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Entropy Pairs and Compensated Compactness for Weakly Asymmetric Systems

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Abstract.

The hyperbolic (Euler) scaling limit of weakly asymmetric Ginzburg–Landau models with a single conservation law is investigated, weak asymmetry means that the microscopic viscosity of the system tends to infinity in a prescribed way during the hydrodynamic limit. The system is not attractive, its potential is a bounded perturbation of a quadratic function. The macroscopic equation reads as $\partial_t \rho + \partial_x S'(\rho) = 0$, where S is a convex function. The Tartar - Murat theory of compensated compactness is extended to microscopic systems, we prove weak convergence of the scaled density field to the set of weak solutions. In the attractive case of a convex potential this set consists of the unique entropy solution. Our main tool is the logarithmic Sobolev inequality of Landim, Panizo and Yau for continuous spins.

§1. Introduction and Main Result

In the last fifteen years a great progress has been made in the theory of hydrodynamic limits. Although the first papers [2] and [22] concern hyperbolic problems, most results are related to diffusive systems, see e.g. [13,29,30] and the monograph [15] with historical notes and further references. The main difficulty in hyperbolic problems comes from the breakdown of regularity and uniqueness of macroscopic solutions. In a smooth regime the relative entropy method of Yau [32] works well in quite general situations. Beyond shocks, however, only some attractive

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systems like asymmetric exclusions, zero range and stick processes are tractable, see [1,4,21,23,24] and also [16,31] on entropy and large deviations for asymmetric exclusion processes. The specific structure of these models is very important, and PDE techniques play an essential role in the proofs. The main purpose of this paper is to develop a general method for hyperbolic problems: we are going to extend the Tartar - Murat theory of compensated compactness to microscopic (stochastic) systems, see [26] and [19] for the first ideas, [14] or [25] for a systematic treatment of these advanced PDE techniques. Compensated compactness yields weak convergence of the scaled empirical density to the set of weak solutions to the macroscopic equations. A first exposition of the main ideas for stochastic systems was given in [9] in the case of asymmetric exclusions, and also for a lattice gas with two conservation laws. Here we study an asymmetric Ginzburg–Landau model with a single conservation law in details; we wanted to demonstrate that this method is really applicable. Another model, a two-component lattice gas with collisions is to be discussed in a forthcoming paper [11], see also [27,28] for a large class of two-component models. To have convergence of the scaled microscopic process to a well specified macroscopic solution, one has to supplement compensated compactness with the entropy condition of Lax and Kružkov implying the uniqueness of the limiting macroscopic solution. Unfortunately, we can only prove this condition for attractive Ginzburg–Landau models by adapting the coupling method of Rezakhanlou [21]. In another paper [10] we investigate non-attractive lattice gas models with a single conservation law. The structure of these systems allows us to verify also the entropy condition, thus we get convergence to a single entropy solution specified by its initial value.

Let $\eta_k(t) \in \mathbb{R}$ for $t \geq 0$, $k \in \mathbb{Z}$, and consider the following infinite system of stochastic differential equations as the evolution law of this continuous spin model. Given a potential $V(y) = y^2/2 + U(y)$ such that U, U', U'' are bounded,

$$(1.1) \qquad d\eta_k = \tfrac{1}{2}\,(V'_{k-1} - V'_{k+1})\,dt \\ + \sigma(\varepsilon)\,(V'_{k+1} + V'_{k-1} - 2V'_k)\,dt + \sqrt{2\sigma(\varepsilon)}\,(dw_{k-1} - dw_k)\,,$$

where w_k, $k \in \mathbb{Z}$ is a family of independent Wiener processes, $\sigma = \sigma(\varepsilon) > 1/2$ is the coefficient of microscopic viscosity; abbreviations like $V'_k := V'(\eta_k)$ are widely used also later on. The scaling parameter, $0 < \varepsilon \to 0$ of the hydrodynamic limit is interpreted as the spacing of the lattice in macroscopic units, hyperbolic scaling means that time is speeded up by a factor of $1/\varepsilon$. We shall let σ depend on ε during the limiting procedure in such a way that $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, thus the effect of the second difference, $\sigma\,(V'_{k+1}+V'_{k-1}-2V'_k)$ diminishes as $\varepsilon \to 0$. A technical condition, $\varepsilon\sigma^2(\varepsilon) \to \infty$ will be explained later.

Since the drift is Lipschitz continuous in a weighted ℓ^2 space, Ω of doubly infinite sequences $\eta = (\eta_k : k \in \mathbb{Z})$ with weights $e^{-|k|}$, for instance, the existence of unique strong solutions to (1.1) follows by a standard iteration procedure, and Ω carries a large class of probability measures, see e.g. [9] for further references. Let $\mathcal{F}_{k,l}$ denote the σ-field generated by $\eta_{k,l} := (\eta_j : k - l < j \le k)$, μ_t is the distribution of the evolved configuration $\eta(t)$, and $\mu_{t,k,l}$ denotes the distribution of $\eta_{k,l}(t)$. Short hand notation is to be used later in case of $k = n$ and $l = 2n$.

The total spin $\sum \eta_k$ is formally preserved by the evolution, and certain product measures λ_z with one dimensional marginal densities g_z, $z \in \mathbb{R}$,

$$g_z(y) := \exp\bigl(zy - V(y) - F(z)\bigr), \quad F(z) := \log \int_{-\infty}^{\infty} e^{zy - V(y)}\, dy$$

are all stationary states. As a reference measure, $\lambda := \lambda_0$ will be used; we may and do assume that $F(0) = F'(0) = 0$. A converse statement on stationary states in a much stronger form will be needed, our main tool is the logarithmic Sobolev inequality of [17]; that is why we are assuming that V is a bounded perturbation of a quadratic function. The model is attractive if V is convex; we are interested in the general case when an effective coupling is not available.

Due to the asymmetric part $(1/2)(V'_{k-1} - V'_{k+1})$ of the drift, the model admits a hyperbolic scaling as specified below. In the absence of the stochastic term $\sqrt{2\sigma}\,(dw_k - dw_{k-1})$, (1.1) looks like a lattice approximation procedure for solving $\partial_t\rho + \partial_x V'(\rho) = 0$; the viscid part $\sigma\,(V'_{k+1} + V'_{k-1} - 2V'_k)$ is needed even in this deterministic situation to stabilize the algorithm, see [14,18,25]. However, in regions of the phase space where V is concave, the viscid correction plays an opposite role; the convexity of V is very important in the deterministic case. Moreover, the value of σ may depend on the initial condition.

The behavior of the stochastic model is similar, but more complex. For $\varepsilon > 0$ interpreted as the macroscopic spacing of the lattice, let $\rho_\varepsilon(t,x) := \eta_k(t/\varepsilon)$ if $|x - k\varepsilon| < \varepsilon/2$ denote the empirical process; we are interested in its limiting behavior as the scaling parameter $\varepsilon \to 0$. In view of the principle of local equilibrium, the true distribution, $\mu_{t/\varepsilon}$ of our process is close to a product measure with marginal densities g_z such that z does depend on space and time. Since $\lambda_z(\eta_k) = F'(z) = \rho$ is the expectation of η_k with respect to λ_z, while $\lambda_z(V'_k) = z = S'(\rho)$ if $\rho = F'(z)$, where $S(\rho) := \sup_z \{z\rho - F(z)\}$, $\partial_t\rho + \partial_x S'(\rho) \approx$

$\varepsilon\sigma\partial_x^2 S'(\rho)$ is expected for the asymptotic mean of ρ_ε as $\varepsilon \to 0$. This was proven in [7] for $\sigma = \sigma_0/\varepsilon$ and small U by means of the parabolic perturbation technique of [6], see also [12,4] on the weakly asymmetric exclusion process. Heuristic considerations of this kind suggest that the macroscopic equation becomes $\partial_t\rho + \partial_x S'(\rho) = 0$ if $\varepsilon\sigma \to 0$ during the hydrodynamic limit. This can be proven by means of the relative entropy method [32] when the macroscopic solution is smooth, even if $\sigma > 0$ is fixed. In case of an incompressible limit (perturbation of equilibrium) the initial configuration is changed during the scaling limit, see [20,24,28].

In a regime of shocks some new methods are needed, this is the subject of the present paper. Unfortunately, we are able to control oscillations of the empirical process only if the microscopic viscosity, σ goes to infinity as $\varepsilon \to 0$. More precisely, we are assuming that $\varepsilon\sigma \to 0$ but $\varepsilon\sigma^2 \to +\infty$ as $\varepsilon \to 0$. For example, $\sigma(\varepsilon) := \sqrt{\varepsilon}\log(1/\varepsilon)$ is an allowed choice. Let us remark that the concept of microscopic viscosity is plausible in many other cases, and conditions on its growth rate are the same as above, see [9,10,11], but Dittrich [4] only needs $\varepsilon\sigma^3 \to +\infty$ in case of asymmetric exclusions. If the generator, $\mathfrak{L}$ of a conservative process decomposes as $\mathfrak{L} = \mathfrak{L}_0 + \sigma\mathfrak{G}$, where $\mathfrak{G}$ is symmetric, then the parameter $\sigma > 0$ may be interpreted as the microscopic viscosity of the model. The paper [28] investigates perturbation of the equilibrium for a class of two-component hyperbolic models in a smooth regime; the order of microscopic viscosity of these models is the same as here. Also in such situations we use the term *weakly asymmetric system*; perhaps the phrase *large microscopic viscosity limit* would be more correct.

A locally square integrable $\rho \in L^2_{\text{loc}}(\mathbb{R}^2_+)$ is a *weak solution* to $\partial_t\rho + \partial_x S'(\rho) = 0$ with initial value $\rho_0 \in L^2_{\text{loc}}(\mathbb{R})$ if $\rho(t,x)$ satisfies

$$(1.2) \qquad \int_0^\infty \int_{-\infty}^\infty \left(\rho\psi_t' + S'(\rho)\psi_x'\right) dx\, dt + \int_{-\infty}^\infty \rho_0(x)\psi(0,x)\, dx = 0$$

for all test functions $\psi \in C_c^1(\mathbb{R}^2)$, where $\psi_u' := \partial_u\psi$, $C_c^k(\mathbb{R}^2)$ is the space of compactly supported $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ with k continuous derivatives, $\mathbb{R}^2_+ := [0,\infty)\times\mathbb{R}$, L^2_{loc} is the space of locally square integrable functions. It is easy to check that S'' is bounded, thus the definition above is not a senseless one. In fact, only the local integrability of ρ is needed in (1.2) because S' is linearly bounded, but we prefer an L^2 setting. In case of a single conservation law the *Lax entropy condition* is sufficient for the uniqueness of weak solutions, see [14] or [25]. For $\alpha > 0$ let $\mathcal{H}_\alpha$ denote the set of such couples (h, J) of continuously differentiable real functions that $|h(u)| + |J(u)| = O(|u|^\alpha)$ for large u, and $J' = h'S''$, that is $\partial_t h(\rho) + \partial_x J(\rho) = 0$ along classical solutions; (h, J) is called an *Lax entropy pair.* A weak solution, ρ satisfies the entropy condition if

$$\int_0^\infty \int_{-\infty}^\infty \big(h(\rho)\psi_t' + J(\rho)\psi_x'\big)\,dx\,dt + \int_{-\infty}^\infty h(\rho_0(x))\psi(0,x)\,dx \ge 0 \tag{1.3}$$

for all $0 \le \psi \in C_c^1(\mathbb{R}^2)$ and $(h, J) \in \mathcal{H}_1$ with h convex. An equivalent version of the Lax inequality has been proposed by Kružkov, see [21,25], namely

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \big(|\rho - c|\,\psi_t' + |S'(\rho) - S'(c)|\,\psi_x'\big)\,dx\,dt \\ &\qquad + \int_{-\infty}^\infty |\rho_0(x) - c|\,\psi(0,x)\,dx \ge 0 \end{aligned} \tag{1.4}$$

for all $0 \le \psi \in C_c^1(R^2)$ and $c \in \mathbb{R}$; the flux of $h_c(u) := |u - c|$ can be chosen as $J_c(u) := |S'(u) - S'(c)|$.

The Lax inequality is motivated by the viscous approximation $\partial_t u_\varepsilon + \partial_x S'(u_\varepsilon) = \varepsilon\sigma\partial_x^2\Phi'(u_\varepsilon)$, where $\varepsilon\sigma \to 0$, because

$$\begin{aligned} \partial_t h(u_\varepsilon) + \partial_x J(u_\varepsilon) &= \varepsilon\sigma h'(u_\varepsilon)\partial_x^2\Phi'(u_\varepsilon) \\ &= \varepsilon\sigma\partial_x\big(h'(u_\varepsilon)\Phi''(u_\varepsilon)\partial_x u_\varepsilon\big) - \varepsilon\sigma h''(u_\varepsilon)\Phi''(u_\varepsilon)(\partial_x u_\varepsilon)^2, \end{aligned} \tag{1.5}$$

whence one can derive (1.3) with an appropriate choice of Φ; the most favored one is $\Phi(u) = u^2/2$. We see that there is a freedom in choosing the viscid correction $\varepsilon\sigma\partial_x^2\Phi'$, but $\Phi'' \ge 0$ is very important at this point; the structure of lattice approximation procedures and other numerical schemes is similar. The viscous correction must be elliptic in all cases. Stochastic models are structured in a more canonical way because they must have a stationary state; the form of (1.1) is dictated by this requirement. Calculations at the microscopic level follow the above scheme of the viscous approximation, that is why the strict convexity of V is needed for (1.3) or (1.4).

Several topologies shall be used in the study of the limit distribution of the empirical process ρ_ε. We shall see that ρ_ε is locally square integrable, thus the weak topology of $L^2_{\text{loc}}(\mathbb{R}^2_+)$ will be the first one; the distribution P_ε of ρ_ε will be considered on this space. We are interested in the limiting behavior of the density field R_ε,

$$R_\varepsilon(\psi) := \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\,\rho_\varepsilon(t,x)\,dx\,dt, \tag{1.6}$$

where $\psi \in C_c(\mathbb{R}^2)$ is compactly supported. The initial conditions are specified in terms of a family $\mu_{\varepsilon,0} : \varepsilon > 0$ of initial distributions. First of all, we have some $\rho_0 \in L^2_{\text{loc}}(\mathbb{R})$ such that

$$(1.7) \qquad \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \varphi(x)\rho_\varepsilon(0,x)\,dx = \int_{-\infty}^{\infty} \varphi(x)\rho_0(x)\,dx$$

in probability for all $\varphi \in C_c(\mathbb{R})$. The second condition tells that the entropy, S of $\mu_{\varepsilon,0}$ relative to $\lambda := \lambda_0$ is extensive. The entropy of a probability measure ν relative to λ is defined as $S[\nu|\lambda] := \nu(\log f)$ if $\nu \ll \lambda$ and $d\nu = f d\lambda$, $S = +\infty$ otherwise. Let $\mu_{\varepsilon,0,n}$ denote the restriction of $\mu_{\varepsilon,0}$ to $\mathfrak{F}_{n,2n}$, and suppose that $f_n := d\mu_{\varepsilon,0,n}/d\lambda$ satisfies

$$(1.8) \qquad S_n[\mu_{\varepsilon,0}|\lambda] := \int f_{\varepsilon,n} \log f_{\varepsilon,n}\,d\lambda \le C_0\, n \quad \forall \varepsilon > 0 \text{ and } n \in \mathbb{N}.$$

Our main result is

Theorem 1.1. *Suppose* (1.7), (1.8) *and specify $\sigma = \sigma(\varepsilon)$ such that $\varepsilon\sigma(\varepsilon) \to 0$ but $\varepsilon\sigma^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Then the family P_ε is tight as $\varepsilon \to 0$, and any limit distribution is concentrated on a set of weak solutions* (1.2) *to the macroscopic equation $\partial_t\rho + \partial_x S'(\rho) = 0$. Moreover, if V is strictly convex, then we have a weak solution $\rho \in L^2_{loc}(\mathbb{R}^2_+)$ such that*

$$\lim_{\varepsilon \to 0} R_\varepsilon(\psi) = R(\psi) := \int_0^{\infty} \int_{-\infty}^{\infty} \psi(t,x)\rho(t,x)\,dx\,dt$$

in probability for all $\psi \in C_c(\mathbb{R}^2)$; this ρ is uniquely specified by its initial value ρ_0 and the entropy condition (1.4).

The paper is organized as follows. Below and in Section 2 we are going to exhibit the main ideas of the argument. Section 3 summarizes some basic facts on the microscopic model, further technical details are added in Section 4. The proof is then completed in the last section.

The first main step of the proof is certainly the replacement of $V'(\rho_\varepsilon)$ with $S'(\rho_\varepsilon)$, this characteristic argument of hydrodynamic limits does not appear in PDE theory. The second step is then to show that the weak limit of $S'(\rho_\varepsilon)$ equals $S'(\rho)$, where ρ is the weak limit of ρ_ε. As we have learned from [13], the replacement of V' with S' can be done at a level of block averages. In case of a diffusive scaling the celebrated two-block estimate allows us to work with macroscopic blocks, thus the weak limit commutes with S'. This step is more difficult if we consider a hyperbolic problem because the two-block lemma extends to blocks of size $l = o(\sqrt{\sigma/\varepsilon})$ only, consequently there is no direct argument to identify the weak limit of $S'(\rho_\varepsilon)$. The concept of measure solution plays an important role at this point, see e.g. [3] on partial differential equations, and [29] on a first application to a microscopic system.

Let Θ denote the set of measurable families $\theta = \theta_{t,x} \, : \, (t,x) \in \mathbb{R}^2_+$ of probability measures on $\mathbb{R}$ such that $\theta_{t,x}(u^2)$ is locally integrable on $\mathbb{R}^2_+$. $\theta \in \Theta$ is a *measure solution* to (1.1) if

$$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \theta_{t,x}(d\rho)\big(\rho\psi'_t + S'(\rho)\psi'_x\big)\,dx\,dt = 0 \tag{1.9}$$

for all $\psi \in C^1_c(\mathbb{R}^2_+)$, the space of $\psi \in C^1(\mathbb{R}^2)$ such that supp ψ is contained in the interior of $\mathbb{R}^2_+$. Notice that the initial value has not been included in this definition. A function $u \in L^2_{\text{loc}}(\mathbb{R}^2_+)$ is represented by a family $\theta \in \Theta$ of Dirac measures such that $\theta_{t,x}$ is concentrated at the actual value $u(t,x)$ of u; this θ is called the *Young representation* of u. Moreover, any $\theta \in \Theta$ can be identified as a locally finite measure m_θ, $dm_\theta := dt\,dx\,\theta_{t,x}(du)$ on $\mathbb{R}^3_+ := \mathbb{R}^2_+ \times \mathbb{R}$; equip Θ with the associated weak topology. Therefore any weak solution is a measure solution, and the existence of measure solutions follows by a direct compactness argument. Compensated compactness is the tool for proving that any measure solution is actually a weak solution. We say that $\theta \in \Theta$ admits a *Tartar factorization* for a couple $(h_1, J_1), (h_2, J_2) \in \mathcal{H}_1$ of entropy pairs, if for almost every $(t,x) \in \mathbb{R}^2_+$ we have

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1)\,. \tag{1.10}$$

In the case of a single conservation law like $\partial_t\rho + \partial_x S'(\rho) = 0$, Tartar's factorization implies that θ is a family of Dirac measures, that is a weak solution. To get uniqueness of weak solutions we need the Kružkov inequality (1.4). The entropy condition can also be stated at the level of measure solutions,

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty \big(\theta_{t,x}(h_c)\psi'_t + \theta_{t,x}(J_c)\psi'_x\big)\,dx\,dt&\\ + \int_{-\infty}^\infty \theta_{0,x}(h)\psi(0,x)\,dx &\geq 0\end{aligned} \tag{1.11}$$

for all $0 \leq \psi \in C^1_c(\mathbb{R}^2)$ and for the Kružkov entropy pairs (h_c, J_c), $c \in \mathbb{R}$, see (1.4); the derivation of (1.11) is easier than that of (1.4). Let us remark that DiPerna [3] proves the uniqueness of measure solutions satisfying (1.11) without any reference to Tartar's factorization, but his initial condition is much stronger than that we do have here. Compensated compactness requires large microscopic viscosity, but it has an advantage from the point of view of uniqueness. Since we have weak solutions, (1.4) is sufficient, i.e. no continuity condition is needed at time zero, see [22].

Entropy pairs constitute additional conservation laws at the macroscopic level, but the microscopic model must be ergodic, thus it can not have any other conservation law than those we are a priori given. Therefore the Lax entropies exhibit rapid oscillations, they should be controlled by means of non-gradient tools as initiated by Varadhan [30].

§2. Compensated Compactness

The proof of Tartar's factorization is based on some functional analytic properties of the Lax entropy production $X := \partial_t h + \partial_x J$, we have to estimate X in various spaces. Let $\|\varphi\|$ denote the uniform norm, $\|\varphi\|_p$ is the L^p norm of $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ for $p \geq 1$. The Sobolev space $H_{+1}(\mathbb{R}^2)$ is defined as the completion of $C_c^1(\mathbb{R}^2)$ with respect to $\|\cdot\|_{+1}$, $\|\varphi\|_{+1}^2 := \|\varphi\|_2^2 + \|\varphi_t'\|_2^2 + \|\varphi_x'\|_2^2$, and $H_{-1}(\mathbb{R}^2)$ is the dual of H_{+1} with respect to $L^2(\mathbb{R}^2)$. Here and below we adopt a convention: if a function u is only defined on $\mathbb{R}_+^2$, then we extend its definition by setting $u(t,x) = 0$ for $t < 0$.

A first version of Tartar's theorem can be stated as follows. Let $(h_i, J_i) \in \mathcal{H}_1$ for $i = 1, 2$, and set $X_{i,\varepsilon} := \partial_t h_i(u_\varepsilon) + \partial_x J_i(u_\varepsilon)$. Suppose that u_ε, $h_i(u_\varepsilon)$ and $J_i(u_\varepsilon)$ are all weakly convergent in $L^2(\mathbb{R}^2)$ as $\varepsilon \to 0$, while the Young representation θ_ε of u_ε tends to some $\theta \in \Theta$. If the set $\{X_{i,\varepsilon} : i = 1, 2; \varepsilon > 0\}$ is relative compact in $H_{-1}(\mathbb{R}^2)$, then (1.10) holds true. The so called Murat lemma on the conditions of Tartar's theorem had certainly been motivated by (1.5). It states that if $h_i(u_\varepsilon)$ and $J_i(u_\varepsilon)$ are bounded in $L^p(\mathbb{R}^2)$ for some $p > 2$, and $X_{i,\varepsilon} = Y_{i,\varepsilon} + Z_{i,\varepsilon}$ such that $Z_{i,\varepsilon}$ is bounded in the space of finite signed measures on $\mathbb{R}^2$, while $Y_{i,\varepsilon}$ belongs to a compact set of $H_{-1}(\mathbb{R}^2)$, then $X_{i,\varepsilon}$ also lies in a compact subset of $H_{-1}(\mathbb{R}^2)$. Since the empirical process does not vanish at infinity, we have to localize the problem by multiplying X with a general $\phi \in C_c^2(\mathbb{R}^2)$; this step is also present in the original papers [26] and [19]. "Compensation" appears at two places. The factorization on the right hand side of (1.10) holds true only for the difference on the left, and $\partial_t h$, $\partial_x J$ alone are only bounded in H_{-1}, their sum does belong to a compact set.

In view of our project, we formulate and prove Tartar's factorization and the entropy inequality at the microscopic level, this will be done in terms of block averages. For any sequence ξ indexed by $\mathbb{Z}$,

$$(2.1)\qquad \bar{\xi}_{l,k} := \frac{1}{l}\sum_{j=0}^{l-1}\xi_{k-j} \quad\text{and}\quad \hat{\xi}_{l,k} := \frac{1}{l^2}\sum_{j=-l}^{l}\big||j|-l\big|\,\xi_{k+j}\,.$$

For example, $\bar{V}'_{l,k}$ refers to the sequence $V'_k = V'(\eta_k)$. The smooth averaging $\hat{\xi}_l$ seems to be convenient in analytic calculations, while the usual one, $\bar{\xi}_l$ is preferred in computing canonical expectations. The size $l = l(\varepsilon)$ of these blocks should be chosen in such a way that

$$(2.2)\qquad \limsup_{\varepsilon\to 0}\frac{\sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)} < +\infty \quad\text{and}\quad \lim_{\varepsilon\to 0}\frac{l(\varepsilon)}{\sigma(\varepsilon)} = 0\,,$$

thus $\varepsilon l^2(\sigma) \to +\infty$ as $\varepsilon \to 0$. Since $\varepsilon\sigma(\varepsilon) \to 0$ and $\varepsilon\sigma^2(\varepsilon) \to +\infty$, $\sigma^2 = o(l^3)$. We see also that $(\sigma/\varepsilon)^{1/3} = O(l) = o(\sigma)$, thus the integer part of $\varepsilon^{-1/4}\sqrt{\sigma(\varepsilon)}$ is an acceptable choice for l. Because of some technical reasons, we modify the empirical process as $\hat{\rho}_\varepsilon(t,x) := \hat{\eta}_{l,k}(t/\varepsilon)$ if $|x - k\varepsilon| < \varepsilon/2$, $\hat{P}_\varepsilon$ denotes its distribution on $L^2_{\text{loc}}(\mathbb{R}^2_+)$; from now on the block size $l = l(\varepsilon)$ is specified according to (2.2). In view of the Young representation, the empirical process $\hat{\rho}_\varepsilon$ can be considered also as a random element $\hat{\theta}_\varepsilon$ of Θ; the distribution, $\hat{P}_{\theta,\varepsilon}$ of $\hat{\theta}_\varepsilon$ is defined on this space. Of course, P_ε, $\hat{P}_\varepsilon$ and $\hat{P}_{\theta,\varepsilon}$ are not really different from each other, just the notion of weak convergence varies.

The microscopic version of entropy production $X = \partial_t h + \partial_x J$ is defined for $\psi \in C^1_c(\mathbb{R}^2_+)$ and $(h,J) \in \mathcal{H}_1$ by

$$(2.3)\qquad X_\varepsilon(\psi,h) := -\int_0^\infty\int_{-\infty}^\infty \big(h(\hat{\rho}_\varepsilon)\psi'_t + J(\hat{\rho}_\varepsilon)\psi'_x\big)\,dx\,dt\,,$$

remember that ψ is compactly supported in the interior of $\mathbb{R}^2_+$. We have

$$(2.4)\qquad \begin{aligned} X_\varepsilon(\psi,h) &= N_\varepsilon(\psi,h) + M_\varepsilon(\psi,h)\\ &\quad+\frac{1}{\varepsilon}\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\big(\mathfrak{L}h(\hat{\rho}_\varepsilon) + \varepsilon\widetilde{\nabla}_\varepsilon J(\hat{\rho}_\varepsilon)\big)\,dx\,dt\,,\end{aligned}$$

where N_ε is a numerical error due to the lattice approximation of the space derivative, M_ε is a stochastic integral coming from the Ito lemma, and $\mathfrak{L} = \mathfrak{L}_0 + \sigma\mathfrak{G}$ is the generator of the microscopic process (1.1). On smooth cylinder functions $\varphi(\eta)$, $\mathfrak{L}_0$ and $\mathfrak{G}$ are acting as

$$\mathfrak{L}_0\varphi := -\sum_{k\in\mathbb{Z}}(\widetilde{\nabla}_1 V'_k)\partial_k\varphi\,,\quad \mathfrak{G}\varphi := \sum_{k\in\mathbb{Z}}(\nabla_1\partial_k - \nabla_1 V'_k)\nabla_1\partial_k\varphi\,,$$

where $\partial_k\varphi := \partial\varphi/\partial\eta_k$, $\nabla_l\xi_k := l^{-1}(\xi_{k+l} - \xi_k)$, $\widetilde{\nabla}_l := (1/2)(\nabla_l - \nabla_l^*)$, $\nabla_l^*\xi_k := l^{-1}(\xi_{k-l} - \xi_k)$, $\Delta_l := -\nabla_l^*\nabla_l$ for $l \in \mathbb{N}$. Note that $\nabla_1\hat{\xi}_l = \nabla_l\bar{\xi}_l$. The formalism is used also for functions as $\varepsilon\nabla_\varepsilon\varphi(x) := \varphi(x+\varepsilon) - \varphi(x)$, $\widetilde{\nabla}_\varepsilon\varphi(x) := (1/2\varepsilon)\big(\varphi(x+\varepsilon) - \varphi(x-\varepsilon)\big)$, and so on.

Mimicking integration by parts, the numerical error becomes

$$N_\varepsilon(\psi, h) = \int_0^\infty \int_{-\infty}^\infty J(\hat{\rho}_\varepsilon)(\widetilde{\nabla}_\varepsilon - \partial_x)\psi(t,x)\,dx\,dt. \tag{2.5}$$

The stochastic equations for $\hat{\eta}$ read as

$$d\hat{\eta}_{l,k} = -\widetilde{\nabla}_1\hat{V}'_{l,k}\,dt + \sigma\Delta_1\hat{V}'_{l,k}\,dt + \sqrt{2\sigma}\,\nabla_1^*\,d\hat{w}_{l,k},$$

thus scaling the noise as $\hat{\zeta}(t,x) := \sqrt{\varepsilon}\,\hat{w}_{l,k}(t/\varepsilon)$ if $|x - k\varepsilon| < \varepsilon/2$,

$$M_\varepsilon(\psi, h) = \sqrt{2\sigma\varepsilon}\int_{-\infty}^\infty \int_0^\infty \psi(t,x)h'(\hat{\rho}_\varepsilon)\nabla_\varepsilon^*\hat{\zeta}_\varepsilon(dt,x)\,dx. \tag{2.6}$$

Splitting $\mathfrak{L}/\varepsilon$ into its asymmetric and symmetric components, we obtain a decomposition $X_\varepsilon = N_\varepsilon + M_\varepsilon + X_{a,\varepsilon} + X_{s,\varepsilon}$, where

$$X_{a,\varepsilon}(\psi, h) := \frac{1}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\big(\mathfrak{L}_0 h(\hat{\rho}_\varepsilon) + \varepsilon\widetilde{\nabla}_\varepsilon J(\hat{\rho}_\varepsilon)\big)\,dx\,dt, \tag{2.7}$$

$$X_{s,\varepsilon}(\psi, h) := \frac{\sigma}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\,\mathfrak{G}h\big(\hat{\rho}_\varepsilon(t,x)\big)\,dx\,dt. \tag{2.8}$$

The main term here is certainly the asymmetric $X_{a,\varepsilon}(\psi, h)$.

Having in mind (1.5) and the Tartar - Murat theorems, we are looking for a decomposition of entropy production $X_\varepsilon(\psi, h) = Y_\varepsilon(\psi, h) + Z_\varepsilon(\psi, h)$ described as follows.

Proposition 2.1. *Let $(h_1, J_1), (h_2, J_2) \in \mathcal{H}_1$, and suppose that we are given some random functionals $Y_\varepsilon(\psi, h_i)$, $Z_\varepsilon(\psi, h_i)$, $A_\varepsilon(\phi)$, $B_\varepsilon(\phi)$ such that $X_\varepsilon = Y_\varepsilon + Z_\varepsilon$, $A_\varepsilon(\phi)$ and $B_\varepsilon(\phi)$ do not depend on ψ, moreover*

$$|Y_\varepsilon(\phi\psi, h_i)| \le A_\varepsilon(\phi)\|\psi\|_{+1}, \quad |Z_\varepsilon(\phi\psi, h_i)| \le B_\varepsilon(\phi)\|\psi\|$$

for each $\psi \in C_c^1(\mathbb{R}_+^2)$, $\phi \in C_c^2(\mathbb{R}^2)$, $i = 1, 2$ and $\varepsilon > 0$. If $\|\phi\rho_\varepsilon\|_2^2 \le B_\varepsilon(\phi)$, $\mathsf{E}A_\varepsilon(\phi) \to 0$ and $\limsup \mathsf{E}B_\varepsilon(\phi) < +\infty$ as $\varepsilon \to 0$, then $\widehat{P}_{\theta,\varepsilon} : \varepsilon > 0$ is tight on Θ, and (1.10) holds true with probability one with respect to any weak limit point $\widehat{P}_\theta$ of $\widehat{P}_{\theta,\varepsilon}$ as $\varepsilon \to 0$.

This is a microscopic (stochastic) synthesis of the fundamental results of L. Tartar and F. Murat on compensated compactness. We postpone its proof to the last section, the main problem is to verify the conditions; that is the content of Sections 3 and 4. The first part of Theorem 1.1 follows from Proposition 2.1 and Lemma 5.1.

For the Lax–Kružkov inequality we do not need bounds that are uniform in ψ, but the viscid term, $\sigma\Delta_1 V'$ of the microscopic evolution must be elliptic as a (nonlinear) operator on the configuration space.

Proposition 2.2. *Suppose all conditions of Theorem 1.1 including the strict convexity of V, then* $\hat{P}_{\theta,\varepsilon} : \varepsilon > 0$ *is a tight family with respect to the weak topology of* Θ, *and its weak limit distributions are concentrated on a set of measure solutions satisfying* (1.11).

The proof of this statement is based on the attractiveness of the microscopic process due to monotonicity of V'. Following [21], it is presented in Section 5. The proof of Theorem 1.1 is then completed by weak uniqueness of entropy solutions. The case of a general (non-convex) potential is a formidable open problem.

§3. The a priori bounds

This section summarizes some estimates based on relative entropy and its rate of production, the fundamental entropy inequality $\nu(\varphi) \leq S[\nu|\lambda] + \log \lambda(e^{\varphi})$ will be used several times. The Donsker–Varadhan rate function of a probability measure $\nu \ll \lambda$ with respect to a self-adjoint generator, $\mathfrak{S}$ of a Markov process in $L^2(\lambda)$ is a Dirichlet form $D[\nu|\lambda, \mathfrak{S}] := -4\lambda(\sqrt{f}\,\mathfrak{S}\,\sqrt{f})$ when $f := d\nu/d\lambda$; for technical details see [13],[15] or [9] with further references. We consider (1.1) with an arbitrary, but fixed value of $\sigma > 1/2$, $\mu_{t,n}$ is the restriction of the evolved measure, μ_t to $\mathfrak{F}_{n,2n}$, and $f_n(t,\eta)$ denotes the λ-density of $\mu_{t,n}$, if any. Set $S_n(t) := S[\mu_{t,n}|\lambda]$, while $D_n(t) := D[\mu_{t,n}|\lambda, \mathfrak{G}_{n,2n}]$, where

$$\mathfrak{G}_{k,l}\,\varphi := \sum_{j=k-l+1}^{k-1} (\nabla_1\partial_j - \nabla_1 V_j')\nabla_1\partial_j\varphi$$

for smooth φ. If $0 < f_n$ is differentiable then

$$D_n(t) = 4\sum_{k=1-n}^{n-1}\int(\nabla_1\partial_k\sqrt{f_n})^2\,d\lambda = \sum_{k=1-n}^{n-1}\int\frac{1}{f_n}(\nabla_1\partial_k f_n)^2\,d\lambda\,.$$

First we derive an explicit bound for S_n and the time integral of D_n.

Lemma 3.1. *If $S_n(0) \le C_0\, n$ then*

$$S_n(t) + \sigma \int_0^t D_n(s)\, ds \le C_1 \left(t + \sqrt{n^2 + \sigma t}\right)$$

for all $n \in \mathbb{N}$, where C_1 is a constant depending only on C_0 and U.

Proof. We follow the proof of Proposition 1 in [8], only the main steps are presented. Remember that λ is preserved by the deterministic process generated by $\mathfrak{L}_0$, i.e. $\lambda(\mathfrak{L}_0\varphi) = 0$, while $\mathfrak{G}$ is symmetric in $L^2(\lambda)$, thus

$$\int \varphi \mathfrak{G} \psi \, d\lambda = -\sum_{k\in\mathbb{Z}} \int (\nabla_1 \partial_k \varphi) \nabla_1 \partial_k \psi \, d\lambda$$

for smooth cylinder functions φ and ψ. If $f_n > 0$ is smooth enough, then by a direct calculation

$$\begin{aligned}
\partial_t S_n &= \int (\partial_t + \mathfrak{L}) \log f_n(t,\eta)\, \mu_t(d\eta) = \int f_{n+1} \mathfrak{L} \log f_n \, d\lambda \\
&= \sum_{k\in\mathbb{Z}} \int f_{n+1} (\tilde{\nabla}_1 V_k') \frac{\partial_k f_n}{f_n}\, d\lambda - \sigma \sum_{k\in\mathbb{Z}} \int (\nabla_1 \partial_k f_{n+1}) \frac{\nabla_1 \partial_k f_n}{f_n}\, d\lambda \\
&= -\sigma\, D_n - \sigma\, D_{\partial,n} + \sum_{k\in\mathbb{Z}} \int (f_{n+1} - f_n)(\tilde{\nabla}_1 V_k') \frac{\partial_k f_n}{f_n}\, d\lambda \\
&\quad - \sigma \sum_{k\in\mathbb{Z}} \int (\nabla_1 \partial_k f_{n+1} - \nabla_1 \partial_k f_n) \frac{\nabla_1 \partial_k f_n}{f_n}\, d\lambda ,
\end{aligned}$$

where

$$D_{\partial,n}(t) := \int \frac{1}{f_n} (\partial_n f_n)^2 \, d\lambda + \int \frac{1}{f_n} (\partial_{1-n} f_n)^2 \, d\lambda$$

and $f_n = f_n(t,\eta)$. Both sums on the right hand side above consist only of boundary terms corresponding to $k = \pm n$, $\lambda(V_k') = 0\ \forall k$, and for $k = n+1$ or $k = -n$ we have

$$\int \varphi_n \partial_k f_{n+1}\, d\lambda = \int \varphi_n V_k' f_{n+1}\, d\lambda$$

whenever φ_n is $\mathfrak{F}_{n,2n}$ measurable. Denoting

$$B_n(t) := \frac{1}{2} \int \left(V_{n+1}' \partial_n f_n - V_{-n}' \partial_{1-n} f_n\right) \frac{f_{n+1}}{f_n}\, d\lambda ,$$

by an easy computation we arrive at

$$\begin{aligned}\partial_t S_n + \sigma\, D_n &= (1+2\sigma)\, B_n - \sigma D_{\partial,n} \\ &= B_n - \sigma \sum_{k=\pm n} \int (\nabla_1 \partial_k f_{n+1})\, \frac{\nabla_1 \partial_k f_n}{f_n}\, d\lambda \\ &\le B_n + \sigma \sqrt{D_{n+1} - D_n}\sqrt{D_{\partial,n}}\ ;\end{aligned} \tag{3.1}$$

at the final step $\nabla_1 \partial_n f_n = -\partial_n f_n$, $\nabla_1 \partial_{-n} f_n = \partial_{1-n} f_n$, the Schwarz inequality and convexity of D were used.

First of all we have to estimate B_n . For any probability measure ν, and $u \in \mathbb{R}$ we have an entropy bound

$$u\,\nu(V_k') \le S[\nu|\lambda] + \log \lambda(e^{uV_k'}) \le S[\nu|\lambda] + \frac{1}{2}\, \|V''\|\, u^2\, ,$$

see (3.4) for the second inequality, whence by setting $u = \pm\sqrt{2S/\|V''\|}$ we obtain that $\nu^2(V_k') \le 2\|V''\|\, S[\nu|\lambda]$. Let $\nu = \mu_t[\cdot|\mathcal{F}_{n,2n}]$, again by Schwarz and convexity we get

$$B_n(t) \le K_0 \sqrt{S_{n+1}(t) - S_n(t)}\, \sqrt{D_{\partial,n}(t)}\, .$$

In view of (3.1) there is nothing to prove if $(1+2\sigma) B_n \le \sigma D_{\partial,n}$, but

$$\sigma D_{\partial,n} \le 4B_n \le 4K_0 \sqrt{(S_{n+1} - S_n) D_{\partial,n}}$$

in the opposite case, whence a system

$$\partial_t S_n + \sigma D_n \le K_1 \big(S_{n+1} - S_n + \sigma \sqrt{S_{n+1} - S_n}\, \sqrt{D_{n+1} - D_n}\, \big)$$

of differential inequalities follows immediately, where K_1 depends only on $\|V''\|$. This system admits an explicit solution, see Lemma 3 in [8], the result is just the bound we have to prove. Since the final statement does not depend on smoothness of f_n any more, this restriction can be removed by a standard regularization. Q.E.D.

As a first consequence, from the entropy bound we get the moment condition $\limsup \mathsf{E}\|\phi\rho_\varepsilon\|_2^2 < +\infty$ of Proposition 2.1 for $\phi \in C_c^2(\mathbb{R}^2)$.

Lemma 3.2. *We have a universal constant C_2 such that*

$$\frac{1}{nt} \sum_{|k|<n} \int_0^t \int \eta_k^2 \, d\mu_s\, ds \le C_2 \Big(1 + \frac{t}{n} + \sqrt{1 + \sigma t/n^2} \Big)\, .$$

Proof. From the basic entropy inequality, $\nu(\varphi) \le S[\nu|\lambda] + \log\lambda(e^{\varphi})$, for any $\beta > 0$ we get

$$\frac{1}{n}\sum_{|k|<n}\mu_t(\eta_k^2) \le \frac{1}{\beta n}S_n(t) + \frac{2}{\beta}\log\lambda(e^{\beta\eta_k^2}).$$

To estimate $\lambda(e^{\eta_k^2})$, let E_g denote expectation with respect to an $N(0,2\beta)$ variable ζ, then $e^{\beta\eta_k^2} = \mathsf{E}_g e^{\zeta\eta_k}$, thus $\lambda(e^{\zeta\eta_k}) = e^{F(\zeta)}$, and $F(\zeta) \le (1/2)\|F''\|\,\zeta^2$ as $F(0) = F'(0) = 0$ by assumption. Since $F''(z)$ is just the variance of η_k under λ_z, $F''(z) \le \lambda_z((\eta_k - y)^2)$ e.g. if $z = V'(y)$. However, $(\eta_k - y)^2 \le a + b(V_k' - z)^2$ because $V''(x)$ is strictly positive for large $|x|$, while $\lambda_z((V_k' - z)^2) = \lambda_z(V_k'')$, we have $\|F''\| \le a + b\|V''\| < +\infty$. Finally,

$$\log\mathsf{E}_g e^{\gamma\zeta^2} = -\log\sqrt{1 - 4\gamma\beta} \le 4\gamma\beta \quad \text{whenever } 8\gamma\beta \le 1, \tag{3.2}$$

which completes the proof via Lemma 3.1. Q.E.D.

The following lemma summarizes some results of [17]. For any linearly bounded $h \in C(\mathbb{R})$, and $\alpha_j \in \mathbb{R} : 0 \le j < l$ set $\hat{h}(\rho) := \lambda_z(h(\eta_k))$,

$$\phi_{l,k}(h,\alpha) := \sum_{j=0}^{l-1}\alpha_j\big(h(\eta_{k-j}) - \hat{h}(\bar{\eta}_{l,k})\big),$$

and $\Phi_h(\rho,u) := \log\lambda_z(e^{uh(\eta_k) - u\hat{h}(\rho)})$, where $z := S'(\rho)$.

Lemma 3.3. *We have positive constants l_0 and C_3 depending only on U such that if $l > l_0$, then any probability measure, ν on $\mathfrak{F}_{k,l}$ satisfies*

$$\beta\int\phi_{l,k}(h,\alpha)\,d\nu \le C_3\left(1 + l^2 D[\nu|\lambda,\mathfrak{G}_{k,l}]\right) + \frac{1}{2}\log\int\exp\Big(\sum_{j=0}^{l-1}\Phi_h(\bar{\eta}_{l,k},2\beta\alpha_j)\Big)\,d\nu.$$

Proof. Given $\bar{\eta}_{l,k} = \rho$, denote $\bar{\nu}_{l,\rho}$ and $\bar{\lambda}_{l,\rho}$ the conditional distributions of $\eta_{k,l}$ under ν and λ, respectively. In view of LSI, which is Theorem 2.2 in [17], we have $S[\bar{\nu}_{l,\rho}|\bar{\lambda}_{l,\rho}] \le C_3' l^2 D[\bar{\nu}_{l,\rho}|\bar{\lambda}_{l,\rho},\mathfrak{G}_{k,l}]$ for all ν and ρ with the same C_3', thus from the entropy bound

$$\beta\bar{\nu}_{l,\rho}(\phi_{l,k}) \le C_3'\, l^2 D[\bar{\nu}_{l,\rho}|\bar{\lambda}_{l,\rho},\mathfrak{G}_{k,l}] + \log\bar{\lambda}_{l,\rho}(e^{\beta\phi_{l,k}}). \tag{3.3}$$

Let $\bar{\lambda}_{l,m,\rho}$ denote the restriction of $\bar{\lambda}_{l,\rho}$ to $\mathfrak{F}_{k,m}$. If l is large enough, $z = S'(\rho)$ and $1 < m \le 1 + l/2$, then $d\bar{\lambda}_{l,m,\rho}/d\lambda_z$ is uniformly bounded in view Corollary 5.5 of [17]. Splitting $\phi_{l,k}(h,\alpha)$ into two pieces, by means of the Schwarz inequality we obtain that

$$\log \bar{\lambda}_{l,\rho}(e^{\beta\phi_{k,l}}) \leq \log C_3'' + \frac{1}{2}\log \lambda_z(e^{2\beta\phi_{l,k}}) \quad \text{if } z = S'(\rho)\,.$$

Since $\mathfrak{G}_{k,l}\,\bar{\eta}_{l,k} = 0$, we can integrate (3.3) with respect to ν; notice that $D[\nu|\lambda_z, \mathfrak{G}_{k,l}]$ does not depend on z. Q.E.D.

From now on we are assuming that $l > l_0$ in all statements. On the rate of convergence to local equilibrium we have

Lemma 3.4. *There exists a universal constant C_4 such that*

$$\frac{1}{nl}\sum_{|k|<n}\int_0^t\int \left(\bar{V}_{l,k}' - S'(\bar{\eta}_{l,k})\right)^2 d\mu_s\,ds \leq C_4\,C_{t,n}(\sigma,l)\,,$$

where $C_{t,n}(\sigma,l) := t/l^2 + (l/\sigma)(1 + tn^{-1} + \sigma tn^{-2})$.

Proof. We apply Lemma 3.3 with $h = V'$, $\alpha_j = 1/l$ and $\beta = \beta_0 l$; for brevity we let $\phi = \phi_{l,k}(V',\alpha)$ and $\Phi(\rho,u) = \Phi_{V'}(\rho,u)$. First we show that $\lambda_z(e^{\beta\phi^2}) \leq C_4'$ if $z = S'(\bar{\eta}_{l,k})$ and β_0 is small. Since $e^{\beta\phi^2} = \mathsf{E}_g e^{\zeta\phi}$ if ζ is an $N(0,2\beta)$ variable, and $\lambda_z(e^{\beta\phi}) = \mathsf{E}_g e^{l\Phi(\rho,\zeta/l)}$, the statement follows in the usual way by (3.2). Indeed, $\Phi(\rho,u) \leq \frac{1}{2}\|V''\|\,u^2$ for all $y \in \mathbb{R}$ because $\Phi(z,0) = 0$ and, integrating by parts, we obtain a bound

$$\Phi_u'(\rho,u) = u\int V''(\eta_k)\exp\bigl(uV_k' - uz - \Phi(\rho,u)\bigr)\,d\lambda_z\,, \tag{3.4}$$

that is $|\Phi_u'(\rho,u)| \leq |u|\,\|V''\|$, whence $C_4' = O(\beta_0)$, thus

$$\beta_0 l\int \bigl(\bar{V}_{l,k} - S'(\bar{\eta}_{l,k}\bigr)\,d\mu_s \leq C_3 + C_3 l^2 D[\mu_{s,k,l}|\lambda, \mathfrak{G}_{k,l}] + C_4'\,.$$

Doing summation for k and integrating with respect to time, the statement follows from Lemma 3.1 by subadditivity of D. Q.E.D.

Differences of various block averages are estimated by means of

Lemma 3.5. *Let $\alpha_j \in \mathbb{R}$ for $0 \leq j < l$ such that $\sum \alpha_j = 0$, $\sum \alpha_j^2 \leq 1/l$, and set $\phi_{l,k}(1,\alpha) := \phi_{l,k}(h,\alpha)$ when $h(y) \equiv y$. We have a universal C_5 such that*

$$\frac{1}{nl}\sum_{|k|<n}\int_0^t\int \phi_{l,k}^2(1,\alpha)\,d\mu_s\,ds \leq C_5\,C_{t,n}(\sigma,l)\,.$$

Proof. It is essentially the same as that of Lemma 3.4 with the only difference that at the final step, in the exponent we have

$$\sum_{j=0}^{l-1}\big(F(\alpha_j\zeta)-\alpha_j\zeta F'(z)-F(z)\big)=\frac{1}{2}\sum_{j=0}^{l-1}F''(\gamma_j)\alpha_j^2\zeta^2\le\frac{1}{2l}\|F''\|\zeta^2\,,$$

which completes the proof as $\|F''\|<+\infty$. Q.E.D.

The following lemma is essentially the two-block estimate of [13]. In particular, choosing $l=2r$ we obtain a bound for $(\nabla_r\bar{V}'_{r,k})^2$.

Lemma 3.6. *We have a universal C_6 such that for $2r\le l$,*

$$\frac{1}{nl}\sum_{|k|<n}\int_0^t\int(\bar{V}'_{r,k}-\bar{V}'_{l,k})^2\,d\mu_s\,ds\le C_6\,C_{t,n}(\sigma,l,r)\,,$$

where $C_{t,n}(\sigma,l,r):=t/rl+(l/\sigma)(1+tn^{-1}+\sigma tn^{-2})$.

Proof. This is a consequence of the previous lemma, but integrating by parts on the left hand side, it can directly be estimated by the Dirichlet form via the Schwarz inequality without any reference to LSI, see e.g. [8] for details. Q.E.D.

Now we are in a position to verify all conditions of Proposition 2.1.

§4. The Lax entropy production

We start with the explicit decomposition $X_\varepsilon=N_\varepsilon+M_\varepsilon+X_{a,\varepsilon}+X_{s,\varepsilon}$ of entropy production, see (2.3) and (2.5), (2.6), (2.7), (2.8). To get $X_\varepsilon=Y_\varepsilon+Z_\varepsilon$ as needed in Proposition 2.1, we split some terms into new ones, and each of them will be casted into one of two categories named by Y and Z according to the bound it satisfies. More precisely, a random functional $\Gamma_\varepsilon(\psi)$ is of type Y if for each $\phi\in C_c^2(\mathbb{R})$ we have a random bound $A_\varepsilon(\phi)$ such that $A_\varepsilon(\phi)$ does not depend on ψ,

$$|\Gamma_\varepsilon(\phi\psi)|\le A_\varepsilon(\phi)\|\psi\|_{+1}\quad\text{and}\quad\lim_{\varepsilon\to0}\mathsf{E}A(\phi)=0.$$

Similarly, Γ is of type Z if

$$|\Gamma_\varepsilon(\phi\psi)|\le A_\varepsilon(\phi)\|\psi\|\quad\text{and}\quad\limsup_{\varepsilon\to0}\mathsf{E}A(\phi)<+\infty.$$

In case of terms of type Z we also indicate if the bound does, or does not vanish.

Throughout this section we deal with an entropy pair $(h, J) \in \mathcal{H}_1$ such that h' and h'' are bounded. All calculations are done at the microscopic level, thus the integral mean

$$\psi_k(t) := \frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} \psi(t\varepsilon, k\varepsilon + x)\phi(t\varepsilon, k\varepsilon + x)\, dx \tag{4.1}$$

appears at several places; the notation $H_k(t) := H(\hat{\eta}_{l,k}(t))$ shall also be used for functions $H \in C(\mathbb{R})$ like h, J, h', S'' and so on. $\phi \in C_c^2(\mathbb{R}^2)$ plays an explicit role only in

Lemma 4.1. *The stochastic integral M_ε is of type Y.*

Proof. This is the only case where we estimate the H_{-1} norm in a direct way by using Fourier transform; the underlying generalized function is just

$$m_\varepsilon(t, x) := \sqrt{2\varepsilon\sigma}\, h'(\hat{\rho}_\varepsilon(t, x))\phi(t, x)\partial_t \nabla_\varepsilon^* \hat{\zeta}_\varepsilon(t, x)\,.$$

In view of $|M_\varepsilon(\psi, h)| \le \|m_\varepsilon\|_{-1}\, \|\psi\|_{+1}$, we have to show that

$$\lim_{\varepsilon \to 0} \mathsf{E}\|m_\varepsilon\|_{-1}^2 = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathsf{E}|\tilde{m}_\varepsilon(\tau, \omega)|^2}{1 + \tau^2 + \omega^2}\, d\tau\, d\omega = 0\,, \tag{4.2}$$

where $\tilde{m}_\varepsilon$ denotes the Fourier transform of m_ε. In microscopic variables

$$\tilde{m}_\varepsilon(\tau, \omega) = \frac{\varepsilon\sqrt{2\sigma}}{l} \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t, \tau, \omega) h_k'(t) (d\bar{w}_{l,k-1} - d\bar{w}_{l,k+l-1})\,,$$

where $\psi_k(t, \tau, \omega)$ is defined by (4.1) with $\psi = (2\pi)^{-1} \exp(\imath t\tau + \imath x\omega)$. The sum of the integrands can be rewritten as a sum like $\sum \bar{\xi}_{l,k}\, dw_k$, thus a simple Ito calculus and $(\bar{\xi}_{l,k})^2 \le (\bar{\xi^2})_{l,k}$ result in

$$\begin{aligned} \mathsf{E}|\tilde{m}_\varepsilon(\tau, \omega)|^2 &\le \frac{4\sigma\varepsilon^2}{l^2} \sum_{k \in \mathbb{Z}} \int_0^\infty |\psi_k(t)\, h_k'(t)|^2\, dt \\ &\le \frac{4\sigma\varepsilon^2}{l^2} \|h'\|^2 \sum_{k \in \mathbb{Z}} \int_0^\infty |\psi_k(t)|^2\, dt\,. \end{aligned}$$

Of course, $\psi_k(t)$ is bounded, and it is zero if one of $|\varepsilon k|$ or εt exceeds some threshold depending on the support of ϕ. For large values of $|\omega|$ another bound of

$$\psi_k(t, \tau, \omega) = \frac{e^{\imath\tau t + \imath\omega k}}{2\pi\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} e^{\imath x\omega}\, \phi(\varepsilon t, \varepsilon k + x)\, dx$$

is needed. Integrating by parts we get

$$\begin{aligned}|\psi_k(t,\tau,\omega)| &\le \frac{1}{2\pi\varepsilon\omega}\left|\int_{-\varepsilon/2}^{\varepsilon/2} e^{\imath x\omega}\,\phi'_x(\varepsilon t,\varepsilon k+x)\,dx\right| \\ &+\frac{1}{2\pi\varepsilon\omega}\left|e^{\imath\varepsilon\omega/2}\phi(\varepsilon t,\varepsilon k+\varepsilon/2)-e^{-\imath\varepsilon\omega/2}\phi(\varepsilon t,\varepsilon k-\varepsilon/2)\right| \\ &\le \frac{3}{4\pi|\omega|}\,\|\phi'_x\|+\frac{|\sin(\omega\varepsilon/2)|}{2\pi\varepsilon|\omega|}\,\|\phi\|\,,\end{aligned}$$

thus we have a constant, K_1 depending only on ϕ such that

$$|\psi_k(t,\tau,\omega)|^2 \le K_1\Psi_\varepsilon(\omega)\,, \quad \text{where} \quad \Psi_\varepsilon(\omega):=\min\left\{1,(\varepsilon\omega)^{-2}\right\}$$

and $0<\varepsilon<1$. Comparing the bounds above, we see that

$$\mathsf{E}|\tilde{m}_\varepsilon(\tau,\omega)|^2 \le K_2\|h'\|^2\,\frac{\sigma}{l^2}\,\Psi_\varepsilon(\omega)\,,$$

thus integrating (4.2) with respect to τ,

$$\mathsf{E}\|m_\varepsilon\|_{-1}^2 \le \frac{K_3\sigma}{l^2}\int_{-\infty}^{\infty}\frac{\Psi_\varepsilon(\omega)\,d\omega}{\sqrt{1+\omega^2}}$$

follows immediately, where K_3 is a new constant depending only on ϕ and $\|h'\|$. Integrating over the domain $|\omega|<1/\varepsilon$, the trivial bound $\Psi_\varepsilon(\omega)\le 1$ is sufficient, while $\Psi_\varepsilon(\omega)\le(\varepsilon\omega)^{-2}$ is used in the opposite case to conclude

$$\mathsf{E}|\tilde{m}_\varepsilon(\tau,\omega)|^2 \le K_4\,\frac{\sigma}{l^2}\,(1-\log\varepsilon)\,.$$

In view of (2.2) and its consequences we have $\sigma=o(l^{3/2})$ and $1/\varepsilon=o(l^2)$, thus the right hand side vanishes as $\varepsilon\to 0$. Q.E.D.

From now on we may suppress the dependence of our functionals on ϕ. In practice this simply means that we put $\phi\equiv 1$ and suppose that the support of ψ is contained in a rectangle $(-1,T)\times(-L,L)$, thus we need the estimates of Section 3 for $n<L/\varepsilon$ and $t<T/\varepsilon$ only. Introduce

$$Q_\varepsilon^* := \frac{\varepsilon}{l}\sum_{|k|<L/\varepsilon}\int_0^{T/\varepsilon} Q_k(t,l)\,dt\,, \tag{4.3}$$

where $Q_k(t,l):=(l\nabla_l\bar{\eta}_{k,l})^2+(\hat{\eta}_{l,k+l}-\hat{\eta}_{l,k})^2+(\hat{\eta}_{l,k}-\bar{\eta}_{l,k})^2+(l\nabla_l\bar{V}'_{l,k})^2$, and

$$Z_\varepsilon^* := \frac{\varepsilon}{l}\sum_{|k|<L/\varepsilon}\int_0^{T/\varepsilon}\left(\bar{V}'_{l,k}(\eta(t))-S'(\bar{\eta}_{l,k}(t)\right)^2 dt\,. \tag{4.4}$$

Moreover, set $C_\varepsilon(\sigma,l) := C_{t,n}(\sigma,l)$ when $t = T/\varepsilon$ and $n = L/\varepsilon$. In the rest of the paper we assume (2.2), thus $C_\varepsilon = O(l/\sigma)$ goes to 0 as $\varepsilon \to 0$. In view of the a priori bounds, $\mathsf{E}Q^*$ and $\mathsf{E}Z_\varepsilon^*$ are of order $C_\varepsilon(\sigma,l)$.

It is a bit surprising that a two-block lemma is needed to treat N_ε.

Lemma 4.2. *The numerical error N_ε is of type Y.*

Proof. Let $\psi_k(t)$ be as in (4.1) with $\phi \equiv 1$, then

$$N_\varepsilon(\psi,h) = \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \big(\widetilde{\nabla}_1\psi_k - \varepsilon\, \widetilde{\nabla}_{\varepsilon/2}\psi(t\varepsilon,k\varepsilon)\big) J_k(t)\, dt\,.$$

Since $\widetilde{\nabla}_1 = \nabla_1 - (1/2)\Delta_1$, the integrand turns into $(1/2)(\nabla_1\psi_k)\nabla_1 J_k + \varphi_k \nabla_1^* J_k$, where $\varphi_k := \psi_k - \psi(t\varepsilon, k\varepsilon - \varepsilon/2)$ is an integral of ψ_x'. By the Schwarz inequality

$$\varphi_k(t) = \frac{1}{2\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} (\varepsilon - 2x)\psi_x'(t\varepsilon, k\varepsilon + x)\, dx = O(\sqrt{\varepsilon})\|1_{\varepsilon,k}\,\psi_x'(t\varepsilon,\cdot)\|_2\,,$$

where $1_{\varepsilon,k}(x)$ is the indicator of the interval $(k\varepsilon - \varepsilon/2, k\varepsilon + \varepsilon/2)$; the L^2 norm refers to space. Similarly,

$$\nabla_1\psi_k(t) = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} (\varepsilon - |x|)\psi_x'(t\varepsilon, k\varepsilon + x + \varepsilon/2)\, dx\,,$$

thus $\nabla_1\psi_k$ satisfies the same bound that φ_k does.

On the other hand, $\nabla_1^* J_k = J'(\gamma_k)\nabla_l^*\bar\eta_{l,k+l-1}$ with some intermediate value γ_k. Since $J' = h'S''$ is bounded, separating φ_k and $\nabla_1^* J_k$ by means of the Schwarz inequality, and doing the same with $\nabla_1\psi_k$ and $\nabla_1 J_k$, we obtain that $N_\varepsilon^2 = O(\varepsilon)\,\|\psi_x'\|_2^2\, Q_\varepsilon^*$, that is N_ε is of type Y, and $\sqrt{\varepsilon l/\sigma}$ is its order. Q.E.D.

The next step is the only one where LSI is really needed.

Lemma 4.3. *The asymmetric functional, $X_{a,\varepsilon}$ reads as $X_{a,\varepsilon} = Y_{a,\varepsilon} + Z_{a,\varepsilon} + Q_{a,\varepsilon}$, where $Q_{a,\varepsilon}$ and $Z_{a,\varepsilon}$ are of type Z with a vanishing bound, $Y_{a,\varepsilon}$ is of type Y.*

Proof. Using earlier notation we have

$$X_{a,\varepsilon}(\psi,h) = \frac{\varepsilon}{2}\sum_{k\in\mathbb{Z}}\int_0^\infty (\psi_k + \psi_{k+1})(\nabla_1 J_k - h_k'\,\nabla_l \bar V_{l,k}')\, dt\,,$$

and $\nabla_1 J_k = h_k'\, S_k''\, \nabla_l\bar\eta_{l,k} + \frac{1}{2}\, J''(\gamma_k)(\nabla_l\bar\eta_{l,k})^2$ with some intermediate value γ_k. Moreover, $S_k''\,\nabla_l\,\bar\eta_{l,k} = \nabla_l S'(\bar\eta_{l,k}) + S'''(\tilde\eta_k'')(\hat\eta_{l,k} - \tilde\eta_k')\nabla_l\bar\eta_{l,k}$, where $\tilde\eta_k'$ is a convex combination of $\bar\eta_{l,k+l}$ and $\bar\eta_{l,k}$, i.e. $|\hat\eta_{l,k} - \tilde\eta_k'| \le$

$|\hat\eta_{l,k} - \bar\eta_{l,k}| + |\hat\eta_{l,k} - \bar\eta_{l,k-l}|$. Summarizing the calculations above, we get $X_{a,\varepsilon} = X^*_{a,\varepsilon} + Q_{a,\varepsilon}$, where

$$X^*_{a,\varepsilon} := \frac{\varepsilon}{2} \sum_{k\in\mathbb{Z}} \int_0^\infty (\psi_k + \psi_{k+1})\, h'_k\, \nabla_l (S'(\bar\eta_{l,k}) - \bar V'_{l,k})\, dt\,,$$

while the remainder, $Q_{a,\varepsilon}$ is a bilinear form of differences of block averages of size at most $2l+1$. Since $J'' = h''S'' + h'S'''$, and S''' is bounded in view of Lemma 5.1 in [17], the coefficients of $Q_{a,\varepsilon}$ are all uniformly bounded, consequently $Q_{a,\varepsilon} = O(\|\psi\|)\, Q^*_\varepsilon$. This means that $Q_{a,\varepsilon}$ is of type Z with a vanishing order of $C_\varepsilon(\sigma, l) = O(l/\sigma)$.

On the other hand, from $\nabla^*_l(\xi_k\xi'_k) = (\nabla^*_l\xi_k)\xi'_k + \xi_{k-l}\nabla^*_l\xi'_k$ we get $X^*_{a,\varepsilon} = Y_{a,\varepsilon} + Z_{a,\varepsilon}$, where

$$Y_{a,\varepsilon} := \frac{\varepsilon}{2} \sum_{k\in\mathbb{Z}} \int_0^\infty (\nabla^*_l\psi_k + \nabla^*_l\psi_{k+1})\, h'_k\, (S'(\bar\eta_{l,k}) - \bar V'_{l,k})\, dt\,,$$

$$Z_{a,\varepsilon} := \frac{\varepsilon}{2} \sum_{k\in\mathbb{Z}} \int_0^\infty (\psi_{k-l} + \psi_{k+1-l})\, (\nabla^*_l h'_k)\, (S'(\bar\eta_{l,k}) - \bar V'_{l,k})\, dt\,.$$

From the estimate of Lemma 4.2 for $\nabla_1\psi$ it follows by convexity that $(\nabla_l\psi_k(t))^2 = O(l\varepsilon)\|1_{l\varepsilon,k}\,\psi'_x(t\varepsilon,\cdot)\|_2^2$. Separating the space gradients of ψ from $h'\,(S' - \bar V'_l)$ by means of the Schwarz inequality, we obtain that $|Y_{a,\varepsilon}|^2 \le \varepsilon^2\|h'\|\,\|\psi'_x\|_2^2\, Z^*_\varepsilon$, thus $Y_{a,\varepsilon}$ is of type Y, and $\sqrt{l\varepsilon}\, C_\varepsilon^{1/2}(\sigma, l)$ is the order of its bound. Finally, $\nabla^*_l h'_k = h''(\gamma'_k)\nabla^*_l\hat\eta_{l,k}$, whence $|Z_{a,\varepsilon}|^2 \le \|h''\|\,\|\psi\|\, Q^*_\varepsilon\, Z^*_\varepsilon$, that is $Z_{a,\varepsilon}$ is of type Z with a vanishing bound of order $C_\varepsilon(\sigma, l)$. Q.E.D.

The symmetric form decomposes as $X_{s,\varepsilon} = X_{s1,\varepsilon} + X_{s2,\varepsilon}$, where

$$X_{s1,\varepsilon}(\psi, h) : = -\sigma\varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \nabla_1(\psi_k h'_k)\, (\nabla_l \bar V'_{l,k})\, dt\,,$$

$$X_{s2,\varepsilon}(\psi, h) : = \frac{\sigma\varepsilon}{l^2} \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k h''_k\, (d\bar w_{l,k-1} - d\bar w_{l,k+l-1})^2 = \frac{2\sigma\varepsilon}{l^3} \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k h''_k\, dt\,.$$

In case of $X_{s1,\varepsilon}$ we write $\nabla_1(\psi_k h'_k) = \psi_k \nabla_1 h'_k + h'_{k+1} \nabla_1 \psi_k$ to get $X_{s1,\varepsilon} = Y_{s,\varepsilon} - Z_{s,\varepsilon}$, where

$$Z_{s,\varepsilon} := \sigma\varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k \,(\nabla_1 h'_k)\, \nabla_l \bar{V}'_{l,k}\, dt\,,$$

$$Y_{s,\varepsilon} := -\sigma\varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty (\nabla_1 \psi_k) h'_{k+1} \nabla_l \bar{V}'_{l,k}\, dt\,.$$

The symmetric part of entropy production is handled by means of

Lemma 4.4. *We have $X_{s,\varepsilon} = Y_{s,\varepsilon} + X_{s2,\varepsilon} - Z_{s,\varepsilon}$, where $Y_{s,\varepsilon}$ is of type Y, $X_{s2,\varepsilon}$ is of type Z, and $\sigma/\varepsilon l^3$ is the order of its bound. $Z_{a,\varepsilon}$ is also of type Z, but its bound does never vanish.*

Proof. Since h'' is bounded, $X_{s2,\varepsilon} = \|\psi\|\, O(\sigma/\varepsilon l^3)$ is of type Z. From $2xy \le x^2 + y^2$, and $\nabla_l h'_k = h''(\gamma''_k) \nabla_l \bar{\eta}_{l,k}$ we get

$$|Z_{s,\varepsilon}| \le \frac{\sigma}{2l} \|h''\|\, \|\psi\| (Q^*_\varepsilon + Z^*_\varepsilon)\,,$$

see (4.3) and (4.4) for the definition of Q^*_ε and Z^*_ε. Therefore $Z_{s,\varepsilon}$ is of type Z, and the bound does not vanish. Finally, applying the Schwarz inequality as we did many times before, we have

$$|Y_{s,\varepsilon}|^2 \le \frac{\sigma^2 \varepsilon}{l} \|h'\|^2\, \|\psi'_x\|_2^2\, Q^*_\varepsilon\,,$$

consequently $Y_{s,\varepsilon}$ is of type Y as $\varepsilon\sigma^2 l^{-1}\, C_\varepsilon(\sigma, l) = O(\varepsilon\sigma)$. Q.E.D.

§5. Completion of the proofs

Proposition 2.1, is a more or less direct consequence of the results of Section 4.

Proof of Proposition 2.1: Suppose first that $(h_i, J_i) \in \mathcal{H}_\alpha$, where $0 < \alpha < 1$, and h', h'' are bounded, then $\limsup \mathsf{E}\|\phi\rho_\varepsilon\|_2^2 < +\infty$ implies $h, J \in L^p(\mathbb{R}^2_+)$ with some $p > 2$. More precisely, the distributions of $h_i(\hat{\rho}_\varepsilon)$ and $J_i(\hat{\rho}_\varepsilon)$ are tight in the weak topology of $L^p_{\mathrm{loc}}(\mathbb{R}^2_+)$. Similarly, the distributions of the functionals Y_ε and Z_ε are tight with respect to the weak local topology of H_{-1} and the space of measures, respectively. In view of the Skorohod embedding theorem, see Theorem 1.8 in Chapter 3 of [5], we can realize the associated weak convergence of probability measures as a.s. convergence on a suitably constructed probability space. In this setting the theorems of Tartar and Murat apply directly, so we have Tartar factorization for entropy pairs from $\mathcal{H}_\alpha$. The final statement follows by a direct approximation procedure. Q.E.D.

Tartar's factorization property is the input of

Lemma 5.1. *Let* $h_1(\rho) := \rho$, $J_1(\rho) := S'(\rho)$, $h_2(\rho) := S'(\rho)$ *and define* J_2 *by* $J_2(0) = 0$ *and* $J_2'(\rho) := S''^2(\rho)$. *If this couple of entropy pairs satisfies* (1.10), *then* $\theta_{t,x}$ *is almost everywhere a Dirac measure.*

Proof. The trivial case of a quadratic V can be excluded, thus there is no such interval where S'' is constant because S is analytic. Rearranging (1.10) we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(u,v)\,\theta_{t,x}(du)\,\theta_{t,x}(dv) = 0 \quad \text{a.s. on } \mathbb{R}_+^2\,,$$

where $Q(u,v) := (u-v)\big(J_2(u) - J_2(v)\big) - \big(S'(u) - S'(v)\big)^2$. Since

$$J_2(u) - J_2(v) = (u-v)\int_0^1 S''^2(tu + (1-t)v)\,dt\,,$$

$$S'(u) - S'(v) = (u-v)\int_0^1 S''(tu + (1-t)v)\,dt\,,$$

$Q(u,v) > 0$ follows by the Schwarz inequality if $u \neq v$, which proves the Dirac property of θ. Q.E.D.

As a consequence, we have (1.2) with probability one with respect to any weak limit distribution of $\hat\rho_\varepsilon$. In view of Lemma 3.5 the same statement holds also true for the usual averages $\bar\rho_\varepsilon$ defined by $\bar\rho_\varepsilon(t,x) := \bar\eta_k(t/\varepsilon)$ if $|x - k\varepsilon| < \varepsilon/2$, and even $l = o(1/\varepsilon)$ is allowed; the lower bound $l \geq (\sigma/\varepsilon)^{1/3}$ is the relevant one.

To prove Proposition 2.2, we have to show that the contribution of terms $(\nabla_1 h(\hat\eta_{l,k}))\nabla_l \bar V'_{l,k}$ is not negative if h is convex. Despite of Lemma 3.4, this is not quite obvious. Fortunately, the Lax-Kružkov inequality does not require uniform bounds as compensated compactness does, weak limiting arguments are sufficient. Nevertheless, convexity of V seems to be essential at this point.

Proof of Proposition 2.2: Since V is convex by assumption, following [21] we can exploit the attractiveness of the process, see also [15] for some technical details. Let ζ denote the equilibrium process with initial distribution λ_z such that $F'(z) = c$. The original process, η is coupled to ζ simply by identifying the Wiener processes in their stochastic equations (1.1); the initial distribution is $\mu_{\varepsilon,0} \times \lambda_z$. It is remarkable that this coupled process admits a comparison principle: the set

$$\{(\eta,\zeta) : (\eta_{k+1} - \eta_k)(\zeta_{k+1} - \zeta_k) \geq 0\ \forall k \in \mathbb{Z}\}$$

is preserved by time. Introduce

$$W_\varepsilon(\eta,\zeta,\psi) := \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_t'(t,\varepsilon k)|\eta_k(t/\varepsilon) - \zeta_k(t/\varepsilon)|\,dx\,dt\,,$$

$$W_\varepsilon^*(\eta,\zeta,\psi) := \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_x'(t,\varepsilon k)|V'(\eta_k(t/\varepsilon)) - V'(\zeta_k(t/\varepsilon))|\,dx\,dt\,,$$

$$H_\varepsilon(\eta,\psi) := \int_0^\infty \int_{-\infty}^\infty \psi_t'(t,x)|\hat\rho_\varepsilon(t,x) - c|\,dx\,dt\,,$$

$$H_\varepsilon^*(\eta,\psi) := \int_0^\infty \int_{-\infty}^\infty \psi_t'(t,x)|S'(\hat\rho_\varepsilon(t,x)) - S'(c)|\,dx\,dt\,;$$

we have to show that for all $c \in \mathbb{R}$ and $0 \le \psi \in C_c^1(\mathbb{R}_+^2)$ we have

$$\lim_{\varepsilon\to 0} \big(W_\varepsilon(\eta,\zeta,\psi) - H_\varepsilon(\eta,\psi)\big) = 0\,, \tag{5.1}$$

$$\lim_{\varepsilon\to 0} \big(W_\varepsilon^*(\eta,\zeta,\psi) - H_\varepsilon^*(\eta,\psi)\big) = 0\,, \tag{5.2}$$

$$\liminf_{\varepsilon\to 0} \big(W_\varepsilon(\eta,\zeta,\psi) + W_\varepsilon^*(\eta,\zeta,\psi)\big) \ge 0 \tag{5.3}$$

in the sense of stochastic convergence, see Section 3 of [21]; the crucial point is (5.3). To prove it, observe first that $\eta_k - \zeta_k$ is differentiable, and $V_{k-1}' - V_{k+1}' = 2V_{k-1}' - 2V_k' - V_{k-1}' - V_{k+1}' + 2V_k'$, thus

$$\begin{aligned}\partial_t|\eta_k - \zeta_k| &= \operatorname{sign}(\eta_k - \zeta_k)\big(V'(\eta_{k-1}) - V'(\zeta_{k-1}) - V'(\eta_k) + V'(\zeta_k)\big)\\ &+ (\sigma - 1/2)\operatorname{sign}(\eta_k - \zeta_k)\big(V'(\eta_{k-1}) - V'(\zeta_{k-1}) - V'(\eta_k) + V'(\zeta_k)\big)\\ &+ (\sigma - 1/2)\operatorname{sign}(\eta_k - \zeta_k)\big(V'(\eta_{k+1}) - V'(\zeta_{k+1}) - V'(\eta_k) + V'(\zeta_k)\big)\,.\end{aligned}$$

Let $\chi_k(\eta,\zeta) := \operatorname{sign}(\eta_k - \zeta_k)\operatorname{sign}(\eta_{k+1} - \zeta_{k+1})$, by an elementary computation

$$\begin{aligned}\partial_t|\eta_k - \zeta_k| \le \chi_{k-1}&|V'(\eta_{k-1}) - V'(\zeta_{k-1})| - \chi_k|V'(\eta_k) - V'(\zeta_k)|\\ &+ (\sigma - 1/2)\,\chi_{k-1}\big(|V'(\eta_{k-1}) - V'(\zeta_{k-1})| - |V'(\eta_k) - V'(\zeta_k)|\big)\\ &- (\sigma - 1/2)\,\chi_k\big(|V'(\eta_k) - V'(\zeta_k)| - |V'(\eta_{k+1}) - V'(\zeta_{k+1})|\big)\\ &- (\sigma - 1/2)(2 - \chi_{k-1} - \chi_k)|V'(\eta_k) - V'(\zeta_k)|\,.\end{aligned}$$

Hence by rearranging the sums we get

$$W_\varepsilon + W_\varepsilon^* \ge R_\varepsilon(\eta,\zeta,\psi) + \varepsilon\sum_{k\in\mathbb{Z}} \int_0^\infty W_k(t/\varepsilon)\,dt\,,$$

where R_ε is a numerical error term, t is the macroscopic time, and

$$\begin{aligned}W_k(t) :&= \psi'_x(\varepsilon t, \varepsilon k)(\chi_k - 1)|V'(\eta_k) - V'(\zeta_k)|\\ &+ (\sigma - 1/2)\psi'_x(t, \varepsilon k)\,\chi_k\,\nabla_1|V'(\eta_k) - V'(\zeta_k)|\\ &+ (1/\varepsilon)(\sigma - 1/2)\psi(\varepsilon t, \varepsilon k)(1 - \chi_k)|V'(\eta_k) - V'(\zeta_k)|\,.\end{aligned}$$

Since $\psi \in C^2(\mathbb{R}^2)$ may be assumed, R_ε goes to zero as $\varepsilon \to 0$, and $\chi_k\nabla_1$ on the second lime above can be replaced with $(\chi_k - 1)\nabla_1$, we see that the last nonnegative terms dominate the rest. Indeed, $\varepsilon|\psi'_x| = o(\psi)$ is certainly true if $\psi > 0$ vanishes in a suitable way as $|x| \to \infty$, whence the general case follows by a direct approximation procedure.

The proofs of (5.1) and (5.2) also follow [21]; but they turn out to be much simpler in our case. The formal generator of the coupled process reads as $\mathfrak{L}_{\eta,\zeta} := \mathfrak{L}_{0,\eta} + \mathfrak{L}_{0,\zeta} + \sigma\,\mathfrak{G}_{\eta,\zeta}$, where $\mathfrak{L}_{0,\eta}$ and $\mathfrak{L}_{0,\zeta}$ are identical copies of $\mathfrak{L}_0$ acting on the η and ζ components, respectively, while $\mathfrak{G}_{\eta,\zeta}$ is the generator of the coupled process (η, ζ) defined by

$$d\eta_k = \Delta_1 V'(\eta_k)\,dt + \sqrt{2}\nabla_1^*\,dw_k\,, \quad d\zeta_k = \Delta_1 V'(\zeta_k)\,dt + \sqrt{2}\nabla_1^*\,dw_k$$

with identical Wiener processes for both systems. In view of the Kolmogorov equation, for smooth cylinder functions

$$\mathsf{E}\phi(\eta(t), \zeta(t)) = \mathsf{E}\phi(\eta(0), \zeta(0)) + \mathsf{E}\int_0^t \mathfrak{L}_{\eta,\zeta}\phi(\eta(s), \zeta(s))\,ds\,.$$

Let $\bar{\nu}_\varepsilon$ denote the time average of the joint distribution of η and ζ from $t = 0$ to $t = 1/\varepsilon$. In view of the L^2 moment condition coming from Lemma 3.2, this family is tight, thus dividing the Kolmogorov equation by σ/ε, we see that its weak limit points are all stationary measures for the coupled process generated by $\mathfrak{G}_{\eta,\zeta}$. Performing a simultaneous averaging also in space, we obtain translation invariant limit distributions $\bar{\nu}$ that are stationary with respect to $\mathfrak{G}_{\eta,\zeta}$, and also satisfy the moment conditions $\bar{\nu}(\eta_k^2 + \zeta_k^2) = K < +\infty$. These statements follow immediately also from Theorem 1 of [8] without any averaging in space. The evaluation of W and W^* should be based on such a joint distribution $\bar{\nu}$, see [13] and [22].

To prove that $\chi_k = 1$ $\bar{\nu}$-a.s. for all $k \in \mathbb{Z}$, consider now the coupled process defined by $\mathfrak{G}_{\eta,\zeta}$, with $\bar{\nu}$ as its initial distribution. By elementary calculation we get

$$\begin{aligned}\partial_t|\eta_k - \zeta_k| &= -\nabla_1\mathrm{sign}(\eta_k - \zeta_k)\cdot\nabla_1\big(V'(\eta_k) - V'(\zeta_k)\big)\\ &+ \mathrm{sign}(\eta_{k+1} - \zeta_{k+1})\,\nabla_1\big(V'(\eta_k) - V'(\zeta_k)\big)\\ &- \mathrm{sign}(\eta_k - \zeta_k)\,\nabla_1\big(V'(\eta_{k-1}) - V'(\zeta_{k-1})\big)\,,\end{aligned}$$

where both sides are of mean zero with respect to $\bar{\nu}$ because of its stationarity. Summing for $k \in (-n, n)$ we see that the last two terms cancel each other, only two of them survives at the boundary. Therefore the translation invariance of $\bar{\nu}$ implies

$$\int \nabla_1 \mathrm{sign}(\eta_k - \zeta_k) \cdot \nabla_1 \big(V'(\eta_k) - V'(\zeta_k)\big)\, d\bar{\nu} = 0$$

for all $k \in \mathbb{Z}$, that is $\mathrm{sign}(\eta_k - \zeta_k)$ is a constant $\bar{\nu}$-a.s. This means that $\bar{\nu}[\chi_k = 1] = 1$ for all $k \in \mathbb{Z}$, thus we can get rid of the absolute values under the sums in the expressions of W and W^*. First we replace η and ζ in W_ε and W_ε^* with their large microscopic block averages $\bar{\eta}_r$ and $\bar{\zeta}_r$. Letting $\varepsilon \to 0$ first, and $r \to +\infty$ at the second step, we get c as the limit of $\bar{\zeta}_r$. Finally, Lemma 3.6 allows us to replace $\bar{\eta}_r$ with $\bar{\eta}_l$, where $l = l(\varepsilon)$ is the intermediate block size of (2.2). The replacement of $V'(\eta_k)$ with $S'(\bar{\eta}_{l,k})$ is the same, thus we can pass to (1.11) along subsequences. Q.E.D.

Proof of Theorem 1.1: In view of Skorohod's embedding, Lemma 5.1 and the a priori bounds, the empirical processes, $\hat{\rho}_\varepsilon$ and $\bar{\rho}_\varepsilon$ converge almost surely, and also in $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ to the same $\rho \in L^2(\mathbb{R}^2_+)$ along subsequences. At the same time, $V'(\rho_\varepsilon)$ has the same weak limits as $S'(\bar{\rho}_\varepsilon)$ does, thus we have convergence to the set of weak solutions.

The uniqueness part is now a direct consequence of Proposition 2.2 and weak uniqueness of entropy solutions, see Kružkov's result, Theorem 2.3.5 in [25]. Although the proof there is written for bounded solutions only, the essential condition is bounded propagation, that is $\|S''\| < +\infty$. By means of the local L^2 bound we have, the argument extends to our case without any essential change. On the other hand, we have already derived from Proposition 2.1 and Lemma 5.1 that the measure solutions involved in (1.11) are all weak solutions, thus we have (1.4), too. Therefore any limit distribution of the empirical process $\hat{\rho}_\varepsilon$ is concentrated on the unique entropy solution specified by its initial value. In this way we have shown that if $\varepsilon \to 0$ then

$$\hat{R}_\varepsilon(\psi) := \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\hat{\rho}_\varepsilon(t,x)\, dx\, dt$$

converges in probability for each $\psi \in C_c(\mathbb{R}^2)$ to $R(\psi)$ defined in Theorem 1.1. However, $R_\varepsilon(\psi)$ has the same limit. Q.E.D.

Concluding remarks: We are trying to present a brief and heuristic description of situations of hyperbolic scaling in which the method proposed here might apply, several principal open problems are also mentioned. We consider a microscopic Markov evolution generated by

$\mathfrak{L} = \mathfrak{L}_0 + \sigma(\varepsilon)\mathfrak{G}$ such that both $\mathfrak{L}_0$ and $\mathfrak{G}$ are Markov generators, and the conservative observables and the associated (equilibrium) Gibbs states of $\mathfrak{G}$ are all conserved also by $\mathfrak{L}_0$. The main component, $\mathfrak{L}_0$ is asymmetric (but not necessarily antisymmetric), while $\mathfrak{G}$ is symmetric with respect to the equilibrium states. The scaling parameter $\varepsilon > 0$ denotes the macroscopic unit of distance in space, $\sigma(\varepsilon) > 0$ is interpreted as the coefficient of microscopic viscosity, $\sigma(\varepsilon) \to +\infty$ and $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

First of all we are assuming that $\mathfrak{L}_0$ admits Euler scaling with a resulting hyperbolic system of macroscopic conservation laws, that is we speed up time by a factor of $1/\varepsilon$, see e.g. [27] for a class of such models. In the absence of the symmetric stabilization $\sigma\mathfrak{G}$, these equations can be derived in a smooth regime only. In general, there is a good reason to expect that the effect of $(1/\varepsilon)\sigma(\varepsilon)\mathfrak{G}$ diminishes as $\varepsilon \to 0$ because $\varepsilon^{-2}\mathfrak{G}$ is the proper scaling of the symmetric $\mathfrak{G}$. In other words,

$$\varepsilon^{-1}\mathfrak{L} = \varepsilon^{-1}\mathfrak{L}_0 + (\varepsilon\sigma(\varepsilon))\,\varepsilon^{-2}\mathfrak{G}$$

resembles the scheme of small viscosity limit as $\varepsilon \to 0$; $\varepsilon\sigma(\varepsilon)$ is the coefficient of macroscopic viscosity.

Independently of the number of conservation laws, once we have LSI for $\mathfrak{G}$, there is a good chance to derive Tartar's factorization property for the limiting Young measures; $\varepsilon\sigma^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ seems to be a general condition at this point, see [9,10,11]. It is not clear this time if this condition could be relaxed, or not. In the case of a single conservation law Tartar factorization is usually sufficient for the identification of measure solutions as weak solutions by using an argument like that of Lemma 5.1. The problem of two conservation laws is more delicate, a very nice model is discussed in [11]. In other cases we have to do something more for proving that measure solutions are weak solutions. Although there is a general theory of hyperbolic and genuinely nonlinear systems of two conservation laws in one space dimension initiated by DiPerna, additional difficulties emerge when we are working on stochastic models. Indeed, this theory requires at the very beginning that the limiting Young measure is compactly supported. Moreover, most physically motivated models have some singularities in the phase space of the macroscopic equations, general methods fail at such points. In PDE theory these difficulties are ruled out by restricting the initial values to singularity-free compact invariant regions, if any. However, it is not easy to establish the existence of such invariant regions in the case of microscopic systems. Coupling is an effective tool, but attractive evolutions do not allow two conservation laws.

Anyway, compensated compactness yields convergence of the empirical process to a set of weak solutions in several cases, so the next question is the uniqueness of the limit. If we have a single conservation law, the Lax–Kružkov entropy condition is sufficient for uniqueness, and coupling based on attractiveness is not the only way of proving it. For example, if $\mathfrak{G}$ is acting on the conservative observables like a discrete Laplacian, that is a *linear elliptic operator*, then the derivation of the Lax inequality (1.3) is only a question of direct computations. This is the case when $\mathfrak{L}_0$ describes interacting exclusions because then $\mathfrak{G}$ can be chosen as the generator of stirring, see [10,11]. There is a conflict of $\mathfrak{L}_0$ and $\mathfrak{G}$ if the cardinality of the individual phase space is bigger than three. For instance, the easy way mentioned above is only available for the trivial, linear Ginzburg–Landau model. It is not clear if attractiveness of $\mathfrak{G}$ were sufficient for the Lax–Kružkov inequality. Uniqueness for two conservation laws is certainly very hard, even in the simplest cases Oleinik type entropy conditions were needed for the Riemann invariants. The derivation of such one-sided uniform bounds on space gradients is really problematic for stochastic models.

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