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Diffusive Behaviour of the Equilibrium Fluctuations in the Asymmetric Exclusion Processes

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Abstract.

We consider the asymmetric simple exclusion process in dimension $d \geq 3$. We review some results concerning the equilibrium bulk fluctuations and the asymptotic behaviour of a second class particle.

§1. Introduction

The asymmetric simple exclusion process is the simplest model of a driven lattice gas. This model is given by the dynamics of infinitely many particles moving on $\mathbb{Z}^d$ as asymmetric random walks with an exclusion rule: when a particle attempts to jump on a site occupied by another particle the jump is suppressed. Of course we consider initial configurations where there is at most one particle per site. We denote the configurations by $\eta \in \{0,1\}^{\mathbb{Z}^d}$ so that $\eta(x) = 1$ if site x is occupied for the configuration η and $\eta(x) = 0$ if site x is empty. The number of particles is conserved and the Bernoulli product measures $\{\nu_\alpha,\ \alpha \in [0,1]\}$ are the ergodic invariant measures.

Rezakhanlou, in [11], proved that the empirical field of particles, after a hyperbolic rescaling of space and time by a parameter ϵ

$$\pi_t^\epsilon = \epsilon^d \sum_x \eta_{t\epsilon^{-1}}(x)\delta_{\epsilon x} \tag{1.1}$$

converges weakly, as $\epsilon \to 0$, to the (entropic) solution of the Burgers equation

$$\begin{cases} \partial_t \rho \ + \ \gamma \cdot \nabla[(1-\rho)\rho] \ = \ 0\ , \\ \rho(0,\cdot) = \rho_0(\cdot)\ . \end{cases}$$

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Here ρ_0 is the initial asymptotic profile, i.e.

$$\pi_0^\epsilon \underset{\epsilon\to 0}{\rightharpoonup} \rho_0 \qquad \text{in probability,} \tag{1.2}$$

while γ is the average velocity of a single particle. In fact in [11] the initial distribution of the particles is assumed to be an inhomogeneous product measure with a slowly varing profile $\nu_{\rho_0(\epsilon\cdot)}$, such that $\nu_{\rho_0(\epsilon\cdot)}(\eta(x)) = \rho_0(\epsilon x)$.

Suppose now that we start the system with a stationary measure ν_α, for a certain density $\alpha \in]0,1[$. The invariant measure ν_α has macroscopic Gaussian uncorrelated fluctuations, i.e. the fluctuation field

$$\xi_0^\epsilon = \epsilon^{d/2} \sum_{x\in\mathbb{Z}^d} \delta_{\epsilon x}(\eta_0(x) - \alpha) \tag{1.3}$$

converges in law to a white noise field ξ with covariance

$$\mathbb{E}(\xi(q)\xi(q')) = \alpha(1-\alpha)\delta(q-q') \tag{1.4}$$

It is not hard to prove in this context that the macroscopic evolution of these fluctuations, at time $t\epsilon^{-1}$, will converge to the solution of the linearized equation

$$\partial_t \xi + (1-2\alpha)\gamma \cdot \nabla \xi = 0 \tag{1.5}$$

Of course this equation (and the following ones) is to be intended in the weak sense, since ξ is only a distribution-valued field on $\mathbb{R}^d$. This means that an initial fluctuation will evolve macroscopically by a deterministic translation with velocity $(1-2\alpha)\gamma$. In simple exclusion processes there is a simple way to keep track of density fluctuations. Let us condition the stationary measure ν_α to have the site 0 empty and put in this site a *second class particle*, i.e. a particle that has the same jump rates as the other particles but when a normal (*first class*) particle attempts to jump in the site occupied by a second class particle, the particles exchange sites. Then this second class particles evolves like a density fluctuation (cf. section 6) and equation (1.5) corresponds to a law of large numbers for the position X_t of the second class particle

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} (1-2\alpha)\gamma\ . \tag{1.6}$$

The natural question is about the fluctuation around this law of large numbers. The answer is that, in dimension $d \geq 3$, the recentered position of the second class particle $X_t - (1-2\alpha)\gamma t$ behaves diffusively and

$\epsilon(X_{t\epsilon^{-2}} - (1-2\alpha)\gamma t\epsilon^{-2})$ converges in law to a Brownian motion with a diffusion matrix $D(\alpha)$ (cf. Theorem 6.2).

In dimension $d = 1$ and 2 the asymptotic behavior of the second class particle is superdiffusive (cf. [9, 14]), and one of the most interesting and challenging problem is to determine the corresponding limit process in these cases.

We review in this article results concerning the dimension $d \geq 3$. The diffusive behaviour of the density fluctuations is then stated in the following way. Consider the recentered density fluctuation field at diffusive scaling

$$Y_t^\epsilon = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \delta_{\epsilon x - vt\epsilon^{-1}}(\eta_{t\epsilon^{-2}}(x) - \alpha) \tag{1.7}$$

where $v = (1-2\alpha)\gamma$. We show in section 4 that, as a process with values on the distributions on $\mathbb{R}^d$, it converges (in law) to the solution of the linear stochastic partial differential equation

$$\partial_t Y = \nabla \cdot D(\alpha)\nabla Y + \sqrt{2\alpha(1-\alpha)}\sqrt{D(\alpha)}\nabla \cdot W \tag{1.8}$$

where $W(x,t) = (W_1, \ldots, W_d)$ are independent standard white noises on $\mathbb{R}^{d+1}$.

This result was proved in [1] for the asymmetric exclusion process in $d \geq 3$ and in finite macroscopic volume. The purpose of this article is to give a simpler proof in infinite volume based on the fluctuation-dissipation theorem as stated in [8]. The fluctuation-dissipation theorem, which was first proved in this context in [7], is in fact the the core of the proof for the macroscopic density fluctuation, as we explain in section 4. This theorem states that the *space-time* fluctuations of the current associated to the density are equivalent to to the fluctuations of a gradient of the density times the matrix $D(\alpha)$, in the sense that the variance of the difference is asymptotically small (cf. Theorem 3.2 in section 3).

This fluctuation-dissipation theorem was also applied in order to study the diffusive incompressible limit (cf. [3]) and the first order corrections to the hydrodynamic limit (cf. [5]). It has also been used to study corresponding results for a lattice system of particles with exclusion rule with conservation of number of particles, velocity and energy (cf. [2] and references therein).

The fluctuation-dissipation theorem for the exclusion processes can be proved by using the duality properties of these models (as explained in [8]), that permits to control the size of these fluctuation by estimates on the Green functions of simple random walks (see also [12] where this method is applied in the study of the diffusive behaviour of a tagged particle). At the moment there are not any result of this type for systems that do not have such nice duality.

§2. Density fluctuations

Fix a probability distribution $p(\cdot)$ supported on a finite subset of $\mathbb{Z}^d_* = \mathbb{Z}^d \backslash \{0\}$ and denote by L the generator of the simple exclusion process on $\mathbb{Z}^d$ associated to $p(\cdot)$. L acts on local functions f on $\mathcal{X}_d = \{0,1\}^{\mathbb{Z}^d}$ as

$$(2.1) \quad (Lf)(\eta) = \sum_{x,y\in\mathbb{Z}^d} p(y-x)\eta(x)\{1-\eta(y)\}[f(\sigma^{x,y}\eta) - f(\eta)] ,$$

where $\sigma^{x,y}\eta$ stands for the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x,\, y \ , \\ \eta(x) & \text{if } z = y \ , \\ \eta(y) & \text{if } z = x \ . \end{cases}$$

Denote by $s(\cdot)$ and $a(\cdot)$ the symmetric and the anti-symmetric parts of the probability $p(\cdot)$:

$$s(x) = (1/2)[p(x) + p(-x)] , \quad a(x) = (1/2)[p(x) - p(-x)]$$

and denote by L^s, L^a the symmetric part and the anti-symmetric part of the generator L. L^s, L^a are obtained by replacing p by s, a in the definition of L.

For α in $[0,1]$, denote by ν_α the Bernoulli product measure on $\mathcal{X}$ with $\nu_\alpha[\eta(x) = 1] = \alpha$. Measures in this one-parameter family are stationary and ergodic for the simple exclusion dynamics and in the symmetric case, i.e. $p(x) = p(-x)$, these measures are reversible. Expectation with respect to ν_α is represented by $< \cdot >_\alpha$ and the scalar product in $L^2(\nu_\alpha)$ by $< \cdot, \cdot >_\alpha$.

Fix $\varepsilon > 0$. For a configuration η, denote by $\pi^\varepsilon = \pi^\varepsilon(\eta)$ the empirical measure associated to η. This is the measure on $\mathbb{R}^d$ obtained assigning mass ε^d to each particle of η :

$$\pi^\varepsilon = \varepsilon^d \sum_{x\in\mathbb{Z}^d} \eta(x)\delta_{x\varepsilon} ,$$

where δ_u stands for the Dirac measure concentrated on u. For a measure π on $\mathbb{R}^d$ and a continuous function G: $\mathbb{R}^d \to \mathbb{R}$, denote by $< \pi, G >$ the integral of G with respect to π. In particular, $< \pi^\varepsilon, G >$ is equal to $\varepsilon^d \sum_{x\in\mathbb{Z}^d} G(x\varepsilon)\eta(x)$.

Consider a sequence of probability measures μ^ε on the configuration space $\mathcal{X}_d$ and assume that, under μ^ε, π^ε converges in probability to an absolutely continuous measure $\rho_0(u)du$. This means that for every continuous function $G\colon \mathbb{R}^d \to \mathbb{R}$ and every $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mu^\varepsilon \Big\{ \Big| < \pi^\varepsilon, G > - \int G(u)\rho_0(u)\, du \Big| \; > \; \delta \Big\} \; = \; 0 \, . \tag{2.2}$$

Consider the hyperbolic equation

$$\begin{cases} \partial_t \rho \; + \; \gamma \cdot \nabla \left[\rho(1-\rho)\right] \; = \; 0 \, , \\ \rho(0,\cdot) = \rho_0(\cdot) \, . \end{cases} \tag{2.3}$$

In this formula γ stands for the drift : $\gamma = \sum_y yp(y)$. For $t > 0$, denote by π_t^ε the empirical measure associated to the state of the process at time t : $\pi_t^\varepsilon = \pi^\varepsilon(\eta_t)$. Rezakhanlou [11] proved that, starting from a inhomogeneous product measure $\nu_{\rho(\varepsilon\cdot)}$ satisfying (2.2), then, for every $t \geq 0$, $\pi_{t\varepsilon^{-1}}^\varepsilon$ converges in probability to the measure $\rho(t,u)du$, where the density ρ is the entropy solution of the hyperbolic equation (2.3). More precisely, for a measure μ on $\mathcal{X}_d$, denote by $\mathbb{P}_\mu$ the measure on the path space $D(\mathbb{R}_+, \mathcal{X}_d)$ induced by the Markov process η_t and the measure μ. Then, for every $t \geq 0$, every continuous function G and every $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_{\nu_{\rho(\varepsilon\cdot)}} \Big\{ \Big| < \pi_{t\varepsilon^{-1}}^\varepsilon, G > - \int G(u)\rho(t,u)\, du \Big| \; > \; \delta \Big\} \; = \; 0 \, ,$$

where the density ρ is the entropy solution of the hyperbolic equation (2.3). Notice the Euler rescaling of time in the previous formula.

Let us now consider, for a fixed density $\alpha \in (0,1)$, the system in equilibrium with distribution $\mathbb{P}_{\nu_\alpha}$ on the path space. Let $Y_\cdot^\varepsilon$ be the density fluctuation field that acts on smooth functions H as

$$Y_t^\varepsilon(H) \; = \; \varepsilon^{d/2} \sum_{x\in\mathbb{Z}^d} H(x\varepsilon - vt\varepsilon^{-1})(\eta_{t\varepsilon^{-2}}(x) - \alpha) \, , \tag{2.4}$$

where $v = \gamma(1-2\alpha)$. Notice the diffusive rescaling of time on the right hand side of this identity.

For any $k \geq 0$ and $f, g \in \mathcal{C}^\infty(\mathbb{R}^d)$ consider the scalar product

$$(g,f)_k = \int_{\mathbb{R}^d} g(q) \left(|q|^2 - \Delta\right)^k f(q)\, dq \tag{2.5}$$

and denote by $\mathcal{H}_k$ the corresponding closure. For any positive k we denote by $\mathcal{H}_{-k}$ its dual space with respect to the $L^2(\mathbb{R}^d) \equiv \mathcal{H}_0$ scalar product.

We will show in section 5 that the probability distribution Q^ϵ of $Y^\varepsilon_\cdot$ under $\mathbb{P}_{\nu\alpha}$, is supported and is tight in $D([0,T],\mathcal{H}_{-k})$ for any $k > d+1$. Observe that in [1] the tightness is proved for any $k > d/2+1$. This difference is due to the fact that here we are dealing with distribution on the infinite volume $\mathbb{R}^d$, while in [1] we were working in the finite d-dimensional torus.

We state now the main theorem. Assume that $d \geq 3$. Let $D_{i,j}(\alpha)$ a strictly positive symmetric matrix. Denote by $\mathcal{A}$, $\mathcal{B}$ the differential operators defined by $\mathcal{A} = \sum_{1\leq i,j\leq d} D_{i,j}(\alpha)\partial^2_{u_i,u_j}$, $\mathcal{B} = \sqrt{2\alpha(1-\alpha)}\,\sigma\,\nabla$ (where the matrix σ is the positive square root of D). Denote by $\{T_t,\, t \geq 0\}$ the semigroup in $L^2(\mathbb{R}^d)$ associated to the operator $\mathcal{A}$. Fix a positive integer $k_0 > d+1$. Let Q be the probability measure concentrated on $C([0,T],\mathcal{H}_{-k_0})$ corresponding to the stationary generalized Ornstein–Uhlenbeck process with mean 0 and covariance

$$E_Q\Big[Y_t(H)Y_s(G)\Big] \;=\; \chi(\alpha)\int_{\mathbb{R}^d} du\,(T_{|t-s|}H)(u)\,G(u)$$

for every $0 \leq s \leq t$ and H, G in $\mathcal{H}_{k_0}$. Here $\chi(\alpha)$ stands for the static compressibility given by $\chi(\alpha) = \mathbf{Var}_{\nu_\alpha}[\eta(0)] = \alpha(1-\alpha)$.

Theorem 2.1. *There exists a strictly positive symmetric matrix $D_{i,j}(\alpha)$ such that the sequence Q^ε converges weakly to the probability measure Q of the corresponding Ornstein–Uhlenbeck process.*

Formally, Y_t is the solution of the stochastic differential equation

$$dY_t \;=\; \mathcal{A}Y_t dt \;+\; dB^\nabla_t\;,$$

where B^∇_t is a mean zero Gaussian field with covariances given by

$$E_Q\Big[B^\nabla_t(H)B^\nabla_s(G)\Big] \;=\; 2\chi(\alpha)(s\wedge t)\int_{\mathbb{R}^d}(\nabla H)\cdot D\,(\nabla G)\,.$$

In section 6 we show that the viscosity matrix $D(\alpha)$ is identified as

(2.6)
$$D_{i,j}(\alpha) = \frac{1}{\alpha(1-\alpha)}\lim_{t\to\infty}\frac{1}{t}\sum_{x\in\mathbb{Z}^d} x_i x_j \mathbb{E}_{\nu_\alpha}\left((\eta_t(x-vt)-\alpha)(\eta_0(0)-\alpha)\right),$$

which is also the asymptotic variance of the second class particle. Other expressions for this matrix can be given (cf. [6]), while in [8] we prove that $D_{i,j}(\alpha)$ are smooth functions of the density $\alpha \in [0,1]$.

§3. The Fluctuation-Dissipation Theorem

We recall in this section some results proved in [8] (cf. also [7]). For local functions u, v, define the scalar product $\ll \cdot , \cdot \gg$ by

$$\ll u, v \gg = \sum_{x \in \mathbb{Z}^d} \{ < \tau_x u, v > - < u >< v > \} , \tag{3.1}$$

where $\{\tau_x, x \in \mathbb{Z}^d\}$ is the group of translations and $< \cdot >$ stands for the expectation with respect to the measure ν_α. That this is in fact an inner product can be seen by the relation

$$\ll u, v \gg = \lim_{V \uparrow \mathbb{Z}^d} \frac{1}{|V|} < \sum_{x \in V} \tau_x(u - < u >), \sum_{x \in V} \tau_x(v - < v >) > .$$

Since $\ll u - \tau_x u , v \gg = 0$ for all x in $\mathbb{Z}^d$, this scalar product is only positive semidefinite. Denote by $L^2_{\ll \cdot,\cdot \gg}(\nu_\alpha)$ the Hilbert space generated by the local functions and the inner product $\ll \cdot, \cdot \gg$.

Denote by L^s the symmetric part of the generator. For two local functions u, v, let

$$\ll u, v \gg_1 = \ll u, (-L^s) v \gg$$

and let H_1 be the Hilbert space generated by local functions and the inner product $\ll \cdot, \cdot \gg_1$. To introduce the dual Hilbert spaces of H_1, for a local function u, consider the semi-norm $\| \cdot \|_{-1}$ given by

$$\|u\|_{-1} = \sup_v \Big\{ 2 \ll u, v \gg - \ll v, v \gg_1 \Big\} ,$$

where the supremum is carried over all local functions v. Denote by H_{-1} the Hilbert space generated by the local functions and the semi-norm $\| \cdot \|_{-1}$.

Notice that the function $\eta(0) - \alpha$ does not belong to H_{-1}. Indeed, due to the translations, $\|\eta(0) - \alpha\|_1 = 0$ so that $\|\eta(0) - \alpha\|_{-1} = \infty$. One can show, however, that these linear functions are essentially the only zero-mean local functions that do not belong to H_{-1}.

To make the previous statement precise, we need to introduce some notation. For a local function f, denote by $\tilde f : [0, 1] \to \mathbb{R}$ the polynomial defined by

$$\tilde f(\alpha) = E_{\nu_\alpha}[f] .$$

Denote by $\mathcal{C}_0 = \mathcal{C}_0(\alpha)$ the collection of local functions such that

$$\tilde f(\alpha) = E_{\nu_\alpha}[f] = 0 , \quad \tilde f'(\alpha) = \frac{d}{d\beta} E_{\nu_\beta}[f] \Big|_{\beta = \alpha} = 0 .$$

It is proved in [8] that all functions in $\mathcal{C}_0$ have finite H_{-1} norm:

$$\|f\|_{-1} < \infty \quad \text{if } f \in \mathcal{C}_0 \ . \tag{3.2}$$

Denote by $\mathcal{C}_1$ the space of local functions f in $\mathcal{C}_0$ which are orthogonal to all linear functions:

$$< f, \eta(x) - \alpha > = 0 \quad \text{for all } x \text{ in } \mathbb{Z}^d.$$

The next result states that a local function f in $\mathcal{C}_1$ has a finite space-time variance in the diffusive scaling.

Theorem 3.1. *Fix $T > 0$, a vector v_0 in $\mathbb{R}^d$, a smooth function $G : \mathbb{R}^d \to \mathbb{R}$ with compact support and a local function f in $\mathcal{C}_1$. There exists a finite constant C_0 such that*

$$\limsup_{\varepsilon \to 0} \mathbb{E}_{\nu_\alpha}\Big[\sup_{0 \le t \le T} \Big(\varepsilon^{d/2+1} \int_0^{t\varepsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\varepsilon[x - rv_0]) f(\tau_x \eta_r)\, dr \Big)^2 \Big]$$
$$\le C_0 T \, \|G\|_{L^2}^2 \, \|f\|_{-1}^2 \, .$$

The theorem is not correct if we replace $\mathcal{C}_1$ by $\mathcal{C}_0$ because $\eta(e_1) - \eta(0)$ has H_{-1} norm equal to 0 and a finite, strictly positive space-time variance.

It is proved in [8] that any local function in $\mathcal{C}_0$ may be approximated in H_{-1} by a local function in the range of the generator. More precisely, for every local function f in $\mathcal{C}_0$ and every $\varepsilon > 0$, there exists a local function u_ε, which may be taken in $\mathcal{C}_1$, such that

$$\|Lu_\varepsilon - f\|_{-1}^2 \le \varepsilon \, . \tag{3.3}$$

The fluctuation-dissipation theorem stated below follows from this result, the estimate stated in Theorem 3.1 and some elementary computations.

Theorem 3.2. *Fix $T > 0$, a vector v_0 in $\mathbb{R}^d$, a local function w in $\mathcal{C}_1$ and a smooth function $G : \mathbb{R}^d \to \mathbb{R}$ with compact support. There exist a sequence of local functions u_m and $D_z(\alpha)$ such that*

$$\limsup_{m \to \infty} \limsup_{\varepsilon \to 0}$$
$$\mathbb{E}_{\nu_\alpha}\Big[\sup_{0 \le t \le T} \Big(\varepsilon^{d/2+1} \int_0^{t\varepsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\varepsilon[x - rv_0]) \tau_x W_m(\eta_s)\, ds \Big)^2 \Big] = 0 \, ,$$

where

$$W_m(\eta) = w - Lu_m + \sum_{z \in \mathbb{Z}^d} a(z) D_z(\alpha) \{\eta(z) - \eta(0)\} \, .$$

The idea of the proof is quite simple. By (3.3), we may approximate w by some local function Lu_m in the range of the generator. However, Lu_m might have linear terms and therefore might not be in $\mathcal{C}_1$. Subtracting expressions of type $\eta(z)-\eta(0)$, we convert Lu_m in a $\mathcal{C}_1$-function and apply Theorem 3.1.

§4. Time evolution of density fluctuations

We show in this section how to deduce from the fluctuation-dissipation theorem the equilibrium fluctuations for the density field defined in (2.4).

Fix a smooth function $G : \mathbb{R}^d \to \mathbb{R}$ with compact support and define the martingale $M_t^{1,\varepsilon}$ by the time evolution equation

$$Y_t^\varepsilon(G) - Y_0^\varepsilon(G) = \int_0^t \left(\partial_s + \varepsilon^{-2}L\right) Y_s^\varepsilon(G)\, ds + M_t^{1,\varepsilon} .$$

An elementary computation shows that

(4.1)
$$\begin{aligned}(\partial_t + \varepsilon^{-2}L)Y_t^\varepsilon(G) = &-\varepsilon^{d/2-1}\sum_x (v\cdot\nabla G)[\varepsilon(x-vt\varepsilon^{-2})](\eta(x)-\alpha)\\ &- \varepsilon^{d/2-2}\sum_x G[\varepsilon(x-vt\varepsilon^{-2})]\sum_y j_{x,y} ,\end{aligned}$$

where $j_{x,y}$, the instantaneous current between sites x and y, is given by

$$\begin{aligned} j_{x,y} &= -s(x-y)[\eta(y)-\eta(x)]\\ &\quad - a(x-y)\Big\{\eta(y)(1-\eta(x)) + \eta(x)(1-\eta(y))\Big\} .\end{aligned}$$

Let $G_t^\varepsilon(x) = G[\varepsilon(x-vt\varepsilon^{-2})]$. Since the current is anti-symmetric ($j_{x,y} = -j_{y,x}$) and since its expectation with respect to ν_α is equal to $2a(y-x)\alpha(1-\alpha)$, an elementary computation gives that

$$\begin{aligned} -\sum_x G_t^\varepsilon(x)\sum_y j_{x,y} = &\sum_{x,y} s(x-y)[G_t^\varepsilon(y) - G_t^\varepsilon(x)][\eta(x)-\alpha]\\ &-\sum_{x,y} a(x-y)[G_t^\varepsilon(x) - G_t^\varepsilon(y)][\eta(y)-\alpha][\eta(x)-\alpha]\\ &+(1-2\alpha)\sum_{x,y} a(x-y)[G_t^\varepsilon(x) - G_t^\varepsilon(y)][\eta(x)-\alpha]\end{aligned}$$

because

$$\begin{aligned}&\eta(y)(1-\eta(x)) \;+\; \eta(x)(1-\eta(y)) \;-\; 2\alpha(1-\alpha)\\ &\quad= -2(\eta(y)-\alpha)(\eta(x)-\alpha) \;+\; (1-2\alpha)\Big\{(\eta(y)-\alpha) + (\eta(x)-\alpha)\Big\} .\end{aligned}$$

Finally, since $a(\cdot)$ is anti-symmetric, a Taylor expansion and a change of variables in the summation permit to conclude that

$$
\begin{aligned}
(4.2) \quad & (\partial_t + \varepsilon^{-2} L) Y_t^\varepsilon(G) \\
&= \varepsilon^{d/2} \sum_x \sum_{i,j} \sigma_{i,j} (\partial_i \partial_j G)[\varepsilon(x - vt\varepsilon^{-2})](\eta(x) - \alpha) \\
&\quad - \varepsilon^{d/2-1} \sum_{x,z} a(z)(z \cdot \nabla G)[\varepsilon(x - vt\varepsilon^{-2})] \Phi_{x,x+z} \; + \; R_\varepsilon(\eta) \,.
\end{aligned}
$$

In this formula, $\sigma_{i,j}$ is the symmetric matrix defined by

$$\sigma_{i,j}^s \;=\; \sum_z s(z) z_i z_j \;,$$

$\Phi_{x,y}$ is the zero-mean local function defined by

$$\Phi_{x,y} \;=\; (\eta(x) - \alpha)(\eta(y) - \alpha)$$

and $R_\varepsilon(\eta)$ is a remainder which vanishes in $L^2(\nu_\alpha)$ as $\varepsilon \downarrow 0$. In fact $< R_\varepsilon(\eta)^2 >= O(\varepsilon^2)$.

Since $\Phi_{x,y}$ belongs to $\mathcal{C}_1$, by Theorem 3.2, there exist a sequence of local functions $\{v_m,\, m \geq 1\}$ in $\mathcal{C}_1$ and constants $D_{z,z'}$ such that

$$
\begin{aligned}
&\lim_{m\to\infty} \lim_{\varepsilon\to 0} \mathbb{E}_{\nu_\alpha}\Big[\sup_{0\le t\le T} \Big| \varepsilon^{d/2-1} \int_0^t \sum_{x,z} a(z)(z\cdot\nabla G)[\varepsilon(x - vt\varepsilon^{-2})] \\
&\qquad \Big\{ \Phi_{0,z} - Lv_m - \sum_{z'} a(z') D_{z,z'} [\eta(z') - \eta(0)] \Big\} (\tau_x \eta_{\varepsilon^{-2}s})\, ds \Big|^2 \Big] \;=\; 0\,.
\end{aligned}
$$

This result shows that we may replace $\Phi_{x,x+z}$ in the second term on the right hand side of (2.4) by $Lv_m - \sum_{z'} a(z') D_{z,z'}[\eta(z') - \eta(0)]$. The difference $\eta(z') - \eta(0)$ enables a second summation by parts which cancels a factor ε^{-1}, while the term Lv_m produces an extra martingale.

Let $F(x) = \sum_z a(z)(z \cdot \nabla G)(x)$. For each $m \geq 1$, consider the martingale $M_t^{2,m,\varepsilon}$ defined by

$$
\begin{aligned}
M_t^{2,m,\varepsilon} =& \varepsilon^{d/2+1} \sum_x \int_0^t F(\varepsilon(x - vs\varepsilon^{-2})) \varepsilon^{-2} Lv_m(\tau_x \eta_{\varepsilon^{-2}s})\, ds \\
&+ \varepsilon^{d/2+1} \sum_x \int_0^t \partial_s F(\varepsilon(x - vs\varepsilon^{-2})) v_m(\tau_x \eta_{\varepsilon^{-2}s})\, ds \\
&- \varepsilon^{d/2+1} \sum_x \Big\{ F(\varepsilon(x - vt\varepsilon^{-2})) v_m(\tau_x \eta_{\varepsilon^{-2}t}) - F(\varepsilon x) v_m(\tau_x \eta_0) \Big\}\,.
\end{aligned}
$$

Since v_m are local functions, it is easy to see that the L^2 norm of the third term of the right hand side vanishes as $\varepsilon \to 0$. The second term is equal to

$$\varepsilon^{d/2} \sum_x \int_0^T (v \cdot \nabla F)(\varepsilon(x - vt\varepsilon^{-2})) v_m(\tau_x \eta_{\varepsilon^{-2}t})\, dt\ .$$

Since v_m belongs to $\mathcal{C}_1$ for each $m \geq 1$, it has finite H_{-1} norm in view of (3.2). In particular, by Theorem 3.1, the expectation of the square goes to 0 as $\varepsilon \to 0$.

In conclusion, we have shown so far that

$$\int_0^t \left(\partial_s + \varepsilon^{-2} L\right) Y_s^\varepsilon(G)\, ds = \int_0^t Y_s^\varepsilon(\mathcal{A}G)\, ds + M_t^{2,m,\varepsilon} + R_{m,\varepsilon}(t)\ ,$$

where

$$\lim_{m\to\infty} \lim_{\varepsilon\to 0} E_{\nu_\alpha}\Big(\sup_{0\leq t\leq T} \Big| R_{m,\varepsilon}(t) \Big|^2 \Big) = 0$$

and the second order differential operator $\mathcal{A}$ is given by

$$\mathcal{A} = \sum_{i,j}^{d} D_{i,j} \partial_i \partial_j\ ,$$

with the matrix $D_{i,j}$ given by

$$D_{i,j} = \sigma_{i,j} \ - \sum_{z,z'} a(z) a(z') z_i z_j' D_{z,z'}\ .$$

We turn now to the calculation of the quadratic variation of the martingale $M_t^{1,\varepsilon} + M_t^{2,n,\varepsilon}$. This is given by

(4.3)
$$\int_0^t \sum_y \sum_z p(z) \eta(y) [1 - \eta(y+z)] \varepsilon^{-2} \left[H_s^\varepsilon(\eta_{s\varepsilon^{-2}}^{y,y+z}) - H_s^\varepsilon(\eta_{s\varepsilon^{-2}}) \right]^2 ds\ ,$$

where

$$H_s^\varepsilon(\eta) = Y_s^\varepsilon(G) - \varepsilon^{d/2+1} \sum_x \int_0^T F(\varepsilon(x - vt\varepsilon^{-2})) v_m(\tau_x \eta_{\varepsilon^{-2}t})\ .$$

Elementary computations show that (4.3) is equal to

(4.4)
$$\int_0^t \varepsilon^d \sum_y \sum_z p(z)\eta(y)[1-\eta(y+z)] \times \Bigg(\varepsilon^{-1}[G_s^\varepsilon(y+z) - G_s^\varepsilon(y)] - \sum_x F_s^\varepsilon(x)\left(v_m(\tau_x\sigma^{y,y+z}\eta_{s\varepsilon^{-2}}) - v_m(\tau_x\eta_{s\varepsilon^{-2}})\right)\Bigg)^2 ds ,$$

where $F_s^\varepsilon(x) = F(\varepsilon(x - vt\varepsilon^{-2}))$.

Since v_m is a local function, the sum inside the square in the above expression extends over a finite number of x depending only on the support of v_m. We can therefore substitute $F_s^\varepsilon(x)$ by $F_s^\varepsilon(y)$, with an error that is small in view of Theorem 3.1. In the same way, we replace the discrete derivative of G by the actual derivative, obtaining that (4.3) is equal to

$$\int_0^t \varepsilon^d \sum_y \sum_z p(z)\eta(y)[1-\eta(y+z)] \times \Bigg[(z\cdot\nabla G)(\varepsilon(y - vt\varepsilon^{-2})) - F_s^\varepsilon(y)\sum_x \left(v_m(\tau_x\sigma^{y,y+z}\eta_{s\varepsilon^{-2}}) - v_m(\tau_x\eta_{s\varepsilon^{-2}})\right)\Bigg]^2 ds$$

plus a remainder $R_\varepsilon(t)$ which vanishes in L^2 as $\varepsilon \downarrow 0$. Recall the definition of F and take the limit as $\varepsilon \to 0$. By the law of large numbers, we obtain that the previous expression converges to

$$t\int dy \sum_z p(z)(z\cdot\nabla G)(y)^2 \times \left\langle \eta(0)(1-\eta(z))\Big(1 - a(z)\Big[\Gamma_{v_m}(\eta^{0,z}) - \Gamma_{v_m}(\eta)\Big]\Big)^2\right\rangle ,$$

where $\Gamma_{v_m}(\eta)$ denotes the formal sum $\sum_x v_m(\tau_x\eta)$.

Since we performed this calculations in equilibrium and since for the invariant product measure the static fluctuations converges to the Gaussian field with covariance operator $\alpha(1-\alpha)(-\Delta)^{-1}$, if

$$b_{i,j} = \lim_{m\to\infty} \sum_z p(z) \times \left\langle \eta(0)(1-\eta(z))\Big(1 - a(z)\Big[\Gamma_{v_m}(\eta^{0,z}) - \Gamma_{v_m}(\eta)\Big]\Big)^2\right\rangle z_i z_j ,$$

the fluctuation-dissipation relation for the limit Ornstein-Uhlenbeck process gives

$$b_{i,j} = \alpha(1-\alpha)D_{i,j} \ .$$

§5. Tightness

Recall that we have defined, for any $k \in \mathbb{R}$, $\mathcal{H}_k$ as the closure of $C^\infty(\mathbb{R}^d)$ with respect to the scalar product

$$(g,f)_k = \int_{\mathbb{R}^d} g(q)\left(|q|^2 - \Delta\right)^k f(q)\, dq$$

It is convenient to represent the scalar product $(\cdot,\cdot)_k$ in the orthonormal basis of the Hermite polynomials, which are the eigenfunctions of $|q|^2 - \Delta$. Let $\vec{n}$ be a multi-index of $(\mathbb{Z}^+)^d$ and $|\vec{n}| = \sum_{i=1}^d n(i)$. We denote by $\lambda_{n(i)} = 2n(i)+1$ for $n(i) \in \mathbb{Z}^+$ and $\lambda_{\vec{n}} = \sum_{i=1}^d \lambda_{n(i)}$. Define $h_{\vec{n}}(q) = \prod_{i=1}^d h_{n(i)}(q_i)$ where h_m is the m^{th} normalized Hermite polynomial of order m in $\mathbb{R}$. We have then for every $k \geq 0$ and $f \in L^2$

$$\|f\|_k^2 = \int_{\mathbb{R}^d} f(q)(|q|^2-\Delta)^k f(q)\, dq = \sum_{\vec{n}\in(\mathbb{Z}^+)^d} \lambda_{\vec{n}}^k \left(\int_{\mathbb{R}^d} f(q) h_{\vec{n}}(q)\, dq\right)^2$$

This is valid also for negative k. So the $\mathcal{H}_{-k}$-norm of a distribution ξ on $\mathbb{R}^d$ can be written as

$$\|\xi\|_{-k}^2 = \sum_{\vec{n}\in(\mathbb{Z}^+)^d} \lambda_{\vec{n}}^{-k}\, \xi(h_{\vec{n}})^2 \tag{5.1}$$

Observe that, for $k' > k$, the injection J of $\mathcal{H}_{-k}$ in $\mathcal{H}_{-k'}$ is compact. In fact it can be approximated by the finite range operators $J_m\xi = \sum_{|\vec{n}|\leq m} \xi(h_{\vec{n}})h_{\vec{n}}$, and it is easy to see that the operator norm of the difference is bounded by

$$\|J - J_m\| \leq (2m+d)^{-(k'-k)}$$

By the compactness of the injections $\mathcal{H}_{-k} \hookrightarrow \mathcal{H}_{-k'}$ for $k < k'$, and standard compactness arguments, the tightness of the distribution of Y_t^ϵ is a consequence of the following proposition.

Proposition 5.1. *For any $k > d+1$ and every $T > 0$, we have that*

1.

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}_\mu \left(\sup_{t\in[0,T]} \|Y_t^\varepsilon\|_{-k}^2 \right) < +\infty$$

2. For any $R > 0$,

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}_\mu \left(\sup_{\substack{t,s\in[0,T]\\ |t-s|\le\delta}} \|Y_t^\varepsilon - Y_s^\varepsilon\|_{-k} > R \right) = 0$$

It is easy to see, by using (5.1) and that $k > d+1$, that Proposition 5.1 is a consequence of the following

Proposition 5.2. *For any smooth function G on $\mathbb{R}^d$ with compact support*

$$\sup_{\varepsilon} \mathbb{E}_\mu \left(\sup_{t\in[0,T]} Y_t^\varepsilon(G)^2 \right) \le TC_G \tag{5.2}$$

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}_\mu \left(\sup_{\substack{t,s\in[0,T]\\ |t-s|\le\delta}} |Y_t^\varepsilon(G) - Y_s^\varepsilon(G)| > R \right) = 0 \tag{5.3}$$

Let just sketch the proof of (5.3), the proof of (5.2) will follow a similar argument (cf. [1] for details).

By the same calculation made in the previous section we have

$$\begin{aligned} Y_t^\varepsilon(G) - Y_s^\varepsilon(G) &= \int_s^t \left(\partial_\tau + \varepsilon^{-2}L\right) Y_\tau^\varepsilon(G)\, d\tau + M_{s,t}^{1,\varepsilon} \\ &= \int_s^t Y_\tau^\varepsilon(\mathcal{A}G)\, d\tau + M_{s,t}^{1,\varepsilon} + M_{s,t}^{2,m,\varepsilon} + R_{m,\varepsilon}(s,t) \ , \end{aligned} \tag{5.4}$$

where $M_{s,t}^{1,\varepsilon}$ and $M_{s,t}^{2,m,\varepsilon}$ are the differences of the corresponding martingales defined in the previous section, and

$$\lim_{m\to\infty} \lim_{\varepsilon\to 0} E_{\nu_\alpha} \Big(\sup_{0\le s\le t\le T} \Big| R_{m,\varepsilon}(s,t) \Big|^2 \Big) = 0$$

The first term is easy to deal since

$$\mathbb{E}_\mu \left(\sup_{\substack{t,s\in[0,T]\\ |t-s|\le\delta}} | \int_s^t Y_\tau^\varepsilon(\mathcal{A}G)\, d\tau|^2 \right) \le \delta T \left\langle Y^\varepsilon(\mathcal{A}G)^2 \right\rangle \le \delta TC\|\mathcal{A}G\|_2^2. \tag{5.5}$$

About the difference martingale $\widetilde{M}^m_{s,t} = M^{1,\varepsilon}_{s,t} + M^{2,m,\varepsilon}_{s,t}$, for any finite m has a bounded quadratic variation given by (4.3) or (4.4), and it is not difficult to show exponential bounds for it. So it follows that for any m

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{P}_\mu \left(\sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} |\widetilde{M}^m_{s,t}| > R \right) = 0.$$

§6. The second class particle

The second class particle is a special particle that has the same jump rates of the other particles (with the exclusion rule), furthermore when a normal (first class) particle attempts to jump in the site occupied by the second class particle, the particles exchange site. Therefore the evolution of the first class particles is unaffected by the presence of the second class particles. We will show here that the asymptotic evolution of the second class particle is closely related to the equilibrium fluctuations of the density.

Let ν^0_α the Bernoulli measure on $\mathbb{Z}^d$ with parameter α conditioned to have the site 0 empty. Let X_t be the position at time t of the second class particle, one starting at time 0 from 0, when the other particles are distributed initially with ν^0_α.

Proposition 6.1.

$$E^{\nu^0_\alpha}(1_{[X_t=x]}) = \frac{1}{\alpha(1-\alpha)} \left[E^{\nu_\alpha}(\eta_t(x)\eta_0(0)) - \alpha^2 \right] \tag{6.1}$$

Proof. Let $\sigma_0\eta(x) = \eta(x)$ if $x \neq 0$ and $\sigma_0\eta(0) = 0$.

$$\begin{aligned} E^{\nu_\alpha}(\eta_t(x)\eta_0(0)) &= \int \nu_\alpha(d\eta)\eta(0)E^\eta(\eta_t(x)) \\ &= \int \nu_\alpha(d\eta)\eta(0) \left[E^\eta(\eta_t(x)) - E^{\sigma_0\eta}(\eta_t(x)) \right] \\ &\qquad + \int \nu_\alpha(d\eta)\eta(0)E^{\sigma_0\eta}(\eta_t(x)) \\ &= \int \nu_\alpha(d\eta)\eta(0)E^\eta(1_{[X_t=x]}) + \int \nu_\alpha(d\xi)\frac{\alpha}{1-\alpha}(1-\xi(0))E^\xi(\eta_t(x)) \end{aligned}$$

(where in the last term we have performed the change of variable $\xi = \sigma_0\eta$)

$$= \alpha E^{\nu^0_\alpha}(1_{[X_t=x]}) + \frac{\alpha}{1-\alpha}\alpha - \frac{\alpha}{1-\alpha}E^{\nu_\alpha}(\eta_t(x)\eta_0(0))$$

Reordering the term one obtains (6.1). □

Theorem 6.2. *$\varepsilon(X_{t\varepsilon^{-2}} - vt\varepsilon^{-2})$ converges in law to a zero mean Gaussian r.w. with covariance matrix $tD^s_{i,j}$, where D^s is the symmetric part of the viscosity matrix D introduced in the previous section.*

Proof. Let $H(y)$ a bounded continuous function on $\mathbb{R}^d$ with compact support. Let $G(y)$ a probability density on $\mathbb{R}^d$ and let ν^y_α the Bernoulli measure conditioned to have the site y empty. Then

$$\varepsilon^d \sum_y G(\varepsilon y) E^{\nu^y_\alpha} \left(H(\varepsilon(X^y_{t\varepsilon^{-2}} - vt\varepsilon^{-2}))\right)$$
$$= \varepsilon^d \sum_y G(\varepsilon y) \sum_x H(\varepsilon(x - vt\varepsilon^{-2})) E^{\nu^y_\alpha}(1_{[X^y_{t\varepsilon^{-2}}=x]}).$$

In this formula, X^y_t stands for the position at time t of a second class particle initially at y. By (6.1) this is equal to

$$\frac{1}{\alpha(1-\alpha)} \varepsilon^d \sum_y \sum_x H(\varepsilon(x - vt\varepsilon^{-2})) G(\varepsilon y) \left[E^{\nu_\alpha}(\eta_{t\varepsilon^{-2}}(x)\eta_0(y)) - \alpha^2\right]$$
$$= \frac{1}{\alpha(1-\alpha)} E^{\nu_\alpha} \Bigg[\varepsilon^d \sum_x H(\varepsilon(x - vt\varepsilon^{-2}))$$
$$\times (\eta_{t\varepsilon^{-2}}(x) - \alpha) \sum_y G(\varepsilon y)(\eta_0(y) - \alpha)\Bigg]$$
$$= \frac{1}{\alpha(1-\alpha)} E^{\nu_\alpha} \left[Y^\varepsilon_t(H) Y^\varepsilon_0(G)\right]$$

and by the convergence result for Y^ε_t proved in the previous section, this converge as $\varepsilon \to 0$ to

$$\int_{\mathbb{R}^d} G(u)\, e^{t\mathcal{A}} H(u)\, du.$$

□

An immediate consequence of the above result is the following formula for the symmetric part of the viscosity matrix:

$$D^s_{i,j} = \lim_{t\to\infty} \frac{1}{t} \frac{1}{\alpha(1-\alpha)} \sum_x x_i x_j \left[E^{\nu_\alpha}(\eta_t(x + tv)\eta_0(0)) - \alpha^2\right] .$$

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