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Appendix: Braiding compatibilities

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§1. Introduction

Let us recall the following basic constructions from [3]: [1] for $\mathcal{S} \in \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$, by taking nearby cycles we obtain $Z(\mathcal{S}) \in \mathrm{Perv}_I(\mathrm{Fl})$. Moreover, for $\mathcal{S}$ as above and $\mathcal{T} \in \mathrm{Perv}_I(\mathrm{Fl})$ we have a perverse sheaf $\mathcal{C}(\mathcal{S},\mathcal{T}) \in \mathrm{Perv}_I(\mathrm{Fl})$ with isomorphisms

$$Z(\mathcal{S}) \star \mathcal{T} \to \mathcal{C}(\mathcal{S},\mathcal{T}) \leftarrow \mathcal{T} \star Z(\mathcal{S}).$$

We will denote the resulting isomorphism $Z(\mathcal{S}) \star \mathcal{T} \to \mathcal{T} \star Z(\mathcal{S})$ by $u_{\mathcal{S},\mathcal{T}}$.

In addition, we will denote by $v_{\mathcal{S}_1,\mathcal{S}_2}$ the morphism $Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \to Z(\mathcal{S}_1 \star \mathcal{S}_2)$ for $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$.

There are 3 properties to check:

1) Let $\mathcal{T}_1, \mathcal{T}_2$ be two I-equivariant perverse sheaves on Fl, and $\mathcal{S}$ be a $G(\widehat{\mathcal{O}})$-equivariant perverse sheaf on Gr. We must have a commutative diagram:

$$\begin{array}{ccc}
Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 & \xrightarrow{u_{\mathcal{S},\mathcal{T}_1} \star \mathrm{id}_{\mathcal{T}_2}} & \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \\
{\scriptstyle u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\mathcal{T}_1} \star u_{\mathcal{S},\mathcal{T}_2}} \\
\mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}) & \xrightarrow{\mathrm{id}} & \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}).
\end{array}$$

2) Let $\mathcal{S}_1, \mathcal{S}_2$ be two $G(\widehat{\mathcal{O}})$-equivariant perverse sheaves on Gr and $\mathcal{T}$–an I-equivariant perverse sheaf on Fl. We must have a commutative diagram:

$$\begin{array}{ccccc}
Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \star \mathcal{T} & \xrightarrow{\mathrm{id}_{Z(\mathcal{S}_1)} \star u_{\mathcal{S}_2,\mathcal{T}}} & Z(\mathcal{S}_1) \star \mathcal{T} \star Z(\mathcal{S}_2) & \xrightarrow{u_{\mathcal{S}_1,\mathcal{T}} \star \mathrm{id}_{Z(\mathcal{S}_2)}} & \mathcal{T} \star Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \\
{\scriptstyle v_{\mathcal{S}_1,\mathcal{S}_2} \star \mathrm{id}_{\mathcal{T}}} \downarrow & & & & \downarrow {\scriptstyle \mathrm{id}_{\mathcal{T}} \star v_{\mathcal{S}_1,\mathcal{S}_2}} \\
Z(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T} & \xrightarrow{u_{\mathcal{S}_1 \star \mathcal{S}_2,\mathcal{T}}} & \mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2) & \xrightarrow{\mathrm{id}} & \mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2).
\end{array}$$

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[1] Our notations follow those of [3].

3) For $\mathcal{S}_1$ and $\mathcal{S}_2$ as above, we must have:

$$\begin{array}{ccc} Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) & \xrightarrow{u_{\mathcal{S}_1, Z(\mathcal{S}_2)}} & Z(\mathcal{S}_2) \star Z(\mathcal{S}_1) \\ {\scriptstyle v_{\mathcal{S}_1,\mathcal{S}_2}} \downarrow & & {\scriptstyle v_{\mathcal{S}_2,\mathcal{S}_1}} \downarrow \\ Z(\mathcal{S}_1 \star \mathcal{S}_2) & \longrightarrow & Z(\mathcal{S}_2 \star \mathcal{S}_1), \end{array}$$

where the bottom arrow comes from the commutativity constraint on the category of spherical perverse sheaves on Gr.

As we will see, 1) and 2) amount to chasing along the diagrams defining u and v, whereas for 3) we will have to consider nearby cycles along a 2-dimensional base.

§2. Verification of Property 1

We will connect the three objects appearing in the commuatative diagram through an intermediate one.

Consider the scheme Fl', classifying the data of

$$(y, \mathcal{F}_G, \mathcal{F}_G \simeq \mathcal{F}^0_G|_{X-\{x,y\}}, \epsilon),$$

and recall that $\mathrm{Fl}'_{X-x} \simeq \mathrm{Gr}_{X-x} \times \mathrm{Fl}$. We cosider the sheaf $\mathcal{A} := \mathcal{S}_{X-x} \boxtimes (\mathcal{T}_1 \star \mathcal{T}_2)$ on it. The intermediate object is $\Psi(\mathcal{A})$. We will show that the three isomorphisms appearing in our commutative diagram come as compositions from isomorphisms

$$\mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \simeq \Psi(\mathcal{A}),\ Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \simeq \Psi(\mathcal{A}),\ \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}) \simeq \Psi(\mathcal{A}).$$

We introduce the schemes Fl^i, $i = 1, 2, 3$ classifying, repsectively, the data of:

$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon', \epsilon'').$$

Let us denote by π^i, $i = 1, 2, 3$ the natural projection of each of these schemes to Fl' (π^i remembers only $\mathcal{F}_G$ and ϵ). In addition, as in [3], on Fl^i_{X-x}, $i = 1, ..., 3$, there exist the perverse sheaves

$$\mathcal{A}^1 := \mathcal{T}_1 \boxtimes \mathcal{S}_{X-x} \boxtimes \mathcal{T}_2,\ \mathcal{A}^2 := \mathcal{S}_{X-x} \boxtimes \mathcal{T}_1 \boxtimes \mathcal{T}_2,\ \mathcal{A}^3 := \mathcal{T}_1 \boxtimes \mathcal{T}_2 \boxtimes \mathcal{S}_{X-x}$$

all having the property that $(\pi^i_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}$ on Fl'_{X-x}. Hence,

$$(\pi^i_x)_!(\Psi(\mathcal{A}^i)) \simeq \Psi(\mathcal{A}).$$

Moreover, by construction,

$$\Psi(\mathcal{A}^1) \simeq \mathcal{T}_1\widetilde{\boxtimes}Z(\mathcal{S})\widetilde{\boxtimes}\mathcal{T}_2,\ \Psi(\mathcal{A}^2) \simeq Z(\mathcal{S})\widetilde{\boxtimes}\mathcal{T}_1\widetilde{\boxtimes}\mathcal{T}_2,\ \Psi(\mathcal{A}^3) \simeq \mathcal{T}_1\widetilde{\boxtimes}\mathcal{T}_2\widetilde{\boxtimes}Z(\mathcal{S}),$$

and hence

$$\begin{aligned}(\pi^1_x)_!(\Psi(\mathcal{A}^1)) &\simeq \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2,\\ (\pi^2_x)_!(\Psi(\mathcal{A}^2)) &\simeq Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2,\\ (\pi^3_x)_!(\Psi(\mathcal{A}^2)) &\simeq \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}).\end{aligned}$$

Therefore, it remains to establish the required relation between these isomorphisms and $\mathrm{id}_{\mathcal{T}_1} \star u_{\mathcal{S},\mathcal{T}_2}$, $u_{\mathcal{S},\mathcal{T}_1} \star \mathrm{id}_{\mathcal{T}_2}$ and $u_{\mathcal{S},\mathcal{T}_1\star\mathcal{T}_2}$. For that we introduce several auxiliary schemes Fl^{12}, Fl^{13}, $\mathrm{Fl}^{23'}$, $\mathrm{Fl}^{23''}$, which classify, respectively, the data of:

$$\begin{aligned}&(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-\{x,y\}}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon')\\ &(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-\{x,y\}}, \epsilon, \epsilon')\\ &(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon')\\ &(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon').\end{aligned}$$

We have the projections $\pi^{ij} : \mathrm{Fl}^{ij} \to \mathrm{Fl}'$ and $\pi^{i,ij} : \mathrm{Fl}^i \to \mathrm{Fl}^{ij}$ with $\pi^{ij} \circ \pi^{i,ij} = \pi^i$. Let $\mathcal{A}^{ij}$ be the following sheaves defined on Fl^{ij}_{X-x} as in [3]:

$$\begin{aligned}&\mathcal{A}^{12} := \mathcal{S}_{X-x} \boxtimes \mathcal{T}_1\widetilde{\boxtimes}\mathcal{T}_2, \mathcal{A}^{13} := \mathcal{T}_1\widetilde{\boxtimes}\mathcal{T}_2 \boxtimes \mathcal{S}_{X-x}\\ &\mathcal{A}^{23'} := \mathcal{S}_{X-x} \boxtimes (\mathcal{T}_1 \star \mathcal{T}_2), \mathcal{A}^{23''} := (\mathcal{T}_1 \star \mathcal{T}_2) \boxtimes \mathcal{S}_{X-x}.\end{aligned}$$

We have $(\pi^{ij}_{X-x})_!(\mathcal{A}^{ij}) \simeq \mathcal{A}$, and $(\pi^{i,ij}_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}^{ij}$.

Note that

$$\begin{aligned}&\Psi(\mathcal{A}^{13}) \simeq \mathcal{C}(\mathcal{S}, \mathcal{T}_1)\widetilde{\boxtimes}\mathcal{T}_2,\ \Psi(\mathcal{A}^{12}) \simeq \mathcal{T}_1\widetilde{\boxtimes}\mathcal{C}(\mathcal{S}, \mathcal{T}_2),\\ &\Psi(\mathcal{A}^{23'}) \simeq Z(\mathcal{S})\widetilde{\boxtimes}(\mathcal{T}_1 \star \mathcal{T}_2),\ \Psi(\mathcal{A}^{23''}) \simeq (\mathcal{T}_1 \star \mathcal{T}_2)\widetilde{\boxtimes}Z(\mathcal{S}).\end{aligned}$$

Therefore, our isomorphism

$$Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \simeq (\pi^2_x)_!(\Psi(\mathcal{A}^2)) \to \Psi(\mathcal{A}) \to (\pi^1_x)_!(\Psi(\mathcal{A}^1)) \simeq \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2$$

can be factored as

$$(\pi_x^2)_!(\Psi(\mathcal{A}^2)) \simeq (\pi_x^{12} \circ \pi_x^{2,12})_!(\Psi(\mathcal{A}^2)) \simeq (\pi_x^{12})_!(\Psi(\mathcal{A}^{12}))$$
$$\simeq (\pi_x^1 \circ \pi_x^{1,12})_!(\Psi(\mathcal{A}^1)) \simeq (\pi_x^1)_!(\Psi(\mathcal{A}^1)),$$

and, therefore, coincides with $u_{\mathcal{S},\mathcal{T}_1} \star \mathrm{id}_{\mathcal{T}_2} : Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \to \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2$, because $\mathcal{T}_2$ enters through the twisted external product construction.

Similarly, the isomorphism

$$\mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \simeq (\pi_x^1)_!(\Psi(\mathcal{A}^1)) \to \Psi(\mathcal{A}) \to (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S})$$

coincides with $\mathrm{id}_{\mathcal{T}_1} \star u_{\mathcal{S},\mathcal{T}_2} : \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \to \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S})$.

Finally, the isomorphism

$$(\mathcal{T}_1 \star \mathcal{T}_2) \star Z(S) \simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq Z(\mathcal{S}) \star (\mathcal{T}_1 \star \mathcal{T}_2)$$

coincides with $u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2}$. Therefore, the isomorphism

$$\mathcal{T}_1 \star \mathcal{T}_2 \star Z(S) \simeq (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \to \Psi(\mathcal{A}) \to (\pi_x^2)_!(\Psi(\mathcal{A}^2)) \simeq Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2,$$

which equals

$$(\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq (\pi_x^2)_!(\Psi(\mathcal{A}^2))$$

induces $u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2} : \mathcal{T}_1 \star \mathcal{T}_2 \star Z(S) \to Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2$.

This establishes the commutativity of the first diagram.

§3. Verification of Property 2

This case is very similar to the previous one. On the scheme Fl'_{X-x} we consider the sheaf $\mathcal{A} := (\mathcal{S}_1 \star \mathcal{S}_2)_{X-x} \boxtimes \mathcal{T}$. We introduce the schemes Fl^i, $i = 1,2,3$, $\mathrm{Fl}^{12}, \mathrm{Fl}^{13}, \mathrm{Fl}^{23'}, \mathrm{Fl}^{23''}$, which classify, respectively, the data of

$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-\{x,y\}}, \epsilon, \epsilon')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-\{x,y\}}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \epsilon, \epsilon')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \epsilon, \epsilon')$$

We have natural proper maps $\pi^i : \mathrm{Fl}^i \to \mathrm{Fl}'$, $\pi^{ij} : \mathrm{Fl}^{ij} \to \mathrm{Fl}'$, and $\pi^{i,ij} : \mathrm{Fl}^i \to \mathrm{Fl}^{ij}$. The corresponding perverse sheaves are

$$\mathcal{A}^1 := (\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}\mathcal{T}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x}, \quad \mathcal{A}^2 := (\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x} \boxtimes \mathcal{T},$$
$$\mathcal{A}^3 := \mathcal{T}\widetilde{\boxtimes}(\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x},$$
$$\mathcal{A}^{12} := (\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}((\mathcal{S}_2)_{X-x} \boxtimes \mathcal{T}), \quad \mathcal{A}^{13} := ((\mathcal{S}_1)_{X-x} \boxtimes \mathcal{T})\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x},$$
$$\mathcal{A}^{23'} := \mathcal{T}\widetilde{\boxtimes}(\mathcal{S}_1 \star \mathcal{S}_2)_{X-x}, \quad \mathcal{A}^{23''} := (\mathcal{S}_1 \star \mathcal{S}_2)_{X-x}\widetilde{\boxtimes}\mathcal{T},$$

$$(\pi^i_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}, \quad (\pi^{ij}_{X-x})_!(\mathcal{A}^{ij}) \simeq \mathcal{A}, \quad (\pi^{i,ij}_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}^{ij}.$$

As in [3], we obtain:

$$\Psi(\mathcal{A}^1) \simeq Z(\mathcal{S}_1)\widetilde{\boxtimes}\mathcal{T}\widetilde{\boxtimes}Z(\mathcal{S}_2), \quad \Psi(\mathcal{A}^2) \simeq Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)\widetilde{\boxtimes}\mathcal{T}$$
$$\Psi(\mathcal{A}^3) \simeq \mathcal{T}\widetilde{\boxtimes}Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2),$$
$$\Psi(\mathcal{A}^{12}) \simeq Z(\mathcal{S}_1)\widetilde{\boxtimes}\mathcal{C}(\mathcal{S}_2, \mathcal{T}), \quad \Psi(\mathcal{A}^{13}) \simeq \mathcal{C}(\mathcal{S}_1, \mathcal{T})\widetilde{\boxtimes}\mathcal{S}_2,$$
$$\Psi(\mathcal{A}^{23'}) \simeq \mathcal{T}\widetilde{\boxtimes}Z(\mathcal{S}_1 \star \mathcal{S}_2), \quad \Psi(\mathcal{A}^{23''}) \simeq Z(\mathcal{S}_1 \star \mathcal{S}_2)\widetilde{\boxtimes}\mathcal{T}.$$

As in the previous section, we obtain that the isomorphism

$$\begin{aligned} Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \star \mathcal{T} &\simeq (\pi^2_x)_!(\Psi(\mathcal{A}^2)) \simeq \Psi(\mathcal{A}) \\ &\simeq (\pi^1_x)_!(\Psi(\mathcal{A}^1)) \simeq Z(\mathcal{S}_1) \star \mathcal{T} \star Z(\mathcal{S}_2) \end{aligned}$$

coincides with $\mathrm{id}_{Z(\mathcal{S}_1)} \star u_{\mathcal{S}_2,\mathcal{T}}$, and

$$\begin{aligned} Z(\mathcal{S}_1) \star \mathcal{T} \star Z(\mathcal{S}_2) &\simeq (\pi^1_x)_!(\Psi(\mathcal{A}^1)) \simeq \Psi(\mathcal{A}) \\ &\simeq (\pi^3_x)_!(\Psi(\mathcal{A}^3)) \simeq \mathcal{T} \star Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \end{aligned}$$

coincides with $u_{\mathcal{S}_1,\mathcal{T}} \star \mathrm{id}_{Z(\mathcal{S}_2)}$.

The isomorphisms

$$\mathcal{T} \star Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \simeq (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq \Psi(\mathcal{A}) \\ \simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq \mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2)$$

and

$$Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \star \mathcal{T} \simeq (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq \Psi(\mathcal{A}) \\ \simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq Z(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T}$$

coincide with $\mathrm{id}_{\mathcal{T}} \star v_{\mathcal{S}_1,\mathcal{S}_2}$ and $v_{\mathcal{S}_1,\mathcal{S}_2} \star \mathrm{id}_{\mathcal{T}}$, respectively, and the isomorphism

$$\mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2) \simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq \Psi(\mathcal{A}) \\ \simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq Z(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T}$$

coincides with $u_{\mathcal{S}_1 \star \mathcal{S}_2, \mathcal{T}}$.

This establishes Property 2.

§4. Nearby cycles along a two-dimensional base

To verify the remaining property 3 we need to make a brief digression on the notion of nearby cycles along a 2-dimensional base.

We do not have a good definition. We will, however, introduce some functor Υ, which will suffice for our purposes. But we do not claim neither that it is exact nor that it commutes with the Verdier duality.

Let $\mathcal{Y}$ be a scheme over $X \times X$ and $\mathcal{A}$ be a sheaf on $\mathcal{Y}_{(X-x)\times(X-x)}$. We would like to compare two sheaves on $\mathcal{Y}_x$: one, which we denote by $\Psi_\Delta(\mathcal{A})$ and the other by $\Psi \circ \Psi(\mathcal{A})$:

The functor $\mathcal{A} \mapsto \Psi_\Delta(\mathcal{A})$ is defined as $\Psi_X(\mathcal{A}|_{\mathcal{Y}_{\Delta(X-x)}})[-1]$, where $\mathcal{Y}_{\Delta(X-x)}$ is the preimage of the diagonal $X - x$ in $\mathcal{Y}$.

The functor $\mathcal{A} \mapsto \Psi \circ \Psi(\mathcal{A})$ is iterated nearby cycles: we first take nearby cycles of $\mathcal{A}$ with respect to the projection $\mathcal{Y} \to X \times X \xrightarrow{p_1} X$ and obtain a perverse sheaf on $\mathcal{Y}_{x\times(X-x)}$, and then take the nearby cycles of the latter with respect to $\mathcal{Y}_{x\times X} \to X$.

We claim that $\Psi_{\mathcal{Y}_\Delta}(\mathcal{A})$ and $(\Psi \circ \Psi)_{\mathcal{Y}}(\mathcal{A})$ are connected by a *correspondence*. I.e., there exists a third functor $\mathcal{A} \mapsto \Upsilon(\mathcal{A}) \in \mathrm{D}^b(\mathcal{Y}_x)$ and functorial maps

$$\Psi_\Delta(\mathcal{A}) \leftarrow \Upsilon(\mathcal{A}) \rightarrow \Psi \circ \Psi(\mathcal{A}).$$

In case when both maps are isomorphisms we will denote by $w_{\mathcal{A}}$ the resulting map

$$\Psi_\Delta(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A}).$$

The construction of $\Upsilon(\mathcal{A})$ is sketched below. Here are its two basic properties:

1. If $\mathcal{Y}$ is $\mathcal{Y}_1 \times \mathcal{Y}_2$ and $\mathcal{A} = \mathcal{A}^1 \boxtimes \mathcal{A}^2$, then both maps $\Upsilon(\mathcal{A}) \to \Psi_\Delta(\mathcal{A})$ and $\Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A})$ are isomorphisms and the resulting isomorphism $w_{\mathcal{A}} : \Psi_\Delta(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A})$ coincides with

$$\Psi_\Delta(\mathcal{A}) \simeq \Psi(\mathcal{A}_1 \boxtimes \mathcal{A}_2|_{\mathcal{Y}_1 \underset{X}{\times} \mathcal{Y}_2}) \simeq \Psi(\mathcal{A}_1) \otimes \Psi(\mathcal{A}_2) \simeq \Psi \circ \Psi(\mathcal{A}),$$

where the second arrow is the "Künneth" formula for nearby cycles, cf. [1].

2. For a map $\pi : \mathcal{Y} \to \mathcal{Y}'$ and $\mathcal{A} \in \mathrm{Perv}(\mathcal{Y}_{(X-x)\times(X-x)})$ there is a canonical isomorphism

$$\pi_x(\Upsilon(\mathcal{A})) \simeq \Upsilon((\pi_{(X-x)\times(X-x)})_!(\mathcal{A}))$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\Psi_\Delta((\pi_{(X-x)\times(X-x)})_!(\mathcal{A})) & \xrightarrow{\sim} & (\pi_x)_!(\Psi_\Delta(\mathcal{A})) \\
\uparrow & & \uparrow \\
\Upsilon((\pi_{(X-x)\times(X-x)})_!(\mathcal{A})) & \xrightarrow{\sim} & (\pi_x)_!(\Upsilon(\mathcal{A})) \\
\downarrow & & \downarrow \\
\Psi \circ \Psi((\pi_{(X-x)\times(X-x)})_!(\mathcal{A})) & \xrightarrow{\sim} & (\pi_x)_!(\Psi \circ \Psi(\mathcal{A})).
\end{array}$$

Here is the construction of $\Upsilon(\mathcal{A})$. (As in [2], we will treat only the unipotent part of nearby cycles.)

Let f denote the structure map $Y \to X \times X$. Let j denote the embedding $\mathcal{Y}_{(X-x)\times(X-x)} \hookrightarrow \mathcal{Y}$, and let i_x denote the embedding of $\mathcal{Y}_x$ into $\mathcal{Y}$.

Following [2], let $\mathcal{E}_n$ be an n-dimensional local system on $X - x$, whose monodromy around x is an n-dimensional nilpotent Jordan block (we can assume that X is affine, hence such a local system exists). The $\mathcal{E}_n$'s form a directed system as $n \in \mathbb{N}$.

Consider the sheaf $\mathcal{F}_{n,m} := j_*(\mathcal{A} \otimes f^*(\mathcal{E}_n \boxtimes \mathcal{E}_m))$. We set $\Upsilon(\mathcal{A})$ to be the direct homotopy limit $\underrightarrow{lim}\ i_x^*(\mathcal{F}_{n,m}[-2])$.

Let us now construct the maps $\Upsilon(\mathcal{A}) \to \Psi_\Delta(\mathcal{A})$ and $\Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A})$.

Let $j_{\Delta(X-x)}$ be the embedding of $\mathcal{Y}_{\Delta(X-x)}$. We have a canonical map

$$\mathcal{F}_{n,m}|_{\mathcal{Y}_{\Delta(X)}} \to (j_{\Delta(X-x)})_*(\mathcal{A}|_{\mathcal{Y}_{\Delta(X-x)}} \otimes f^*(\mathcal{E}_n \otimes \mathcal{E}_m)).$$

In addition, we have the maps $\mathcal{E}_n \otimes \mathcal{E}_m \to \mathcal{E}_k$, where $k = \max\{m,n\}$, and by composing we obtain a map

$$\mathcal{F}_{n,m}|_{\mathcal{Y}_{\Delta(X)}}[-1] \to (j_{\Delta(X-x)})_*(\mathcal{A}|_{\mathcal{Y}_{\Delta(X-x)}}[-1] \otimes f^*(\mathcal{E}_k)).$$

By applying the functor i_x^* to both sides and passing to the direct limit, we obtain the required map.

Now, let j_1 be the embedding of $\mathcal{Y}_{x\times(X-x)}$ into $\mathcal{Y}$. Again, we have a natural map

$$\mathcal{F}_{n,m}|_{\mathcal{Y}_{x\times X}} \to (j_1)_*(j_*(\mathcal{A} \otimes (p_1 \circ f)^*(\mathcal{E}_n))|_{\mathcal{Y}_{x\times X}} \otimes (p_2 \circ f)^*(\mathcal{E}_m)).$$

Note that the direct limit of $j_*(\mathcal{A} \otimes (p_1 \circ f)^*(\mathcal{E}_n))|_{\mathcal{Y}_{x\times X}}[-1]$ with respect to n is the 1-step nearby cycles $\Psi(\mathcal{A}) \in \mathrm{Perv}(\mathcal{Y}_{x\times(X-x)})$. Threfeore, by applying i_x^* and taking the direct limit with respect to m as well, we obtain the required map

$$\Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A}).$$

§5. Verification of Property 3

Recall the Beilinson-Drinfeld scheme Gr'' over $X \times X$. By definition, it classifies the data of $(y_1, y_2, \mathcal{F}_G, \mathcal{F}_G \simeq \mathcal{F}_G^0|_{X-\{y_1,y_2\}})$. We have:

$$\mathrm{Gr}''_{X\times X-\Delta} \simeq (\mathrm{Gr}_X \times \mathrm{Gr}_X)_{X\times X-\Delta},$$
$$\mathrm{Gr}''_{\Delta} \simeq \mathrm{Gr}_X,$$

where $\Delta \subset X \times X$ denotes the diagonal.

Starting with $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$ we consider the corresponding sheaf $(\mathcal{S}_1)_{X-x} \boxtimes (\mathcal{S}_2)_{X-x}$ on $\mathrm{Gr}''_{X\times X-\Delta}$ and denote by $\mathcal{B}$ its Goresky-MacPherson extension to the diagonal. It is known that the extension is acyclic and $\mathcal{B}|_{\mathrm{Gr}''_\Delta}$ identifies with $(\mathcal{S}_1 \star \mathcal{S}_2)_X \simeq (\mathcal{S}_2 \star \mathcal{S}_1)_X$, which is in fact the definition of the commutativity constraint on $\mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$.

Consider now the scheme Fl'' fibered over $X \times X$: A point of Fl'' is a data of

$$(y_1, y_2, \mathcal{F}_G, \mathcal{F}_G \simeq \mathcal{F}_G^0|_{X-\{y_1,y_2\}}, \epsilon),$$

where ϵ is as usual a reduction of $\mathcal{F}_G|_x$ to B. We have:

$$\mathrm{Fl}''_{(X-x)\times(X-x)} \simeq (\mathrm{Gr} \times \mathrm{Gr})_{(X-x)\times(X-x)} \times G/B,$$

and we define the sheaf $\mathcal{A}$ on it equal to $\mathcal{B} \boxtimes \delta_1$.

We introduce two auxiliary schemes Fl^1 and Fl^2 over $X \times X$, which classify the data of $(y_1, y_2, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y_2}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y_1}, \epsilon, \epsilon')$, $(y_1, y_2, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y_1}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y_2}, \epsilon, \epsilon')$, respectively.

We have the natural proper projections $\pi^i : \mathrm{Fl}^i \to \mathrm{Fl}''$ for $i = 1, 2$, which induce isomorphisms

$$\mathrm{Fl}^i_{X \times X - \{x \times X, X \times x, \Delta\}} \simeq \mathrm{Gr}''_{X \times X - \{x \times X, X \times x, \Delta\}} \times G/B \times G/B.$$

Let $\mathcal{A}^i$ be the corresponding perverse sheaves on $\mathrm{Fl}^i_{(X-x)\times(X-x)}$, i.e. we take $\mathcal{B} \boxtimes \delta_1 \boxtimes \delta_1$ on $\mathrm{Fl}^i_{X \times X - \{x \times X, X \times x, \Delta\}}$ and extend it minimally to the preimage of $\Delta(X - x)$.

We obtain a commutative diagram:

$$\begin{array}{ccc}
(\pi^1_x)_!(\Psi_\Delta(\mathcal{A}^1)) & \xrightarrow{w_{\mathcal{A}^1}} & (\pi^1_x)_!(\Psi \circ \Psi(\mathcal{A}^1)) \\
\sim\downarrow & & \sim\downarrow \\
\Psi_\Delta(\mathcal{A}) & \xrightarrow{w_{\mathcal{A}}} & \Psi \circ \Psi(\mathcal{A}) \\
\sim\downarrow & & \sim\downarrow \\
(\pi^2_x)_!(\Psi_\Delta(\mathcal{A}^2)) & \xrightarrow{w_{\mathcal{A}^2}} & (\pi^2_x)_!(\Psi \circ \Psi(\mathcal{A}^2)).
\end{array}$$

As in [3], the terms of this diagram identify, respectively, with

$$\begin{array}{ccc}
(\pi^1_x)_!(\Psi_{\mathrm{Fl}^1_\Delta}((\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x})) & \xrightarrow{w_{\mathcal{A}^1}} & (\pi^1_x)_!(Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)) \\
\sim\downarrow & & \sim\downarrow \\
\Psi(\mathcal{A}|_\Delta[-1]) & \xrightarrow{w_{\mathcal{A}}} & \mathcal{C}(\mathcal{S}_1, Z(\mathcal{S}_2)) \\
\sim\downarrow & & \sim\downarrow \\
(\pi^2_x)_!(\Psi_{\mathrm{Fl}^2_\Delta}((\mathcal{S}_2)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_1)_{X-x})) & \xrightarrow{w_{\mathcal{A}^2}} & (\pi^2_x)_!(Z(\mathcal{S}_2)\widetilde{\boxtimes}Z(\mathcal{S}_1)).
\end{array}$$

Note first, that we have isomorphisms

$$Z(\mathcal{S}_1 \star \mathcal{S}_2) \simeq \Psi(\mathcal{A}|_\Delta[-1]) \simeq Z(\mathcal{S}_2 \star \mathcal{S}_1),$$

whose composition, by definition, comes from the commutativity constraint on $\mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$.

The morphism

$Z(\mathcal{S}_1 \star \mathcal{S}_2) \simeq \Psi(\mathcal{A}|_\Delta[-1]) \simeq$

$(\pi^1_x)_!(\Psi_{\mathrm{Fl}^1_\Delta}((\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x})) \to (\pi^1_x)_!(Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)) \simeq Z(\mathcal{S}_1) \star Z(\mathcal{S}_2)$

coincides with the isomorphism $v_{\mathcal{S}_1,\mathcal{S}_2} : Z(\mathcal{S}_1 \star \mathcal{S}_2) \to Z(\mathcal{S}_1) \star Z(\mathcal{S}_2)$ of [3], by the construction of the latter, and **2** above.

Similarly, the morphism

$$\begin{aligned} Z(\mathcal{S}_2 \star \mathcal{S}_1) &\simeq \Psi(\mathcal{A}|_\Delta[-1]) \\ &\simeq (\pi^2_x)_!(\Psi_{\mathrm{Fl}^2_\Delta}((\mathcal{S}_2)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_1)_{X-x})) \to (\pi^2_x)_!(Z(\mathcal{S}_2)\widetilde{\boxtimes}Z(\mathcal{S}_1)) \\ &\simeq Z(\mathcal{S}_2) \star Z(\mathcal{S}_1) \end{aligned}$$

coincides with $v_{\mathcal{S}_2,\mathcal{S}_1}$.

Finally, the composed isomorphism

$$\begin{aligned} Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) &\simeq (\pi^1_x)_!(Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)) \simeq \mathcal{C}(\mathcal{S}_1, Z(\mathcal{S}_2)) \\ &\simeq (\pi^2_x)_!(Z(\mathcal{S}_2)\widetilde{\boxtimes}Z(\mathcal{S}_1)) \simeq Z(\mathcal{S}_2) \star Z(\mathcal{S}_1) \end{aligned}$$

coincides with $u_{\mathcal{S}_1,Z(\mathcal{S}_2)}$, which is what we had to prove.

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