

On tensor categories attached to cells in affine Weyl groups

Roman Bezrukavnikov

Abstract.

This note is devoted to Lusztig's bijection between unipotent conjugacy classes in a simple complex algebraic group and 2-sided cells in the affine Weyl group of the Langlands dual group; and also to the description of the reductive quotient of the centralizer of the unipotent element in terms of convolution of perverse sheaves on affine flag variety of the dual group conjectured by Lusztig in [L4]. Our main tool is a recent construction by Gaiety (based on an idea of Beilinson and Kottwitz), the so-called sheaf-theoretic construction of the center of an affine Hecke algebra (see [Ga]). We show how this remarkable construction provides a geometric interpretation of the bijection, and allows to prove the conjecture.

§1. Introduction.

Let G be a split simple algebraic group. Let W be the corresponding extended affine Weyl group; thus W is the semi-direct product of the Weyl group W_f and the lattice of coweights of G . Let J be the corresponding asymptotic affine Hecke algebra [L2]. Recall that J is an algebra over \mathbb{Z} , and it comes with a basis t_w parametrized by W . The group W is the union of two-sided cells [L1], and the algebra J is the direct sum of algebras, $J = \bigoplus J_{\underline{c}}$, where \underline{c} runs over the set of two-sided cells, and $J_{\underline{c}}$ is the span of t_w , $w \in \underline{c}$. Let also $W^f \subset W$ be the set of

Received February 13, 2002.

Revised November 12, 2002.

Acknowledgement. I am much indebted to M. Finkelberg and D. Gaiety for patiently explaining to me various mathematics; this paper owes its existence to them. I also thank A. Beilinson and V. Drinfeld for helpful comments, and V. Ostrik for stimulating interest. The results were obtained (in a preliminary form) at IAS during the special year on Geometric Methods in Representation Theory (98/99). The author is supported by the Clay Institute, and NSF grant DMS0071967.

minimal length representatives of double cosets $W_f \backslash W / W_f$. Let J^f be the span of t_w , $w \in W^f$, and $J_{\underline{c}}^f = J^f \cap J_{\underline{c}}$. It follows from the result of [LX] that $J^f \subset J$, $J_{\underline{c}}^f \subset J_{\underline{c}}$ are subalgebras.

Let ${}^L G$ be the Langlands dual group (over an algebraically closed field of characteristic zero). In [L3] Lusztig constructed a bijection between two-sided cells in W and unipotent conjugacy classes in ${}^L G$; for a 2-sided cell \underline{c} we will denote by $N_{\underline{c}} \in {}^L G$ a representative of the unipotent conjugacy class corresponding to \underline{c} . In [L4] the based algebras $J_{\underline{c}}$, $J_{\underline{c}}^f$ are realized as Grothendieck groups of certain semisimple monoidal (tensor without commutativity) categories, which we denote respectively by $A_{\underline{c}}$, $A_{\underline{c}}^f$ (thus $A_{\underline{c}}^f$ is a monoidal subcategory in $A_{\underline{c}}$). The categories $A_{\underline{c}}$, $A_{\underline{c}}^f$ are defined as subcategories of the category of (semisimple) perverse sheaves on the affine flag manifold of G ; the monoidal structure is provided by the truncated convolution, see *loc. cit.*

The following conjectural description of the monoidal categories $A_{\underline{c}}$, $A_{\underline{c}}^f$ (and hence of based rings $J_{\underline{c}}$, $J_{\underline{c}}^f$) was proposed in [L4], §3.2. Let $Z = Z_{{}^L G}(N_{\underline{c}})$ be the centralizer of $N_{\underline{c}}$ in ${}^L G$. For any two-sided cell \underline{c} there should exist a finite set X with a Z action, such that $A_{\underline{c}}$ is equivalent to the category $\text{Vect}_{ss}^Z(X \times X)$ of semisimple Z -equivariant sheaves on $X \times X$, with monoidal structure given by convolution. The finite set X should contain a preferred point $x \in X$ fixed by Z ; thus the monoidal category $\text{Vect}_{ss}^Z(X \times X)$ contains a monoidal subcategory $\text{Vect}_{ss}^Z(\{(x, x)\}) = \text{Rep}^{ss}(Z)$ (the category of semisimple finite dimensional representations of Z). This subcategory should be identified with $A_{\underline{c}}^f \subset A_{\underline{c}}$.

In this note we (extend and) prove the part of the above conjecture which asserts the equivalence $A_{\underline{c}}^f = \text{Rep}^{ss}(Z)$ (see [BO] for further results in this direction). The proof is based on a recent construction by Gaitsgory (following an idea of Beilinson and Kottwitz), see [Ga].

Recall that the link between representations of ${}^L G$ and perverse sheaves on affine Grassmanian of G is provided by the so-called geometric version of the Satake isomorphism (the idea going back to [L0] is developed in [Gi] and [MV]; see also [BD]). The classical Satake isomorphism is an isomorphism between the spherical Hecke algebra \mathbb{H}_{sph} of $G(O)$ -biinvariant functions on $G(F)$, and the ring $\mathcal{R}({}^L G)$, where \mathcal{R} stands for the representation ring, and $F \supset O$ is a non-archimedean local field and its ring of integers. Its geometric (or categorical) version is an equivalence of tensor categories between $\text{Rep}({}^L G)$ and the category of “spherical” perverse sheaves on the affine Grassmanian (the more precise statement is recalled in section 3.1 below).

Further, a theorem of Bernstein (see e.g. [L0], Proposition 8.6) asserts that the center of the Iwahori-Matsumoto Hecke algebra \mathbb{H} is also isomorphic to $\mathcal{R}({}^L G)$. The main result of [Ga] provides a geometric (categorical) counterpart of this isomorphism. More precisely, it defines a monoidal functor from $Rep({}^L G)$ to the Iwahori-equivariant derived category of l -adic sheaves on affine flags; the corresponding map of Grothendieck groups induces the imbedding $\mathcal{R}({}^L G) = Z(\mathbb{H}) \hookrightarrow \mathbb{H}$. This functor enjoys various favorable properties; it also carries a canonical unipotent automorphism \mathfrak{M} (the monodromy).

The idea of the present note is that one can identify certain subquotient categories of perverse sheaves on affine flags related to a 2-sided cell \underline{c} with $Rep(H)$ for a subgroup $H \subset Z(N_{\underline{c}})$, in such a way that the canonical action of $Rep({}^L G)$ constructed in [Ga] is identified with the tautological action of $Rep({}^L G)$ on $Rep(H)$

$$V : W \mapsto Res_H^{L G}(V) \otimes W;$$

moreover this requirement fixes the identification with $Rep(H)$ uniquely. Then the monodromy automorphism \mathfrak{M} provides (by Tannakian formalism) a unipotent element $N \in {}^L G$, commuting with H . This element lies in the conjugacy class attached to \underline{c} by Lusztig.

With these tools in hand, Lusztig’s conjecture becomes an exercise in Tannakian formalism (at least modulo some powerful Theorems of Lusztig on the structure of asymptotic Hecke algebras).

I want to point out that at several places in this paper we use “as a black box” results of Lusztig on asymptotic Hecke algebras to check a categorical property of perverse sheaves on affine flags. In Remarks 4, 7 we discuss a possible plan for replacing some (though not all) of these uses by a direct geometric argument (in other words, for providing a geometric proof of Lusztig’s results).

§2. Preliminaries on tensor categories

In this section we collect some (more or less standard) technicalities needed in the main argument.

2.1. Central functors

Definition 1. Let \mathcal{A} be a monoidal category, and \mathcal{B} be a tensor (symmetric monoidal) category. A *central functor* from \mathcal{B} to \mathcal{A} is a monoidal functor $F : \mathcal{B} \rightarrow \mathcal{A}$ together with an isomorphism

$$(1) \quad \sigma_{X,Y} : F(X) \otimes Y \cong Y \otimes F(X)$$

fixed for all $X \in \mathcal{B}$, $Y \in \mathcal{A}$, subject to the following compatibilities.

i) $\sigma_{X,Y}$ is functorial in X, Y ;

ii) For $X, X' \in \mathcal{B}$ the isomorphism $\sigma_{X,F(X')}$ coincides with the composition

$$F(X) \otimes F(X') \cong F(X \otimes X') \cong F(X' \otimes X) \cong F(X') \otimes F(X)$$

(where the middle isomorphism comes from the commutativity constraint in \mathcal{B} , and the other two from the tensor structure on F).

iii) For $Y_1, Y_2 \in \mathcal{A}$ and $X \in \mathcal{B}$ the composition

$$F(X) \otimes Y_1 \otimes Y_2 \xrightarrow{\sigma_{X,Y_1 \otimes Y_2}} Y_1 \otimes F(X) \otimes Y_2 \xrightarrow{Y_1 \otimes \sigma_{X,Y_2}} Y_1 \otimes Y_2 \otimes F(X)$$

coincides with $\sigma_{X,Y_1 \otimes Y_2}$.

iv) For $Y \in \mathcal{A}$ and $X_1, X_2 \in \mathcal{B}$ the composition

$$\begin{aligned} & F(X_1 \otimes X_2) \otimes Y \cong F(X_1) \otimes F(X_2) \otimes Y \\ & \xrightarrow{F(X_1) \otimes \sigma_{X_2,Y}} F(X_1) \otimes Y \otimes F(X_2) \xrightarrow{\sigma_{X_1,Y \otimes F(X_2)}} \\ & Y \otimes F(X_1) \otimes F(X_2) = Y \otimes F(X_1 \otimes X_2) \end{aligned}$$

coincides with $\sigma_{X_1 \otimes X_2, Y}$.

Remark 1. ¹ For a monoidal category \mathcal{A} its *center* $\mathcal{Z}(\mathcal{A})$ is defined as the category of pairs (X, σ) , where $X \in \mathcal{A}$, and $\sigma = (\sigma_Y)$ is a collection of isomorphisms $\sigma_Y : X \otimes Y \cong Y \otimes X$, subject to certain compatibilities (see e.g. [Ka], XIII.4). Then $\mathcal{Z}(\mathcal{A})$ turns out to carry a natural structure of a braided monoidal category. If \mathcal{A} is the category of representations of a Hopf algebra A , then $\mathcal{Z}(\mathcal{A})$ is identified with the category of representations of Drinfeld's double of A .

One can check that a central functor from a (symmetric) tensor category \mathcal{B} to a monoidal category \mathcal{A} is the same as a monoidal functor from \mathcal{B} to $\mathcal{Z}(\mathcal{A})$, which intertwines commutativity in \mathcal{B} with the braiding in $\mathcal{Z}(\mathcal{A})$.

2.2. Reconstructing a subgroup from the restriction functor

Proposition 1. *Let k be an algebraically closed field. Let \mathcal{A} be a k -linear abelian monoidal category with a unit object $\mathbb{1}$ such that $\text{End}(\mathbb{1}) = k$. We assume that the product in \mathcal{A} is exact in each variable.*

Let G be an algebraic group over k , and $\text{Rep}(G)$ be the category of its finite dimensional algebraic representations. Let $F : \text{Rep}(G) \rightarrow \mathcal{A}$ be

¹I thank Drinfeld who pointed out the content of this remark to me.

an exact central functor. Suppose that any object $Y \in \mathcal{A}$ is isomorphic to a subquotient of $F(X)$ for some $X \in \text{Rep}(G)$.

Assume also² that k is uncountable and $\text{Hom}(X, Y)$ is finite dimensional for $X, Y \in \mathcal{A}$. Then there exists an algebraic subgroup $H \subset G$, and an equivalence of monoidal categories $\Phi : \text{Rep}(H) \xrightarrow{\sim} \mathcal{A}$, such that

$$F \cong \Phi \circ \text{Res}_H^G.$$

The subgroup $H \subset G$ is defined uniquely up to conjugation.

Proof. The group G acts on itself by left translations, making the space $\mathcal{O}(G)$ of regular functions on G an algebraic G -module, thus an ind-object of $\text{Rep}(G)$. Let $\underline{\mathcal{O}}_G$ denote this ind-object. It is a ring ind-object, i.e. we have a multiplication morphism $m : \underline{\mathcal{O}}_G \otimes \underline{\mathcal{O}}_G \rightarrow \underline{\mathcal{O}}_G$ satisfying the usual commutative ring axioms. For a ring ind-object \mathcal{J} we will write $m_{\mathcal{J}}$ for the multiplication morphism $m_{\mathcal{J}} : \mathcal{J} \otimes \mathcal{J} \rightarrow \mathcal{J}$. Let $\mathcal{J} \subsetneq F(\underline{\mathcal{O}}_G)$ be a maximal left ideal subobject, i.e. \mathcal{J} is a maximal proper sub ind-object in $F(\underline{\mathcal{O}}_G)$ satisfying

$$(2) \quad m_{F(\underline{\mathcal{O}}_G)}(F(\underline{\mathcal{O}}_G) \otimes \mathcal{J}) \subset \mathcal{J}.$$

Then \mathcal{J} is also a right ideal subobject, i.e. we have

$$(3) \quad m(\mathcal{J} \otimes F(\underline{\mathcal{O}}_G)) \subset \mathcal{J};$$

indeed, commutativity of $\underline{\mathcal{O}}_G$, and property (ii) in the definition of a central functor show that

$$m_{F(\underline{\mathcal{O}}_G)} \circ \sigma_{\underline{\mathcal{O}}_G, F(\underline{\mathcal{O}}_G)} = m_{F(\underline{\mathcal{O}}_G)},$$

and property (i) yields the equality

$$m_{F(\underline{\mathcal{O}}_G)}|_{\mathcal{J} \otimes F(\underline{\mathcal{O}}_G)} \circ \sigma_{\underline{\mathcal{O}}_G, \mathcal{J}} = m_{F(\underline{\mathcal{O}}_G)}|_{F(\underline{\mathcal{O}}_G) \otimes \mathcal{J}},$$

which implies (3).

Set $\underline{\mathcal{O}}_H = F(\underline{\mathcal{O}}_G)/\mathcal{J}$. (2), (3) imply that $\underline{\mathcal{O}}_H$ is a ring ind-object of \mathcal{A} . Thus the category of $\underline{\mathcal{O}}_H$ -module (ind)objects in \mathcal{A} is well-defined. We will denote this category by $\underline{\mathcal{O}}_H\text{-mod}$, call its objects $\underline{\mathcal{O}}_H$ -modules, and write $\text{Hom}_{\underline{\mathcal{O}}_H}$ instead of $\text{Hom}_{\underline{\mathcal{O}}_H\text{-mod}}$.

Then $\underline{\mathcal{O}}_H\text{-mod}$ is an abelian category, and $\underline{\mathcal{O}}_H \in \underline{\mathcal{O}}_H\text{-mod}$ is a simple object. Thus $K = \text{End}_{\underline{\mathcal{O}}_H}(\underline{\mathcal{O}}_H)$ is a division algebra, and $V \mapsto V \otimes_K \underline{\mathcal{O}}_H$ is an equivalence between the category of (right) finite

²These assumptions are not necessary, and are imposed to shorten the proof.

K -modules and the full subcategory in $\underline{\mathcal{O}}_H - \text{mod}$ generated by $\underline{\mathcal{O}}_H$ under finite direct sums and subquotients; we denote the latter category by $\underline{\mathcal{O}}_H - \text{mod}^{fr}$.

Lemma 1. *We have $K = \text{Hom}_{\mathcal{A}}(\mathbb{1}, \underline{\mathcal{O}}_H) = k$.*

Proof. $\underline{\mathcal{O}}_H$ is a unital ring object since $\underline{\mathcal{O}}_G$ is, i.e. the unit $\iota : \mathbb{1} \hookrightarrow \underline{\mathcal{O}}_H$ is fixed. The map $\phi \mapsto \phi \circ \iota$ provides an isomorphism $\text{End}_{\underline{\mathcal{O}}_H}(\underline{\mathcal{O}}_H) \cong \text{Hom}_{\mathcal{A}}(\mathbb{1}, \underline{\mathcal{O}}_H)$, with inverse isomorphism given by $i \mapsto m_{\underline{\mathcal{O}}_H} \circ (i \circ \text{Id}_{\underline{\mathcal{O}}_H})$. Thus the first equality is clear.

To check the second one, notice that the cardinality of a basis of a division algebra K over an algebraically closed field k is not less than the cardinality of k (indeed, for $x \in K$, $x \notin k$ the elements $(x - \lambda)^{-1}$, $\lambda \in k$ are linearly independent). However, $\underline{\mathcal{O}}_G$ is a countable union of objects of $\text{Rep}(G)$, hence $\text{Hom}_{\mathcal{A}}(\mathbb{1}, F(\underline{\mathcal{O}}_G))$ is at most countable dimensional. \square

Corollary 1. *a) For any $X \in \mathcal{A}$ we have $X \otimes \underline{\mathcal{O}}_H \in \underline{\mathcal{O}}_H - \text{mod}^{fr}$, i.e. $X \otimes \underline{\mathcal{O}}_H \cong V \otimes_k \underline{\mathcal{O}}_H$ for some finite dimensional k -vector space V .*

b) The functor $\Phi_H : \mathcal{A} \rightarrow \text{Vect}$ given by $\Phi_H : X \mapsto \text{Hom}(\mathbb{1}, X \otimes \underline{\mathcal{O}}_H)$ is exact, admits a structure of a monoidal functor, and we have a canonical isomorphism of monoidal functors

$$(4) \quad \Phi_G \cong \Phi_H \circ F,$$

where $\Phi_G : \text{Rep}(G) \rightarrow \text{Vect}$ is the fiber functor.

Proof. a) Let first $X = F(Y)$ for $Y \in \text{Rep}(G)$. We have an isomorphism of $\underline{\mathcal{O}}_G$ -modules $Y \otimes \underline{\mathcal{O}}_G \cong \Phi_G(Y) \otimes_k \underline{\mathcal{O}}_G$, hence an isomorphism of $F(\underline{\mathcal{O}}_G)$ -modules

$$X \otimes F(\underline{\mathcal{O}}_G) \cong \Phi_G(Y) \otimes_k F(\underline{\mathcal{O}}_G).$$

Replacing each side of the last equality by the maximal quotient on which $F(\underline{\mathcal{O}}_G)$ acts through $\underline{\mathcal{O}}_H$ we get an isomorphism of $\underline{\mathcal{O}}_H$ -modules:

$$F(Y) \otimes \underline{\mathcal{O}}_H \cong \Phi_G(Y) \otimes_k \underline{\mathcal{O}}_H.$$

Since any $X \in \mathcal{A}$ is a subquotient of $F(Y)$ for some $Y \in \text{Rep}(G)$ we see that the $\underline{\mathcal{O}}_H$ -module $X \otimes \underline{\mathcal{O}}_H \in \underline{\mathcal{O}}_H - \text{mod}$ is a subquotient of $\Phi_G(Y) \otimes_k \underline{\mathcal{O}}_H \in \underline{\mathcal{O}}_H - \text{mod}^{fr}$, hence also lies in $\underline{\mathcal{O}}_H - \text{mod}^{fr}$.

Proof of (b). Notice that (a) together with Lemma 1 imply that $X \otimes \underline{\mathcal{O}}_H = \Phi_H(X) \otimes_k \underline{\mathcal{O}}_H$ canonically for $X \in \mathcal{A}$. This shows exactness of Φ_H , and also establishes monoidal structure on Φ_H , because we have

$$X \otimes Y \otimes \underline{\mathcal{O}}_H = X \otimes (\Phi_H(Y) \otimes_k \underline{\mathcal{O}}_H) = \Phi_H(X) \otimes_k \Phi_H(Y) \otimes_k \underline{\mathcal{O}}_H.$$

Finally, the isomorphism

$$X \otimes \underline{\mathcal{O}}_G \cong \Phi_G(X) \otimes_k \underline{\mathcal{O}}_G$$

for $X \in \text{Rep}(G)$ yields (by applying F , and taking the maximal quotient on which $\underline{\mathcal{O}}_G$ acts through $\underline{\mathcal{O}}_H$) an isomorphism

$$F(X) \otimes \underline{\mathcal{O}}_H \cong \Phi_G(X) \otimes_k \underline{\mathcal{O}}_H,$$

hence an isomorphism (4). □

We can now finish the proof of Proposition 1. According to §2 of [DM], a functor Φ_H as in Corollary 1 (b) above yields a bialgebra A , an equivalence of monoidal categories $\Psi_H : \text{Comod}_A \cong \mathcal{A}$, a morphism of bialgebras $\phi : \mathcal{O}(G) \rightarrow A$ and an isomorphism of monoidal functors

$$F \cong \Psi_H \circ \phi_*$$

Since any object of \mathcal{A} is a subquotient of $F(X)$ for some $X \in \text{Rep}(G)$, the morphism $\phi : \mathcal{O}(G) \rightarrow A$ is surjective. Thus $A = \mathcal{O}(H)$ for a Zariski closed subsemigroup $H \subset G$. Thus Proposition 1 follows from the next Lemma. □

We state the next Lemma in a slightly greater generality than needed for our application (since the proof is the same).

Lemma 2. *Let G be a group scheme of finite type over a Noetherian ring k . Then a closed subsemigroup scheme $H \subset G$ is a subgroup scheme.*

*Proof.*³ We must check that for any commutative k -algebra R the subsemigroup $H(R) \subset G(R)$ is a subgroup. It is enough to check this for R of finite type over k . For $g \in H(R)$ let $L_g : G_R \rightarrow G_R$ be the (left) multiplication by g (here the subindex R denotes base change to R). Then $L_g(H_R) \subseteq H_R$; we want to check that in fact $L_g(H_R) = H_R$. But otherwise $H_R \supsetneq L_g(H_R) \supsetneq L_g^2(H_R) \supsetneq \dots$ is an infinite decreasing chain of closed subschemes in G_R , which contradicts the fact that G_R is Noetherian. □

Remark 2. In this Remark we outline an alternative argument, which is shorter than the proof of Proposition 1 presented above, but uses a deep Theorem of Deligne [De2], and proves a weaker statement (which is still sufficient for our applications).

In the situation of Proposition 1 assume that $\text{char}(k) = 0$, and also that rigidity on the target category \mathcal{A} is given. (In view of Remark 3

³I thank Dima Arinkin, to whom this proof is due.

below this weaker statement is sufficient for our application.) Then one can show first that there exists a unique commutativity constraint on \mathcal{A} compatible with one in \mathcal{B} ; thus \mathcal{B} is a Tannakian category. Now a Theorem of Deligne ([De2] Theorem 7.1) says that for an algebraically closed field k of characteristic 0, a k -linear Tannakian category \mathcal{A} admits a fiber functor, and is identified with the category of representations of an algebraic group, provided that for any object $X \in \mathcal{A}$ we have $\Lambda^n(X) = 0$ for large n (where Λ^n stands the n -th exterior power). If an object Y is a subquotient of X , then $\Lambda^n(Y)$ is a subquotient of $\Lambda^n(X)$, in particular $\Lambda^n(X) = 0 \Rightarrow \Lambda^n(Y) = 0$. Thus \mathcal{A} satisfies the conditions of Deligne's Theorem, and hence $\mathcal{A} = \text{Rep}(H)$ for an algebraic group H by that Theorem. The tensor functor $F: \text{Rep}(G) \rightarrow \mathcal{A} = \text{Rep}(H)$ yields by Tannakian formalism a homomorphism $H \rightarrow G$; since any object of $\text{Rep}(H)$ is a subquotient of $F(X)$ for $X \in \text{Rep}(G)$, this homomorphism is injective.

2.3. Auxiliary Lemmas

The next two Lemmas will be used in establishing, respectively, exactness of the product in a monoidal category, and existence of a unit object.

For a category \mathcal{A} (an abelian or a triangulated category) we will write $K(\mathcal{A})$ for the Grothendieck group of \mathcal{A} . For an object $X \in \mathcal{A}$ we will denote its class by $[X] \in K(\mathcal{A})$.

Lemma 3. *Let \mathcal{A} be an abelian category with all objects having finite length. Let \otimes denote a functor $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ which is linear and mid-exact in each variable. For an object $X \in \mathcal{A}$ let X^{ss} denote the semisimplification of X . Let $X, Y \in \mathcal{A}$ be two objects satisfying*

$$(5) \quad [X \otimes Y] = [X^{ss} \otimes Y^{ss}].$$

Then (5) remains true when X, Y are replaced by a subquotient. Moreover, if X', Y' are subquotients of respectively X, Y and $0 \rightarrow X'' \rightarrow X' \rightarrow X''' \rightarrow 0$ is an exact sequence, then the sequence $0 \rightarrow X'' \otimes Y' \rightarrow X' \otimes Y' \rightarrow X''' \otimes Y' \rightarrow 0$ is exact.

Proof. For $\alpha, \beta \in K(\mathcal{A})$ let us write $\alpha \leq \beta$ if $\beta - \alpha$ is a class of actual (as opposed to virtual) object. Then for any short exact sequence $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ we have

$$(6) \quad [X_2 \otimes Y] \leq [X_1 \otimes Y] + [X_3 \otimes Y]$$

for any Y , with equality being true iff the sequence $0 \rightarrow X_1 \otimes Y \rightarrow X_2 \otimes Y \rightarrow X_3 \otimes Y \rightarrow 0$ is exact. Hence the first statement implies the second.

By induction in Jordan-Hoelder series we see that $[X \otimes Y] \leq [X^{ss} \otimes Y^{ss}]$ for any X, Y . Moreover, if for subquotients X', Y' of X, Y the strict inequality $[X' \otimes Y'] < [X'^{ss} \otimes Y'^{ss}]$ holds, then successive use of (6) shows that $[X \otimes Y] < [X^{ss} \otimes Y^{ss}]$, which contradicts (5). \square

Lemma 4. *Let \mathcal{A} be an abelian category, $\mathcal{I} : \mathcal{A} \rightarrow \mathcal{A}$ be a functor, and $\epsilon : \mathcal{I} \xrightarrow{\sim} \mathcal{I} \circ \mathcal{I}$ be an isomorphism. Let us say that an object $X \in \mathcal{A}$ is fixed by \mathcal{I} if there exists an isomorphism $\iota_X : X \xrightarrow{\sim} \mathcal{I}(X)$ such that*

$$(7) \quad \mathcal{I}(\iota_X) = \epsilon|_X.$$

Assume that \mathcal{I} is exact, and $\mathcal{I}(X) \neq 0$ for $X \neq 0$. Then

- a) If all objects of \mathcal{A} are fixed by \mathcal{I} , then $\mathcal{I} \cong Id$.
- b) A subquotient of an object fixed by \mathcal{I} is fixed by \mathcal{I} .

Proof. Since \mathcal{I} is exact, and kills no objects, it is injective on morphisms. It follows that for given X an isomorphism ι_X satisfying (7) is unique if exists. Also, $\iota = (\iota_X : X \rightarrow \mathcal{I}(X))$ defines a morphism of functors $Id \rightarrow \mathcal{I}$ on the full subcategory of \mathcal{I} -fixed objects, because $\mathcal{I}(\iota) = \epsilon$ is a morphism of functors $\mathcal{I} \rightarrow \mathcal{I} \circ \mathcal{I}$. This proves (a). To check (b) it suffices to see that for an \mathcal{I} -fixed object X , and any subobject $j : Y \hookrightarrow X$ the images of imbeddings $\mathcal{I}(j) : \mathcal{I}(Y) \hookrightarrow \mathcal{I}(X)$ and $\iota_X \circ j : Y \hookrightarrow \mathcal{I}(X)$ coincide; indeed, then the induced isomorphisms $\iota_Y : Y \xrightarrow{\sim} \mathcal{I}(Y)$ and $\iota_{X/Y} : X/Y \xrightarrow{\sim} \mathcal{I}(X/Y)$ are respectively sub and quotient of ι_X , hence satisfy (7). But then it is enough to see that images of $\mathcal{I} \circ \mathcal{I}(j)$ and $\mathcal{I}(\iota_X \circ j)$ coincide. Since $\mathcal{I}(\iota_X \circ j) = \epsilon|_X \circ \mathcal{I}(j) = \mathcal{I} \circ \mathcal{I}(j) \circ \epsilon|_Y$ the proof is finished. \square

§3. Notations and recollections

3.1. General notations

Let G be a split simple algebraic group over \mathbb{Z} . Let $F = \mathbb{F}_q((t))$ be a local field of prime characteristic, and $O = \mathbb{F}_q[[t]]$ be its ring of integers.⁴ Let $I \subset G(O) \subset G(F)$ be an Iwahori subgroup. There exist canonically defined group schemes \mathbf{K}, \mathbf{I} over \mathbb{F}_q (of infinite type) such that $\mathbf{K}(\mathbb{F}_q) = G(O)$, $\mathbf{I}(\mathbb{F}_q) = I$; and an ind-group scheme \mathbf{G} with $\mathbf{G}(\mathbb{F}_q) = G(F)$. We also have ind-varieties $\mathcal{F}\ell = \mathbf{G}/\mathbf{I}$ and $\mathcal{G}\tau = \mathbf{G}/\mathbf{K}$. More precisely, $\mathcal{F}\ell, \mathcal{G}\tau$ are direct limits of projective varieties with transition maps being closed imbeddings, and $\mathcal{F}\ell(\mathbb{F}_q) = G(F)/I$, $\mathcal{G}\tau(\mathbb{F}_q) = G(F)/G(O)$.

⁴As usual one could replace \mathbb{F}_q by an algebraically closed field of characteristic zero; then we would have to work with Hodge D -modules instead of mixed sheaves in the proof of Lemma 7 below.

The orbits of \mathbf{I} on $\mathcal{F}\ell$, $\mathcal{G}\mathfrak{r}$ are finite dimensional, and isomorphic to affine spaces; they are sometimes called Schubert cells. As before, W_f is the Weyl group of G , and W is the extended affine Weyl group. Then W is identified with the set of Schubert cells in $\mathcal{F}\ell$. For $w \in W$ (respectively $w \in W/W_f$) let $\mathcal{F}\ell_w$, $\mathcal{G}\mathfrak{r}_w$ be the corresponding Schubert cells.

Let $D^I = D^I(\mathcal{F}\ell)$ be the \mathbf{I} -equivariant derived category of l -adic sheaves on $\mathcal{F}\ell_{\overline{\mathbb{F}}_q}$ ($l \neq p$), and $\mathcal{P}^I(\mathcal{F}\ell) \subset D^I$ be the full subcategory of perverse sheaves (here the subscript $\overline{\mathbb{F}}_q$ denotes extension of scalars from \mathbb{F}_q to the algebraic closure $\overline{\mathbb{F}}_q$).

The convolution product, which we denote by $*$, defines a functor $D^I \times D^I \rightarrow D^I$, which provides D^I with the structure of a monoidal category.

3.2. Central sheaves

Let $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$ be the category of \mathbf{K} equivariant perverse sheaves on $\mathcal{G}\mathfrak{r}$. It is known ([L0], see also [Ga] for an alternative proof and a generalization) that for $X, Y \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$ the convolution $X * Y$ lies in $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$. Further, convolution endows $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$ with the structure of a monoidal category, and it naturally extends to a structure of a commutative rigid tensor category with a fiber functor; the resulting Tannakian category is equivalent to the category $\text{Rep}({}^L G)$ of algebraic representations of the Langlands dual group ${}^L G$ over $\overline{\mathbb{Q}}_l$ (see [Gi], [MV], at least for an analogous statement over \mathbb{C} , and also [BD]). We will identify $\text{Rep}({}^L G)$ with $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$.

In [Ga] a functor $Z : \text{Rep}({}^L G) = \mathcal{P}_{\mathcal{G}\mathfrak{r}} \rightarrow \mathcal{P}^I(\mathcal{F}\ell)$ was constructed. It enjoys the following properties (properties (i), (ii), (iv) are checked in [Ga], and property (iii) is checked in Appendix below).

- i) $\pi_* \circ Z \cong \text{id}$, where $\pi : \mathcal{F}\ell \rightarrow \mathcal{G}\mathfrak{r}$ is the projection.
- ii) (Exactness of convolution) For $\mathcal{F} \in \mathcal{P}_{\mathcal{G}\mathfrak{r}}$, $\mathcal{G} \in \mathcal{P}$ we have $\mathcal{G} * Z(\mathcal{F}) \in \mathcal{P}$.
- iii) (Compatibility with convolution and centrality) Z is a central functor from the tensor category $\mathcal{P}_{\mathcal{G}\mathfrak{r}}$ to the monoidal category D^I in the sense of definition 1 above.
- iv) (Monodromy) A unipotent automorphism \mathfrak{M} of the functor Z is given, $\mathfrak{M}_{Z(\mathcal{F})} \in \text{Aut}(Z(\mathcal{F}))$; it is called the monodromy automorphism (for reasons explained in [Ga]). It satisfies

$$(8) \quad \mathfrak{M}_{Z(\mathcal{F} * \mathcal{F}')} = \mathfrak{M}_{Z(\mathcal{F})} * \mathfrak{M}_{Z(\mathcal{F}')},$$

where we identified $\text{End}(Z(\mathcal{F} * \mathcal{F}')) = \text{End}(Z(\mathcal{F}) * Z(\mathcal{F}'))$ by means of (iii).

3.3. Serre quotient categories

The set of isomorphism classes of irreducible objects in \mathcal{P} is in bijection with W . For $w \in W$ let L_w be the corresponding irreducible object; more precisely, we set

$$(9) \quad L_w = j_{w!*} \left(\overline{\mathbb{Q}_l}[\dim \mathcal{F}l_w] \left(\frac{\dim \mathcal{F}l_w}{2} \right) \right),$$

where j_w denotes the imbedding $\mathcal{F}l_w \hookrightarrow \mathcal{F}l$, and $\overline{\mathbb{Q}_l}$ is the constant sheaf (the Tate twist by $\frac{\dim \mathcal{F}l_w}{2}$ will be essential later when we will work with mixed sheaves).

Recall that a Serre subcategory in an abelian category is a strictly full abelian subcategory closed under extensions and subquotients. If A is an abelian category, and B is a Serre subcategory, then the *Serre quotient* A/B is again an abelian category. If every object in A has finite length, then B is uniquely specified by the set of (isomorphism classes of) irreducible objects of A which lie in B .

Let $\underline{c} \subset W$ be a two-sided cell. Set $W_{<\underline{c}} = \bigcup_{\underline{c}' < \underline{c}} \underline{c}'$; $W_{\leq \underline{c}} = \bigcup_{\underline{c}' \leq \underline{c}} \underline{c}'$; here \leq is the standard partial order on the set of 2-sided cells (see [L1]), and we write $\underline{c}' < \underline{c}$ instead of $\underline{c}' \leq \underline{c}$ & $\underline{c}' \neq \underline{c}$.

For a subset $S \subset W$ let \mathcal{P}_S^I denote the Serre subcategory of \mathcal{P}^I whose set of (representatives for isomorphism classes of) irreducible objects is $\{L_w, w \in S\}$. We abbreviate $\mathcal{P}_{\leq \underline{c}}^I = \mathcal{P}_{W_{\leq \underline{c}}}^I$, $\mathcal{P}_{<\underline{c}}^I = \mathcal{P}_{W_{<\underline{c}}}^I$.

Let $\mathcal{P}_{\underline{c}}^I$ denote the Serre quotient category $\mathcal{P}_{\leq \underline{c}}^I / \mathcal{P}_{<\underline{c}}^I$.

§4. Truncated convolution categories

4.1. Truncated convolution and action of the central sheaves

Let $D_{\leq \underline{c}}^I(\mathcal{F}l)$, $D_{<\underline{c}}^I(\mathcal{F}l)$ be the full triangulated subcategories of $D^I(\mathcal{F}l)$ consisting of complexes with cohomology in, respectively, $\mathcal{P}_{\leq \underline{c}}^I$, $\mathcal{P}_{<\underline{c}}^I$. From the definition of a two-sided cell it follows that $D_{\leq \underline{c}}^I$, $D_{<\underline{c}}^I$ are stable under convolution with any object of D^I , i.e. $X \in D_{\leq \underline{c}}^I$, $Y \in D^I \Rightarrow X * Y \in D_{\leq \underline{c}}^I$, and the same for $D_{<\underline{c}}^I$.

In particular, for $X, Y \in \mathcal{P}_{\leq \underline{c}}^I$ we have $H^i(X * Y) \in \mathcal{P}_{\leq \underline{c}}^I$, and the image of $H^i(X * Y)$ in $\mathcal{P}_{\underline{c}}^I$ depends canonically only on the image of X, Y in $\mathcal{P}_{\underline{c}}^I$. Thus the formula

$$(X, Y) \mapsto H^i(X * Y) \quad \text{mod } \mathcal{P}_{<\underline{c}}^I$$

defines a bilinear functor $\mathcal{P}_{\underline{c}}^I \times \mathcal{P}_{\underline{c}}^I \rightarrow \mathcal{P}_{\underline{c}}^I$.

Recall that for a two-sided cell \underline{c} a non-negative integer $a(\underline{c})$ is defined (see [L1]); and we have $H^i(X * Y) \in \mathcal{P}_{<\underline{c}}^I$ for $i > a(\underline{c})$, $X \in \mathcal{P}_{\leq\underline{c}}^I$, $Y \in \mathcal{P}^I$.

For $X, Y \in \mathcal{P}_{\underline{c}}^I$ we define their truncated convolution $X \bullet Y \in \mathcal{P}_{\underline{c}}^I$ by $X \bullet Y = H^{a(\underline{c})}(X * Y) \bmod \mathcal{P}_{<\underline{c}}^I$. For semisimple X, Y this coincides with the definition in [L4].

Also, for $\mathcal{F} \in \mathcal{P}_{\mathfrak{G}\tau}$ the exact functor $\mathcal{G} \mapsto \mathcal{G} * Z(\mathcal{F})$ preserves $\mathcal{P}_{\leq\underline{c}}^I$ and $\mathcal{P}_{<\underline{c}}^I$, hence induces an exact functor from $\mathcal{P}_{\underline{c}}$ to itself (denoted again by $\mathcal{G} \mapsto \mathcal{G} * Z(\mathcal{F})$).

4.2. The monoidal category $\mathcal{A}_{\underline{c}}$

Proposition 2. *Let $\mathcal{A}_{\underline{c}}$ be the strictly full subcategory of $\mathcal{P}_{\underline{c}}^I$ consisting of all subquotients of $L_w * Z(\mathcal{F})$, $w \in \underline{c}$, $\mathcal{F} \in \mathcal{P}_{\mathfrak{G}\tau}$. Then*

a) *The restriction of \bullet to $\mathcal{A}_{\underline{c}} \times \mathcal{A}_{\underline{c}}$ takes values in $\mathcal{A}_{\underline{c}}$, and is exact in each variable.*

b) *It equips $\mathcal{A}_{\underline{c}}$ with a structure of a monoidal category; the object $\mathbb{I}_{\underline{c}} = \bigoplus_d L_d$, where d runs over the set of Duflo involutions in \underline{c} , is a unit object of $(\mathcal{A}_{\underline{c}}, \bullet)$.*

Remark 3. It seems possible to check that the monoidal category $\mathcal{A}_{\underline{c}}$ is rigid, i.e. for $X \in \mathcal{A}_{\underline{c}}$ the “dual” object X^\sim is defined together with morphisms $ev : X \bullet X^\sim \rightarrow \mathbb{I}_{\underline{c}}$, and $\delta : \mathbb{I}_{\underline{c}} \rightarrow X^\sim \bullet X$ satisfying the usual compatibilities (see e.g. [De2], 2.1.2). Here X^\sim has the following geometric interpretation. The category \mathcal{P}^I has a canonical anti-involution $\iota : \mathcal{P}^I \rightarrow (\mathcal{P}^I)^{op}$ induced (loosely speaking) by the morphism $i : \mathfrak{G} \rightarrow \mathfrak{G}$, $i : g \mapsto g^{-1}$. Then

$$(10) \quad X^\sim \cong \mathbb{V}(\iota(X))$$

canonically, where \mathbb{V} stands for Verdier duality. (We neither use nor prove this fact here).

Proof of the Proposition. a) It follows from the definitions that \bullet is right exact in each variable. We will deduce that it is exact on $\mathcal{A}_{\underline{c}}$ from a result of Lusztig on asymptotic Hecke algebras. (The argument will use this result of Lusztig and mid-exactness of \bullet , but not its right exactness; see also Remark 4 below).

The group $K(\mathcal{P}_{\underline{c}})$ has a natural structure of an associative algebra; the product on $K(\mathcal{A}_{\underline{c}})$ is denoted by \bullet and is defined by

$$[L_{w_1}] \bullet [L_{w_2}] = [L_{w_1} \bullet L_{w_2}]$$

for irreducible objects $L_{w_1}, L_{w_2} \in \mathcal{A}_{\underline{c}}$. Thus $K(\mathcal{A}_{\underline{c}}), \bullet$ is the asymptotic Hecke algebra $J_{\underline{c}}$, cf. [L4]. The classes of simple objects $L_w \in \mathcal{P}_{\underline{c}}$ form

a standard basis of this algebra; the standard notation for elements of this basis is $t_w = [L_w] \in J_{\underline{c}} = K(\mathcal{P}_{\underline{c}})$.

The key step is the next

Lemma 5. For $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{P}_{\mathcal{G}_\tau}$ and simple objects $L_{w_1}, L_{w_2} \in \mathcal{A}_{\underline{c}}$ we have

$$(11) \quad [(Z(\mathcal{G}_1) * L_{w_1}) \bullet (L_{w_2} * Z(\mathcal{G}_2))] = [Z(\mathcal{G}_1) * L_{w_1}] \bullet [L_{w_2} * Z(\mathcal{G}_2)].$$

Proof. The Lemma is a consequence of [L2], 2.4. Let us recall the statement of *loc. cit.*

To formulate it we need some notations. Set $A = \mathbb{Z}[v, v^{-1}]$; thus the affine Hecke algebra \mathcal{H} is an A -algebra (the algebra \mathbb{H} mentioned in the introduction is given by $\mathbb{H} = \mathcal{H} \otimes_A \mathbb{C}$, where $v \mapsto q^{1/2} \in \mathbb{C}$). Let $C_w \in \mathcal{H}$, $w \in W$ be the Kazhdan Lusztig basis of \mathcal{H} . Let also $\mathcal{H}_{\leq \underline{c}}, \mathcal{H}_{< \underline{c}}$ be A submodules in \mathcal{H} generated by C_w with w running over the corresponding subset of W . According to the definition of a two-sided cell the A -submodules $\mathcal{H}_{\leq \underline{c}}, \mathcal{H}_{< \underline{c}}$ are two-sided ideals in \mathcal{H} . Thus $\mathcal{H}_{\underline{c}} := \mathcal{H}_{\leq \underline{c}}/\mathcal{H}_{< \underline{c}}$ is a bimodule over \mathcal{H} . Let $S : \mathcal{H}_{\underline{c}} \rightarrow J_{\underline{c}} \otimes_{\mathbb{Z}} A$ be the isomorphism of A -modules, which sends C_w into $t_w = [L_w] \in J_{\underline{c}} \subset J_{\underline{c}} \otimes A$. Let us transport the structure of an \mathcal{H} -bimodule to $J_{\underline{c}} \otimes A$ by means of S ; denote the resulting action by $h_1 \otimes h_2 : x \mapsto h_1 * x * h_2$, $h_1, h_2 \in \mathcal{H}$, $x \in J_{\underline{c}} \otimes A$.

Fact 1 ([L2], 2.4). *a) Thus defined right (respectively, left) action of \mathcal{H} on $J_{\underline{c}} \otimes A$ commutes with the canonical left (resp., right) action of $J_{\underline{c}}$ on $J_{\underline{c}} \otimes A$. In other words, extending the \bullet product to an A -algebra structure on $J_{\underline{c}} \otimes A$, we get*

$$(12) \quad h_1 * (x \bullet y) * h_2 = (h_1 * x) \bullet (y * h_2).$$

b) The map $\phi_{\underline{c}} : \mathcal{H} \rightarrow J_{\underline{c}} \otimes A$ defined by

$$(13) \quad \phi_{\underline{c}}(h) = h * \left(\sum t_d \right),$$

(where d runs over the set of Duflo involutions in \underline{c}) is an algebra homomorphism. \square

We deduce Lemma 5 from (12). In fact we will use only specialization of (12) at $v = 1$, and only for h in the center of \mathcal{H} .

Fix the homomorphism $A \rightarrow \mathbb{Z}$ sending v to 1, and set $H = \mathcal{H} \otimes_A \mathbb{Z} \cong \mathbb{Z}[W]$. The structure of an \mathcal{H} -bimodule on $J_{\underline{c}} \otimes A$ yields a structure of H -bimodule on $J_{\underline{c}}$, which we also denote by $h_1 \otimes h_2 : x \mapsto h_1 * x * h_2$. Thus (12) holds for $x, y \in J_{\underline{c}}$ and $h_1, h_2 \in H$.

We have a standard isomorphism $K(\mathcal{P}^I) = K(D^I) \cong H = \mathbb{Z}[W]$ compatible with the ring structure, where the product in $K(D^I)$ is defined by means of convolution: $[X] * [Y] = [X * Y]$.

Let $x = t_{w_1} = [L_{w_1}]$, $y = t_{w_2} = [L_{w_2}] \in K(\mathcal{P}_{\underline{c}}^I) = J_{\underline{c}}$; and $h_i = [Z(\mathcal{G}_i)] \in K(\mathcal{P}^I) = H$, $i = 1, 2$; then we claim that the left-hand side of (11) equals the left-hand side of (12), and the right-hand side of (11) equals the right-hand side of (12).

Indeed, the statement about the right-hand sides follows from the definitions.

To check the statement about the left-hand sides we rewrite

$$\begin{aligned} (Z(\mathcal{G}_1) * L_{w_1}) \bullet (L_{w_2} * Z(\mathcal{G}_2)) &= H^a(Z(\mathcal{G}_1) * L_{w_1} * L_{w_2} * Z(\mathcal{G}_2)) \pmod{\mathcal{P}_{<\underline{c}}^I} \\ &= H^a(Z(\mathcal{G}_1) * (L_{w_1} * L_{w_2}) * Z(\mathcal{G}_2)) \pmod{\mathcal{P}_{<\underline{c}}^I}. \end{aligned}$$

By 3.2 (ii) above, convolution with $Z(\mathcal{G})$ is exact for $\mathcal{G} \in \mathcal{P}_{\mathcal{G}\tau}$, thus the right hand side in the last equality equals $Z(\mathcal{G}_1) * H^a(L_{w_1} * L_{w_2}) * Z(\mathcal{G}_2)$, hence its class equals $h_1 * (t_{w_1} \bullet t_{w_2}) * h_2$. \square

The last Lemma together with Lemma 3 imply exactness of $\bullet|_{\mathcal{A}_{\underline{c}} \times \mathcal{A}_{\underline{c}}}$ in each variable, and part (a) of the Proposition.

Remark 4. It may be possible to get an alternative proof of exactness of \bullet product, not appealing to Lusztig's result (12), by establishing rigidity in $\mathcal{A}_{\underline{c}}$ by direct geometric considerations as in Remark 3 above. (Recall that for an abelian tensor category rigidity implies exactness of tensor product, cf. [DM], Proposition 1.16.)

Proof of Proposition 2(b). Associativity of truncated convolution follows from associativity of convolution, and equality $H^i(X * Y) = 0 \pmod{\mathcal{P}_{<\underline{c}}^I}$ for $i > a(\underline{c})$, $X, Y \in \mathcal{P}_{\leq \underline{c}}^I$, because

$$(X \bullet Y) \bullet Z \cong H^{2a}(X * Y * Z) \pmod{\mathcal{P}_{<\underline{c}}^I} \cong X \bullet (Y \bullet Z)$$

canonically. Also, the properties of the functor Z imply that

$$(L_{w_1} * Z(\mathcal{G}_1)) \bullet (L_{w_2} * Z(\mathcal{G}_2)) \cong (L_{w_1} \bullet L_{w_2}) * Z(\mathcal{G}_1 * \mathcal{G}_2)$$

canonically. The right hand side of the last equality lies in $\mathcal{A}_{\underline{c}}$; also, exactness of $\bullet|_{\mathcal{A}_{\underline{c}} \times \mathcal{A}_{\underline{c}}}$ implies that if $X_i \in \mathcal{P}_{\underline{c}}^I$ is a subquotient of $L_{w_i} * Z(\mathcal{G}_i)$, $i = 1, 2$, then $X_1 \bullet X_2$ is a subquotient of $(L_{w_1} \bullet L_{w_2}) * Z(\mathcal{G}_1 * \mathcal{G}_2)$. Thus $\mathcal{A}_{\underline{c}}$ is stable under the \bullet -product. Let us check that $\mathbb{I}_{\underline{c}}$ is a unit object in $\mathcal{A}_{\underline{c}}$. Lusztig proved (see [L4], 2.9) that

$$(14) \quad \mathbb{I}_{\underline{c}} \bullet \mathbb{I}_{\underline{c}} \cong \mathbb{I}_{\underline{c}};$$

$$(15) \quad \mathbb{I}_{\underline{c}} \bullet L_w \cong L_w \cong L_w \bullet \mathbb{I}_{\underline{c}}, \quad w \in \underline{c}.$$

Fix isomorphism (14) arbitrarily; it suffices to show that any object of $\mathcal{A}_{\underline{c}}$ is fixed by the functors $\mathcal{I}_l : X \mapsto \mathbb{I}_{\underline{c}} \bullet X$, and $\mathcal{I}_r : X \mapsto X \bullet \mathbb{I}_{\underline{c}}$ in the sense of Lemma 4 above. We have seen that these functors are exact, and (15) shows that it kills no object. Also, it is easy to see that (15) can be chosen to satisfy (7); hence all semisimple objects of $\mathcal{A}_{\underline{c}}$ are fixed by $\mathcal{I}_l, \mathcal{I}_r$. Finally, (15) yields an isomorphism

$$\mathcal{I}_l(L_w * Z(\mathcal{G})) = \mathbb{I}_{\underline{c}} \bullet (L_w * Z(\mathcal{G})) \cong (\mathbb{I}_{\underline{c}} \bullet L_w) * Z(\mathcal{G}) \cong L_w * Z(\mathcal{G})$$

for $\mathcal{G} \in \mathcal{P}_{\mathcal{G}\tau}$, which is easily seen to satisfy (7) if (15) does, and similarly for \mathcal{I}_r . Thus any object of $\mathcal{A}_{\underline{c}}$ is a subquotient of an object fixed by $\mathcal{I}_l, \mathcal{I}_r$, and Lemma 4 shows that $\mathcal{I}_l, \mathcal{I}_r$ are isomorphic to the identity functor, hence $\mathbb{I}_{\underline{c}}$ is a unit object. \square

4.3. Tannakian category \mathcal{A}_d

Let $d \in \underline{c}$ be a Duflo involution. Then by [L4], 2.9 we have

$$(16) \quad L_d \bullet L_d \cong L_d.$$

Let $\mathcal{A}_d \subset \mathcal{A}_{\underline{c}} \subset \mathcal{P}_{\underline{c}}^I$ be the strictly full subcategory consisting of all subquotients of $L_d * Z(\mathcal{F})$, $\mathcal{F} \in \mathcal{P}_{\mathcal{G}\tau}$. Let a functor $Res_d : Rep(LG) = \mathcal{P}_{\mathcal{G}\tau} \rightarrow \mathcal{A}_d$ be defined by $Res_d(\mathcal{G}) = L_d * Z(\mathcal{G})$.

Lemma 6. a) $\mathcal{A}_d \subset \mathcal{A}_{\underline{c}}$ is a monoidal subcategory, and $L_d \subset \mathcal{A}_d$ is a unit object.

b) Res_d has a natural structure of a central functor.

c) \mathfrak{M} induces a tensor automorphism of Res_d (to be denoted by \mathfrak{M}_d).

Proof. a) The first statement follows from

$$(L_d * Z(\mathcal{G}_1)) \bullet (L_d * Z(\mathcal{G}_2)) \cong (L_d \bullet L_d) * Z(\mathcal{G}_1 * \mathcal{G}_2) \cong L_d * Z(\mathcal{G}_1 * \mathcal{G}_2)$$

and exactness of $\bullet|_{\mathcal{A}_{\underline{c}}}$. In view of (16), in order to check that L_d is a unit object we have only to show that

$$L_d \bullet X \cong X$$

for $X \in \mathcal{A}_d$ (cf. the proof of Proposition 2 (b)). It also follows from [L4], 2.9 that

$$L_{d'} \bullet L_d = 0$$

if $d' \neq d$ are different Duflo involutions in \underline{c} . Hence $L_{d'} \bullet (L_d * Z(\mathcal{G})) \cong (L_{d'} \bullet L_d) * Z(\mathcal{G}) = 0$, and by exactness of $\bullet|_{\mathcal{A}_{\underline{c}}}$ it follows that $L_{d'} * X = 0$

for $X \in \mathcal{A}_d$. Thus for $X \in \mathcal{A}_d$ we have

$$X \cong \mathbb{I}_{\underline{c}} \bullet X = \bigoplus_{d' \in \underline{c}} L_{d'} \bullet X = L_d \bullet X.$$

This proves (a).

(b), (c) follow immediately from, respectively, properties (iii) and (iv) stated in section 3.2. \square

§5. Main result

Recall that if $G_1 \supset G_2$ are algebraic groups over a base field k , and $N \in G_1(k)$ is an element which commutes with G_2 , then N defines a tensor automorphism of the restriction functor $Res_{G_2}^{G_1} : Rep(G_1) \rightarrow Rep(G_2)$. We denote this automorphism by Aut_N .

Theorem 1. *There exists a pair H_d, N_d , where $H_d \subset {}^L G_{\mathbb{Q}_l}$ is an algebraic subgroup, and $N_d \in {}^L G(\mathbb{Q}_l)$ is a unipotent element commuting with H_d ; an equivalence of tensor categories $\Phi_d : Rep(H_d) \cong \mathcal{A}_d$, and an isomorphism $Res_{H_d}^{L^G} \cong \Phi_d \circ Res_d$, which intertwines the tensor automorphisms Aut_{N_d} and \mathfrak{M}_d .*

The pair (H_d, N_d) is unique up to conjugacy.

Remark 5. Rigidity (duality) in \mathcal{A}_d can most probably be interpreted geometrically, and is given by (3) above.

Proof of the Theorem follows directly from Lemma 6 and Proposition 1. \square

Below k will denote an algebraically closed field, $\text{char}(k) = 0$. As before, for an algebraic group H over k we will write $\mathcal{R}(H)$ for its representation ring, and set $\mathcal{R}^k(H) = \mathcal{R}(H) \otimes k$. For $s \in H(k)$ we will denote the corresponding character of $\mathcal{R}(H)$ (or of $\mathcal{R}^k(H)$) by $\chi_s : \mathcal{R}(H) \rightarrow k$, $\chi_s([V]) = Tr(s, V)$.

Recall the bijection between two-sided cells in W and unipotent conjugacy classes in ${}^L G(k)$ constructed by Lusztig in [L3]. For a two-sided cell \underline{c} we let $N_{\underline{c}} \in {}^L G(\mathbb{Q}_l)$ denote a unipotent element in the corresponding conjugacy class.

Theorem 2. *For $d \in \underline{c}$ the conjugacy classes of elements N_d and $N_{\underline{c}}$ coincide.*

Proof. We will need a characterization of the bijection $\underline{c} \leftrightarrow N_{\underline{c}}$.

We set $J_{\underline{c}}^k = J_{\underline{c}} \otimes_{\mathbb{Z}} k$, $\mathcal{H}^k = \mathcal{H} \otimes_{\mathbb{Z}} k$.

For a unipotent element $N \in {}^L G(k)$ fix a homomorphism $\gamma_N : SL(2) \rightarrow {}^L G$ defined over k , such that $N = \gamma(E)$, where $E \in SL(2, k)$

is the standard unipotent element. Let $s_{SL(2)}^v \in SL(2, k[v, v^{-1}])$ be the diagonal matrix with entries v, v^{-1} ; and define an element $s_N^v \in {}^L G(k[v, v^{-1}])$ by

$$s_N^v = \gamma_N(s_{SL(2)}^v).$$

Recall the homomorphism $\phi_{\underline{c}} : \mathcal{H} \rightarrow J_{\underline{c}} \otimes \mathbb{Z}[v, v^{-1}]$, see (13).

We will also make use of the isomorphism (due to Bernstein)

$$(17) \quad B : \mathcal{R}({}^L G)[v, v^{-1}] \xrightarrow{\sim} Z(\mathcal{H}),$$

where $Z(\mathcal{H})$ is the center of \mathcal{H} .

Fact 2. [L3] a) The center of $J_{\underline{c}}^k$, $Z(J_{\underline{c}}^k)$ is isomorphic to the algebra $\mathcal{R}^k(Z_{L_G}(N_{\underline{c}})) = \mathcal{R}^k(Z_{L_G}(\gamma_{N_{\underline{c}}}))$.

b) The homomorphism $\phi_{\underline{c}}$ sends the center $Z(\mathcal{H})$ into $Z(J_{\underline{c}}) \otimes_{\mathbb{Z}} A$.

c) Let s be a semisimple element of $Z_{L_G(k)}(\gamma_{N_{\underline{c}}})$. It defines a character $\chi_s : Z(J_{\underline{c}}) \rightarrow k$, and, by extension of scalars, a character $\chi_s^v : Z(J_{\underline{c}}) \otimes A \rightarrow k[v, v^{-1}]$. The character $\tilde{\chi}_s = \chi_s^v \circ (\phi_{\underline{c}}|_{Z(\mathcal{H})}) : Z(\mathcal{H}) \rightarrow k[v, v^{-1}]$ defines a semisimple conjugacy class $\Omega_s \subset {}^L G(\overline{k(v)})$.

Then $\Omega_s \ni s_{N_{\underline{c}}}^v \cdot s$.

Notice that if N, N' are non-conjugate unipotent elements, then $s_N^v \cdot s$ is not ${}^L G(\overline{k(v)})$ -conjugate to $s_{N'}^v \cdot s'$ for any $s \in Z_{L_G(k)}(N)$, $s' \in Z_{L_G(k)}(N')$.

Thus, in view of Fact 2(d), to prove Theorem 2 it suffices to check that setting $k = \overline{\mathbb{Q}_l}$ we get

$$(18) \quad \Omega_s \ni s_{N_d}^v \cdot s_{\text{const}}$$

for some $s_{\text{const}} \in {}^L G(k)$.

The key step in the proof of Theorem 2 is the next Lemma.

We keep notations of Theorem 1. In particular for a Duflo involution $d \in \underline{c}$ we have a subgroup $H_d \subset {}^L G_{\overline{\mathbb{Q}_l}}$ with an equivalence $\text{Rep}(H_d) \cong \mathcal{A}_d \subset \mathcal{A}_{\underline{c}}$; thus the Grothendieck group $\mathcal{R}(H_d) = K(\mathcal{A}_d)$ is a subalgebra in $J_{\underline{c}} = K(\mathcal{A}_{\underline{c}})$.

For $V \in \text{Rep}({}^L G)$ the nilpotent endomorphism $\log(\mathfrak{M}_d) = \log(N_d)$ yields a filtration (the Jacobson-Morozov-Deligne filtration, see [De1], 1.6, and also [BB], 4.1) on $\text{Res}_d(V) \cong \Phi_d(V)$. Let $gr_i(\text{Res}_{H_d}^{{}^L G}(V)) \in \text{Rep}(H_d)$ denote the i -th associated graded subquotient of this filtration.

Lemma 7. We have an equality in $J_{\underline{c}}[v, v^{-1}]$:

$$(19) \quad \phi_{\underline{c}}(B([V])) = \sum_d \sum_i v^i [gr_i(\text{Res}_{H_d}^{{}^L G}(V))]$$

(where $\phi_{\underline{c}}$ is as in (13)).

Proof of the Lemma. Let $D_{\mathbb{F}_q}^I$ be the \mathbf{I} -equivariant derived category of l -adic sheaves on $\mathcal{F}\ell$, and $D_{mix}^I = D_{mix}^I(\mathcal{F}\ell) \subset D_{\mathbb{F}_q}^I$ be the subcategory of mixed complexes, see [BBD], 5.1.5. Let $\mathcal{P}_{mix}^I \subset D_{mix}^I$ be the full subcategory of perverse sheaves. Then D_{mix}^I is a monoidal category; and we have a natural functor (pull-back under the morphism $\mathcal{F}\ell_{\mathbb{F}_q} \rightarrow \mathcal{F}\ell$) $D_{mix}^I \rightarrow D^I$. The functor Z factors through a functor $Rep(LG) \rightarrow \mathcal{P}_{mix}$, which will be denoted by the same letter; thus for $V \in Rep(LG)$ the perverse sheaf $Z(V)$ is obtained by taking the nearby cycles functor from a weight zero irreducible perverse sheaf.

We have a standard homomorphism of abelian groups $\tau : K(D_{mix}^I) \rightarrow \mathcal{H}$ characterized by $\tau([j_{w!}(\mathcal{F})]) = T_w$ where \mathcal{F} is a weight zero \mathbf{I} -equivariant sheaf on $\mathcal{F}\ell_w$, and T_w is an element of the standard basis of \mathcal{H} . We have also $\tau([L_w]) = C_w$ where the mixed sheaf L_w is defined by (9). It satisfies

$$\tau([X * Y]) = \tau([X]) \cdot \tau([Y]).$$

For $V \in Rep(LG)$ we have $\tau([Z(V)]) = B([V])$ (cf. [Ga], 1.2.1).

Then it follows from the definitions that

$$\phi_{\underline{c}}(B[V]) = S(\tau(Z(V) * \bigoplus_d L_d) \bmod \mathcal{H}_{<\underline{c}}),$$

where $S : \mathcal{H}_{\leq \underline{c}} / \mathcal{H}_{<\underline{c}} \xrightarrow{\sim} J_{\underline{c}} \otimes \mathbb{Z}[v, v^{-1}]$ is the isomorphism sending C_w to t_w .

Any $X \in \mathcal{P}_{mix}^I$ carries the canonical weight filtration (see [BBD]); let $\mathbf{gr}_i(X)$ denote the i -th associated graded subquotient of this filtration. Then $[\mathbf{gr}_i(X)]$ lies in the \mathbb{Z} -span of $v^i C_w$.

In particular,

$$S(\tau(\mathbf{gr}_i(Z(V) * L_d)) \bmod \mathcal{H}_{<\underline{c}}) \in v^i J_{\underline{c}}.$$

Thus to check (19) it is enough to ensure that the filtration on $\Phi_d(V)$ induced by the weight filtration on $Z(V) * L_d$ coincides with the canonical (Deligne) filtration associated to the nilpotent endomorphism $\log N_{\underline{c}} \in \mathit{End}(\Phi_d(V))$.

We claim that a stronger statement holds. Namely, we claim that the canonical filtration on $Z(V) * L_d$ associated to the logarithm of monodormy coincides with the weight filtration (this implies the desired statement, as canonical filtration associated to a nilpotent endomorphism is compatible with passing to a Serre quotient category). Since $Z(V) * L_d$ is obtained by nearby cycles from a pure weight zero perverse sheaf (cf. [Ga], Proposition 6), the latter statement is a particular case of a general Theorem of Gabber et. al. on coincidence of monodormic and

weight filtrations on nearby cycles of a pure sheaf, see [BB], Theorem 5.1.2. \square

Remark 6. The proof of Lemma 7 is the only place in this note where we have to work with l -adic sheaves on a scheme over the finite field \mathbb{F}_q , rather than over $\overline{\mathbb{F}}_q$ (the latter could be safely replaced by constructible sheaves on the corresponding complex variety).

Corollary 2. *Let $\phi_d : Z(\mathcal{H}) \rightarrow \mathcal{R}(H_d) \otimes A \subset J_{\underline{c}} \otimes A$ be the homomorphism given by $z \mapsto z * t_d$. For a semisimple $s \in H_d(k)$ consider the character $\chi_s^v : \mathcal{R}(H_d) \rightarrow k[v, v^{-1}]$ obtained from $\chi_s : \mathcal{R}(H_d) \rightarrow k$ by extension of scalars. The character $\tilde{\chi}_s = \chi_s^v \circ (\phi_d) : Z(\mathcal{H}) = \mathcal{R}({}^L G) \rightarrow k[v, v^{-1}]$ defines a semisimple conjugacy class in ${}^L G(k(v))$.*

This conjugacy class contains the element $s_{N_d}^v \cdot s$. \square

Proof of Theorem 2. Let M be an irreducible $J_{\underline{c}}^k$ module on which the idempotent $t_d \in J_{\underline{c}}$ acts by a nonzero operator. The commutative subalgebra $K(\mathcal{A}_d)^k = \mathcal{R}^k(H_d) \subset t_d J_{\underline{c}}^k t_d$ acts on the finite dimensional vector space $t_d M$, with unit element t_d acting by identity; let $u \in t_d M$ be an eigen-vector. The corresponding character $\chi_u : \mathcal{R}(H_d) \rightarrow k$ comes from an element $s_u \in H_d(k)$; by Corollary 2 the center $Z(\mathcal{H})$ acts on the vector $u \in \phi_{\underline{c}}^*(M)$ by the character $\chi_{s_u \cdot s_{N_d}^v}$, which shows (18). \square

5.1.

We now restrict attention to the unique Duflo involution in $\underline{c} \cap W^f$; we call it d^f .

Theorem 3. *a) The set of simple objects of \mathcal{A}_{d^f} is $\{L_w \mid w \in W^f \cap \underline{c}\}$.*

b) H_{d^f} contains a maximal reductive subgroup of the centralizer $Z_{L_G}(N_d)$.

Remark 7. A statement stronger than that of Theorem 3(b) is proved in [B], Corollary 2: we show there that in fact $H_{d^f} = Z_{L_G}(N_{\underline{c}})$. The argument of [B] does not use the “nonelementary” result about asymptotic Hecke algebras cited in Fact 3 below (the proof of this result in [L3] relies on the theory of character sheaves).

Proof of the Theorem. The set of irreducible objects of \mathcal{A}_d consists of those L_w , which are subquotients of $Z(V) * L_d$ for some $V \in \text{Rep}({}^L G)$. Identifying $J_{\underline{c}} = K(\mathcal{A}_{\underline{c}})$ we get $[Z(V) * L_d] = B(V) * t_d|_{v=1}$; since $B(V) * t_d = t_d * B(V) \in t_{d^f} \cdot J_{\underline{c}} \cdot t_{d^f} = J_{\underline{c}}^f$ we see that indeed any subquotient of $Z(V) * L_d$ has the form L_w , $w \in \underline{c} \cap W^f$.

To check that $L_w \in \mathcal{A}_{d^f}$ for all $w \in W^f \cap \underline{c}$, it suffices to check that for any proper subset $S \subsetneq \underline{c} \cap W^f$ there exists V such that $B(V) * t_d$ does not lie in the span of t_w , $w \in S$. This follows from the next Lemma.

Let $\phi_{\underline{c}}^f : Z(\mathcal{H}) \rightarrow J_{\underline{c}}^f \otimes A$ be the homomorphism $z \mapsto z * t_{df}$.

Lemma 8. *For any proper subalgebra $J' \subset (J_{\underline{c}}^f)^k$ we have $\text{im}(\phi_{\underline{c}}^f) \not\subset J' \otimes k(v)$.*

We will deduce the Lemma from another result of Lusztig (see Remark 7 above).

Fact 3. *Let $J_{\underline{c}}^f \subset J_{\underline{c}}$ be the span of $\{t_w \mid w \in W^f \cap \underline{c}\}$.*

Then $J_{\underline{c}}^f = t_{df} \cdot J_{\underline{c}}^f \cdot t_{df}$ is a subalgebra.

The map $z \mapsto z \bullet t_{df}$ induces an isomorphism between $Z(J_{\underline{c}}) \otimes k$ and $J_{\underline{c}}^f \otimes k$ (recall that k is a characteristic zero field).

Proof of Lemma 8. Comparing Fact 3 and Fact 2(a) we can identify $(J_{\underline{c}}^f) \otimes k = \mathcal{R}^k(Z_{LG}(N_{\underline{c}})) = \mathcal{R}^k(Z_{LG}(\gamma_{N_{\underline{c}}}))$.

If the image of $\phi_{\underline{c}}^f$ is contained in $J' \otimes k[v, v^{-1}]$ for $J' \subsetneq (J_{\underline{c}}^f) \otimes k$, then for some non-conjugate semi-simple elements $s_1, s_2 \in Z_{LG(k)}(N_{\underline{c}})$ we have $\chi_{s_1} \circ \phi_{\underline{c}}^f = \chi_{s_2} \circ \phi_{\underline{c}}^f$. This means that $s_1 \cdot s_{N_{\underline{c}}}^v$ is conjugate to $s_2 \cdot s_{N_{\underline{c}}}^v$ in ${}^L G(\overline{k(v)})$; this is well-known to be impossible. \square

Proof of Theorem 3(b). Comparing Facts 2 and 3 we see that there exists an isomorphism $I_{lus} : (J_{\underline{c}}^f) \otimes k \xrightarrow{\sim} \mathcal{R}^k(Z_{LG}(N_{\underline{c}}))$, such that for $s \in Z_{LG(k)}(N_{\underline{c}})$ we have $\chi_s^v \circ I_{lus} \circ \phi_{df} = \chi_{s_{N_{df}}^v}$.

By part (a) of the Theorem we have also an isomorphism $I_{gaitsg} : J_{\underline{c}}^f \xrightarrow{\sim} \mathcal{R}(H_d)$, and, by Corollary 2 for $s \in H_d(k)$ we have $\chi_s^v \circ I_{gaitsg} \circ \phi_{df} = \chi_{s_{N_{df}}^v}$.

The proof of (a) shows that for two different characters $\chi_1, \chi_2 : J_{\underline{c}}^f \rightarrow k$ we have $\chi_1 \circ \phi_{df} \neq \chi_2 \circ \phi_{df}$; hence for $s \in H_d^{ss}$ we have $\chi_s \circ I_{lus} = \chi_s \cdot I_{gaitsg}$, i.e. the isomorphism $I_{gaitsg} \otimes k \circ I_{lus}^{-1} : \mathcal{R}^k(Z(N_{\underline{c}})) \rightarrow \mathcal{R}^k(H_d)$ coincides with the natural restriction map. Thus statement (b) of the Theorem follows from the next Lemma.

Lemma 9. *If a homomorphism $i : H_1 \rightarrow H_2$ of algebraic groups over an algebraically closed field k of characteristic zero induces an isomorphism $\mathcal{R}^k(H_2) \xrightarrow{\sim} \mathcal{R}^k(H_1)$ then it induces an isomorphism of maximal reductive quotients $H_1^{red} \xrightarrow{\sim} H_2^{red}$.*

Proof. Since $\mathcal{R}^k(H) = \mathcal{R}^k(H^{red})$ for an algebraic group H , we can replace H_1, H_2 by H_1^{red}, H_2^{red} , and assume that H_1, H_2 are reductive. It is clear that i is injective (otherwise i sends a nontrivial semisimple conjugacy class in H_1 to identity of H_2 , which contradicts the fact that characters separate semisimple conjugacy classes).

Let us first check that i induces an isomorphism of connected components of identity $H_1^0 \rightarrow H_2^0$.

It is easy to see that for a (not necessarily connected) reductive group H the connected component of identity in $\text{Spec}(\mathcal{R}^k)$ is described by $\text{Spec}(\mathcal{R}^k)^0 = C/N_H(C)$, where $C \subset H$ is a maximal torus. Hence H_1 and H_2 have a common Cartan T , and the image of the map $N_{H_i}(T)/T \rightarrow \text{Aut}(T)$ does not depend on $i = 1, 2$. It is enough to check that the image of $N_{H_i}(T)/T$ in $\text{Aut}(T)$ does not depend on i either.

So take $x \in N_{H_1}(T)$. It is easy to see that $\text{Ad } x|_T \in \text{im}(N_{H_i^0}(T)/T)$ iff $\text{Ad } x|_{H_i^0}$ is an inner automorphism of H_i^0 . However the latter condition is equivalent to $\dim((H_i/\text{Ad } H_i)^x) = \dim((H_i/\text{Ad } H_i)^0)$, where $(H_i/\text{Ad } H_i)^x$ is the connected component of the class of x . (This is so because $\dim(\mathfrak{g}^F)$, where F is a generic automorphism of a reductive Lie algebra \mathfrak{g} in a fixed coset modulo inner automorphisms, is maximal when the coset is trivial). Since $\dim((H_1/\text{Ad } H_1)^x) = \dim((H_2/\text{Ad } H_2)^x)$ we get the statement.

To finish the proof it remains to see that i induces a bijection on the set of connected components, $i : H_1/H_1^0 \xrightarrow{\sim} H_2/H_2^0$. For this it is enough to prove the statement of the Lemma for finite groups H_1, H_2 . Then it is known as Jordan's Lemma, see e.g. [Se], Lemma 4.6.1. \square

Theorem 3 is proved. \square

Corollary 3. *The semisimple monoidal category $A_{\underline{c}}^f$, whose set of irreducible objects is $\{L_w \mid w \in \underline{c} \cap W^f\}$, and the monoidal structure is provided by truncated convolution (see [L4]), is equivalent to the category of representations of $Z_{L_G}^{\text{red}}(N_{\underline{c}})$, the maximal reductive quotient of $Z_{L_G}(N_{\underline{c}})$.*

Proof. Theorem 3(a) implies that $A_{\underline{c}}^f$ is the category of semisimple objects in \mathcal{A}_{d^f} . The latter is identified with $\text{Rep}(H_{d^f})$ by Theorem 1, thus $A_{\underline{c}}^f \cong \text{Rep}(H_{d^f}^{\text{red}})$. In view of Theorem 3(b) we have $H_{d^f}^{\text{red}} = Z_{L_G}^{\text{red}}(N_{d^f})$, and by Theorem 2 N_{d^f} is conjugate to $N_{\underline{c}}$. \square

The statement of the Corollary was conjectured in [L4], §3.2.

References

- [BB] Beilinson, A., Bernstein, J., *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, 1–50, Adv. Soviet Math., 16, Part 1, Amer. Math. Soc., Providence, RI, 1993.
- [BBD] Beilinson, A., Bernstein, J., Deligne, P., *Faisceaux pervers*, Astérisque **100**, (1982), 5–171.
- [BD] Beilinson, Drinfeld, *Quantization of Hitchin's Integrable System and Hecke Eigensheaves*, preprint, available at: <http://www.math.uchicago.edu/~benzvi/Math.html>.

- [BGS] Beilinson, A., Ginzburg, V., Soergel, W., *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.
- [B] Bezrukavnikov, R., *Perverse sheaves on affine flags and nilpotent cone of the Langlands dual group*, preprint, math.RT/0201256.
- [BO] Bezrukavnikov, R., Ostrik, V., *Tensor categories attached to cells in affine Weyl groups, II*, preprint, math.RT/0102220, submitted to Proceedings of Tokyo conference.
- [De1] Deligne, P., *La conjecture de Weil, II*, Publ. IHES **52** (1980), 137–252.
- [De2] Deligne, P., *Catégorie tannakiennes*, Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., 87, 1990.
- [DM] Deligne, P., Milne, A., *Tannakian Categories*, in “Hodge cycles, Motives and Shimura varieties,” Lecture Notes in Math., 900, Springer Verlag, 1982, pp 101–228.
- [Ga] Gaitsgory, D., *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), no. 2, 253–280.
- [Gi] Ginzburg, V., *Perverse sheaves on a Loop group and Langlands’ duality*, preprint, alg-geom/9511007.
- [Ka] Kassel, C., *Quantum groups*, Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.
- [L0] Lusztig, G., *Singularities, character formulas and a q -analog of weight multiplicities*, Astérisque **101-102** (1983), 208-229.
- [L1] Lusztig, G., *Cells in affine Weyl groups*, in “Algebraic Groups and Related Topics,” Adv. Studies in Pure Math., vol. 6, 1985, pp 255–287.
- [L2] Lusztig, G., *Cells in affine Weyl groups, II*, J. Algebra **109** (1987), no. 2, 536–548.
- [L3] Lusztig, G., *Cells in affine Weyl groups, IV*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36** (1989), no. 2, 297–328.
- [L4] Lusztig, G., *Cells in affine Weyl groups and tensor categories*, Adv. Math. **129** (1997), no. 1, 85–98.
- [LX] Lusztig, G., Xi, N.H., *Canonical left cells in affine Weyl groups*, Adv. in Math. **72** (1988), no. 2, 284–288.
- [MV] Mirkovic, I., Vilonen, K., *Perverse Sheaves on affine Grassmannians and Langlands Duality*, Math. Res. Lett. **7** (2000), no. 1, 13–24.
- [Se] Serre, J.-P., *Topics in Galois Theory*, Research Notes in Mathematics, 1. Jones and Bartlett Publishers, Boston, MA, 1992.

*Department of Mathematics
University of Chicago
Chicago IL 60637, USA*

*Present address:
Department of Mathematics
Northwestern University
Evanston IL 60208, USA*