overline{\mathcal{O}_N})) = 0$ if k is odd. For $N' \in \mathcal{N}_{s,q}$ such that $\mathcal{O}_{N'} \leq \mathcal{O}_N$, write $\mathscr{H}^k_{N'}(IC(\overline{\mathcal{O}_N})) = 0$ for the stalk at any point of $\mathcal{O}_{N'}$ of this sheaf, and define

$$\tilde{K}_{N,N'}(t) = \sum_{j \geq 0} \dim \mathscr{H}^{2k}_{N'}(IC(\overline{\mathcal{O}_N}))t^k \tag{7.2}$$

Now the modules $M_{s,N}$ generally have a top quotient $L_{s,N}$, which is irreducible or zero, and the non-zero modules among the $L_{s,N}$ form a complete set of irreducible $\widehat{H_n^a}(q)$-modules (see [5, Chapter 8] or [43]).

Theorem 7.3 (Ginzburg [5], 8.6.23) (see also [1] and [11]). *We have the following multiplicity formula for the standard modules $M_{s,N}$. Assume $L_{s,N} \neq 0$. Then*

$$[M_{s',N'} : L_{s,N}] = \begin{cases} 0 \textit{ unless } s' \sim_G s \\ \tilde{K}_{N,N'}(1) \textit{ if } s = s' \textit{ and } \mathcal{O}_{N'} \leq \mathcal{O}_N \\ 0 \textit{ if } s = s' \textit{ and } \mathcal{O}_{N'} \not\leq \mathcal{O}_N \end{cases} \tag{7.4}$$

Here $\sim_G$ denotes conjugacy in $G = GL_n(\mathbb{C})$.

Let $V = \mathbb{C}^n$, on which both $G = GL_n(\mathbb{C})$ and $\mathfrak{G}$ act via matrix multiplication. For any linear transformation A of V and element $\xi \in \mathbb{C}$, write $V(A,\xi)$ for the ξ-eigenspace of A in V. For any pair (s, N) as above, write $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ for the s-eigenspace decomposition of V, where $V_i = V(s,\xi_i)$ and the ξ_i are distinct. The relation (7.1) shows that

$$N \cdot V(s,\xi_i) \subset V(s,q^2\xi_i). \tag{7.5}$$

Multiplication by q^2 partitions $\mathbb{C}^\times$ into orbits which are infinite if q^2 is not a root of unity and have cardinality e if q^2 has finite multiplicative order e (say). These orbits are linearly (resp. cyclically) ordered if the order of q^2 is infinite (resp. finite). We may assume that the ξ_i are ordered so that each orbit occurs as a sequence of successive elements in $(\xi_1,\cdots,\xi_k)$ in the specified linear or cyclic order, and that the ξ_j in a given orbit occur as a "block", which we denote by s_i $(i=1,\cdots,r)$, in the matrix s. Let $k_1,\cdots,k_r$ be the sizes of these blocks, so that $k_1+\cdots+k_r=n$. The nilpotent element N correspondingly decomposes into blocks: $N=N_1\oplus\cdots\oplus N_r$, where N_i is of size k_i and $s_i\cdot N_i=q^2N_i$ for $i=1,\cdots,r$. The following facts may be found in [4, 41, 1].

Proposition 7.6. *(i) The standard module $M_{s,N}$ is Grothendieck equivalent to the induced module*

$$Ind_{\widehat{H^a_\lambda}(q)}^{\widehat{H^a_n}(q)}\widetilde{\otimes_{i=1}^r M_{s_i,N_i}} \tag{7.7}$$

where M_{s_i,N_i} is the standard module of $\widehat{H^a_{k_i}}(q)$, $\lambda=(k_1,\cdots,k_r)$ and $\widetilde{\otimes_{i=1}^r M_{s_i,N_i}}$ is the lift from $\otimes_i\widehat{H^a_{\lambda_i}}(q)$ to $\widehat{H^a_{\lambda_i}}(q)$ of $\otimes_{i=1}^r M_{s_i,N_i}$.

(ii) If ρ_i is a composition factor of M_{s_i,N_i} $(i=1,\cdots,r)$, then $Ind_{\widehat{H^a_\lambda}(q)}^{\widehat{H^a_n}(q)}\widetilde{\otimes_{i=1}^r\rho_i}$ is irreducible.

(iii) The standard module $M_{s,N}$ is multiplicity free if and only if each component M_{s_i,N_i} is multiplicity free.

Thus the decomposition of $M_{s,N}$ may be discussed in terms of that of the M_{s_i,N_i}; in this way one reduces to the case where all the eigenvalues ξ_j of s are in the same q^2-orbit in $\mathbb{C}$. This may also be seen from the geometric description of the decomposition numbers given above (see (7.3)) as follows.

The centraliser $Z_G(s)\cong\prod_{i=1}^r GL_{n_i}(\mathbb{C})$, where n_i is the size of the block s_i (or N_i). It is clear that the orbit $\mathcal{O}_N\subset\mathcal{N}_{s,q}$ is isomorphic to the product $\prod_{i=1}^r\mathcal{O}_{N_i}$, where $\mathcal{O}_{N_i}:=Z_{GL_{n_i}(\mathbb{C})}(s_i)\cdot N_i$. It follows from the Kunneth theorem for intersection cohomology, that in the above notation,

$$\tilde{K}_{N,N'}(t)=\prod_i\tilde{K}_{N_i,N_i'}(t), \tag{7.8}$$

where N'_i is the i-block of N', regarded as an element of $\mathfrak{G}_i$ =Lie $GL_{n_i}(\mathbb{C})$.

Thus for the decomposition of $M_{s,N}$, it suffices to consider the case when all eigenvalues of s are in the same q^2-orbit, and hence after multiplication by a scalar, one may assume that they are powers of q^2. The relation (7.5) then shows that the pair (s, N) defines a representation of the cyclic or linear quiver, and this is a convenient context for an explicit discussion of the combinatorial geometric interpretation of our result (6.9)(i).

We shall therefore formulate our result in the language of quiver representations. We start by recalling some elementary facts concerning quiver representations, most of which may be found in [40].

Fix an element $e \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{Q}_e$ be the cyclic quiver of type $\widetilde{A}_{e-1}$. For $e = 0$, this is interpreted as the linear quiver A_∞. If we write $I = \mathbb{Z}/e\mathbb{Z}$, $\mathcal{Q}_e$ has vertex set I and an arrow from i to $i-1$ for each $i \in I$. A *representation* of $\mathcal{Q}_e$ is a pair (V, E), where $V = \oplus_{i\in I} V_i$ is an I-graded $\mathbb{C}$-vector space, and $E = \oplus_{i\in I} E_i \in E_V$, where $E_V = \oplus_{i\in I}\mathrm{Hom}_{i\in I}(V_i, V_{i-1}) \subset \mathrm{End}(V)$. The dimension vector of (V, E) is denoted $\mathbf{d} = (d_1, ..., d_e)$, where d_i =dimV_i, and its total dimension is written $n = \sum_{i=1}^n d_i$. We have a notion of equivalence for representations of $\mathcal{Q}_e$, which is defined in the obvious way. For a fixed graded vector space V, the set of all representations (V, E) is identified with the affine space E_V in the obvious way, and the group $G_V := \prod_{i\in I} GL(V_i)$ acts on this space by conjugation. The representation $E \in E_V$ (and its G_V-orbit) is said to be nilpotent if E is nilpotent as an endomorphism of V; the set of nilpotent representations of $\mathcal{Q}_e$ on V is clearly (Zariski)-closed in E_V, and we denote this variety by $\mathcal{N}_V$.

Suppose henceforth that q^2 has multiplicative order e, where $e = 0$ if q is not a root of unity. Let (s, N) be a pair with $N \in \mathcal{N}_{s,q}$, and suppose all eigenvalues of s are powers of q^2, and hence are parametrised by I (we have seen that the general case may be reduced to this one). Then by (7.5), the s-eigenspace decomposition $V = \oplus_{i\in I} V_i$ of $V = \mathbb{C}^n$ defines, together with N, a nilpotent representation of the quiver $\mathcal{Q}_e$. Conversely, given such a representation, $V = \oplus_{i\in I} V_i$, with a nilpotent endomorphism $E : V_j \to V_{j-1}$ ($j \in I$), define $s \in GL(V)$ as the linear transformation which acts on V_i as multiplication by q^{-2i}. Then $E \in \mathcal{N}_{s,q}$. The following statement is now clear.

Proposition 7.9. *(i) There is a bijection between $GL_n(\mathbb{C})$-conjugacy classes of pairs (s, N) such that the eigenvalues of s are powers of q and $s \cdot N = q^2 N$, and isomorphism classes of nilpotent representations (V, E) of the quiver $\mathcal{Q}_e$, with total dimension n.*

(ii) If (s, N) and (V, E) correspond under this bijection (recall that V comes with its grading), the variety $\mathcal{O}_N$ above is isomorphic to the G_V-orbit of E in $\mathcal{N}_V$.

We next give a parametrisation of the G_V-orbits on $\mathcal{N}_V$. By (7.9)(i), as the pair (V, E) varies, V being an I-grading of $\mathbb{C}^n$ and $E \in \mathcal{N}_V$, these orbits will index the standard modules of $\widehat{H_n^a}(q)$ in (7.9), and by (7.3) and (7.9)(ii), the geometry of any orbit determines the decomposition of the corresponding standard module. Take $E \in \mathcal{N}_V$. Since E is a nilpotent endomorphism of V, it corresponds to a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p(\lambda)} > 0$ of n (we refer to $p(\lambda)$ as the number of parts of the partition λ and to n as its size). Of course in general λ does not determine the G_V-class of E. To describe the latter, we require the following description of the indecomposable nilpotent representations of $\mathcal{Q}_e$; an arbitrary nilpotent representation is uniquely (up to order of the summands) expessible a sum of nilpotent indecomposables. The nilpotent indecomposables are parametrised by pairs (i, ℓ), $i \in I, \ell \in \mathbb{Z}_{>0}$; denote the corresponding representation of $\mathcal{Q}_e$ by $M(i; \ell)$. If $\ell = ae + b$, $(a, b \in \mathbb{Z}_{\geq 0}, 0 \leq b < e)$, the graded vector space $V = \oplus_{j \in I} V_j$ has

$$d_j = \begin{cases} a + 1 \text{ for } j = i, i + 1, \cdots, i + b - 1 \\ a \text{ otherwise.} \end{cases} \tag{7.10}$$

If V_j has basis $\{v_{j,1}, \cdots, v_{j,d_j}\}$, the homomorphisms $E_j : V_{j+1} \to V_j$ are described by

$$v_{j+1,c} \xmapsto{E_j} \begin{cases} v_{j,c} \text{ if } j \neq i \\ v_{j,c+1} \text{ if } j = i \\ \text{where } v_{j,c} \text{ is defined to be 0 if } c > d_j. \end{cases} \tag{7.11}$$

The pair $(i; \ell)$ may be identified with the sequence $(i, i+1, \cdots, i+\ell-1)$ of elements of I, and so is sometimes referred to as a *segment.* Note that the total dimension of $M(i; \ell)$ is ℓ. Any nilpotent representation M of $\mathcal{Q}_e$ of total degree n is equivalent to a sum $M \cong M(i_1; \ell_1) \oplus \cdots \oplus M(i_p; \ell_p)$, where $\sum_j \ell_j = n$. The multiset $\{(i_j, \ell_j) \mid j = 1, \cdots, p\}$ is uniquely determined by the isomorphism class of M and conversely; hence the isomorphism classes are often referred to as *"multisegments"*. If the segments are ordered so that the ℓ_j are in weakly decreasing order, the sequences corresponding to the (i_j, ℓ_j) may be written in an array, which may be thought of as a Young diagram whose boxes are labelled with elements of I, which increase by one across the rows. The dimension vector of the representation M is $\mathbf{d} = (d_0, \cdots, d_{e-1})$, where d_i is the number of entries of the array which are equal to i. If the partition of n which corresponds to the Young diagram is λ, we call the array an $I-$ *labelling* of λ, or simply an I-partition (of n). Clearly λ describes the $GL(V)$-orbit of the nilpotent transformation E.

Remark 7.12. The distinct I-partitions of n parametrise the nilpotent representations of $\mathcal{Q}_e$ of total dimension n, with two I-partitions deemed to be the same if one is obtained from the other by permuting rows (necessarily of the same length). We denote an I-labelling of λ by a symbol (λ, l), where l refers to the labelling. In view of (7.9)(i), corresponding to each I-partition (λ, l), we have a standard module for $\widehat{H_n^a}(q)$, which we denote by $M_{(\lambda,l)}$, which is uniquely defined up to isomorphism; similarly for the top quotient $L_{(\lambda,l)}$ mentioned above. Further, in view of (7.9)(ii), we also have associated to (λ, l) a corresponding variety, which we denote $\mathcal{O}_{(\lambda,l)}$; the closure relation among these varieties defines a partial order among I-partitions, which we denote $\leq$.

Lusztig (see [30, §5 and Appendix]) has given a necessary and sufficient criterion for $L_{(\lambda,l)}$ to be non-zero. Say that (λ, l) is aperiodic if for each $m \in \mathbb{Z}_{>0}$ there exists an element $i \in I$ such that (λ, l) does not have a row of length m starting with i. Then $L_{(\lambda,l)} \neq 0$ if and only if (λ, l) is aperiodic.

We may now express (7.3) in the language of I-partitions as follows. Note that the dimension vector $\mathbf{d}$ of an I-partition determines the "s" in the corresponding pair (s, N).

Corollary 7.13. *Let (λ, l) and (λ', l') be two I-partitions with the same dimension vector such that $(\lambda', l') \leq (\lambda, l)$ in the sense above, and suppose that (λ, l) is aperiodic. Define*

$$\tilde{K}_{(\lambda,l),(\lambda',l')}(t) = \sum_{j\geq 0} dim\, \mathscr{H}^{2k}_{(\lambda',l')}(IC(\overline{\mathcal{O}_{(\lambda,l)}}))t^k. \tag{7.14}$$

Then

$$[M_{(\lambda',l')} \; : \; L_{(\lambda,l)}] = \tilde{K}_{(\lambda,l),(\lambda',l')}(1). \tag{7.15}$$

If (λ, l) and (λ', l') do not have the same dimension vector, $[M_{(\lambda,l)} \; : \; L_{(\lambda',l')}] = 0$.

Our result (6.9) pertains to I-partitions with at most 2 parts (rows). Note that in this case, the criterion (6.3) given in our result (6.2) for $L_{(\lambda,l)}$ to be non-zero coincides with Lusztig's aperiodicity. Denote the set of I-partitions with dimension vector $\mathbf{d}$ by $\mathcal{P}(\mathbf{d})$, and the set of elements of $\mathcal{P}(\mathbf{d})$ with at most 2 parts by $\mathcal{P}^{\leq 2}(\mathbf{d})$. The subsets corresponding to I-partitions (λ, l) with $L_{(\lambda,l)} \neq 0$ will be denoted respectively by $\mathcal{P}_0(\mathbf{d})$ and $\mathcal{P}_0^{\leq 2}(\mathbf{d})$. Clearly if $(\mu, m) \in \mathcal{P}^{\leq 2}(\mathbf{d})$, $((\lambda, l)) \in \mathcal{P}(\mathbf{d})$ and $(\mu, m) \leq (\lambda, l)$, then $(\lambda, l) \in \mathcal{P}^{\leq 2}(\mathbf{d})$. Our (6.9) may now be stated as follows

Theorem 7.16. *Let* $\mathbf{d}$ *be a fixed dimension vector and suppose that* $(\lambda, l), (\lambda', l') \in \mathcal{P}^{\leq 2}(\mathbf{d})$, *and* $(\lambda', l') \leq (\lambda, l)$. *Then*

$$\tilde{K}_{(\lambda,l),(\lambda',l')}(t) = 1. \tag{7.17}$$

For $(\lambda, l) \in \mathcal{P}_0^{\leq 2}(\mathbf{d})$, (7.16) follows immediately from (6.9) and (7.17), because $K_{(\lambda,l),(\lambda',l')}(1) = 1$ implies that $K_{(\lambda,l),(\lambda',l')}(t) = 1$, since $K_{(\lambda,l),(\lambda',l')}(t) \in \mathbb{Z}_{\geq 0}[t]$. But if (λ, l) is not aperiodic, then as observed above $\lambda = (m, m)$ for some m, and so (λ, l) is minimal in $\mathcal{P}^{\leq 2}(\mathbf{d})$, whence $(\lambda, l) = (\lambda', l')$ and (7.17) holds trivially.

This means that the variety $\overline{\mathcal{O}_{(\lambda,l)}}$ is rationally smooth (but not generally smooth) at the points of $\mathcal{O}_{(\lambda',l')}$. A. Henderson has given ([19]) a geometric proof of Theorem (7.16) using an explicit resolution of $\overline{\mathcal{O}_{(\lambda,l)}}$

Note that in the case $e = 1$ (i.e. $q^2 = 1$), $(\lambda, l) = \lambda$ (i.e. the label is irrelevant) and Lusztig has shown ([34]) that

$$\tilde{K}_{\lambda,\lambda'}(t) = t^{n(\lambda')-n(\lambda)} K_{\lambda,\lambda'}(t^{-1}),$$

where $n(\lambda) = \sum_j (j-1)\lambda_j$, and $K_{\lambda,\lambda'}(t)$ is the Kostka-Foulkes polynomial. In general, Lusztig [35, §11] has shown that for any $(\lambda, l) \in \mathcal{P}(\mathbf{d})$, the variety $\overline{\mathcal{O}_{(\lambda,l)}}$ may be embedded as an open subvariety of an affine Schubert variety of type $\tilde{A}_{n-1}$. It follows from this that $\mathcal{H}^k(IC(\overline{\mathcal{O}_{(\lambda,l)}})) = 0$ for k odd. Furthermore, there is is an order preserving injective map $\mathcal{P}(\mathbf{d}) \hookrightarrow W^a$ (notation as in §2 above) such that if w, w' are the respective images of $(\lambda, l), (\lambda', l')$, then $\tilde{K}_{(\lambda,l),(\lambda',l')}(t) = P_{w,w'}(t)$, the usual Kazhdan-Lusztig polynomial associated with the coxeter system of type $\tilde{A}_{n-1}$. Thus our result also implies the triviality of certain special affine Kazhdan-Lusztig polynomials.

We remark finally is that the results of [18] may be used to discuss aspects of the representation theory of the affine Hecke algebra $\widehat{H_n^a}(q)$ over any algebraically closed field of positive characteristic, i.e. the "modular case". This is carried out for the algebras $TLB_n(q, Q)$ in [6].

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