Advanced Studies in Pure Mathematics 39, 2004
Stochastic Analysis on Large Scale Interacting Systems
pp. 1–27

Large Deviations for the Asymmetric Simple Exclusion Process

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Abstract.

We explain the large deviation behavior of the totally asymmetric simple exclusion process in one dimension.

§1. Introduction

So far, large deviations from hydrodynamic scaling have been worked out only for systems under diffusive scaling. Large deviation results are presented here for the Totally Asymmetric Simple Exclusion Process or TASEP in one dimension. This work was carried out by Leif Jensen [2] in his PhD dissertation submitted to New York University in the year 2000 and is available at the website

http://www.math.columbia.edu/~jensen/thesis.html

We will present here a detailed sketch of the derivation of the upper bound and a rough outline of how the lower bound is established.

§2. Hydrodynamic limit of TASEP

The Model.

We have a particle system on the integers $\mathbf{Z}$ or (in the periodic case) on $\mathbf{Z}_N$, the integers moduo N. The configuration is $\eta = \{\eta_x : x \in \mathbf{Z}\}$ or $\{\eta_x : x \in \mathbf{Z}_N\}$. The evolution of $\eta(t) = \{\eta_x(t)\}$ is governed by the generator

$$(\mathcal{L}f)(\eta) = \sum_x \eta_x(1-\eta_{x+1})[f(\eta^{x,x+1}) - f(\eta)]$$

Received December 31, 2002.
Revised May 19, 2003.
Partially Supported by NSF grant 0104343.

where

$$\eta_z^{x,y} = \begin{cases} \eta_z & if \quad z \neq x, y \\ \eta_y & if \quad z = x \\ \eta_x & if \quad z = y \end{cases}$$

This corresponds to the process where the particles independently wait for an exponential time and then jump one step to the right if the site is free. Otherwise they wait for another exponential time. All the particles are doing this simultaneously and independently.

The Scaling.

For each N we consider an initial configuration $\eta_{x,N}$, that may or may not be random. We consider these models for $N \to \infty$. Assume that for some deterministic density function $\rho_0(\xi)$, $0 \le \rho_0(\cdot) \le 1$, and every test function $J(\cdot)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum J(\frac{x}{N})\eta_{x,N} = \int J(\xi)\rho_0(\xi)d\xi$$

The limit is taken in probability in the random case. The class of test functions are continuous functions with compact support in $\mathbf{R}$, if we started with $\mathbf{Z}$ and the periodic unit interval $\mathbf{S}$, if we started with $\mathbf{Z}_N$.

Time is speeded up by a factor of N, i.e. the process is viewed at time Nt or equivalently the generator is multiplied by a factor of N. This introduces in a natural way a probability measure P_N on the space of trajectories $\{\eta_x(t) : x \in Z_N \text{ or } Z, t \ge 0\}$.

Theorem 2.1. *(The law of large numbers.) For any $t > 0$, there exists a deterministic density function $\rho(t, \cdot)$, on $\mathbf{R}$ or $\mathbf{S}$ as the case may be, such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum J(\frac{x}{N})\ \eta_x(t) = \int J(\xi)\ \rho(t,\xi)d\xi$$

in probability for every suitable test function. The density $\rho(t, x)$ is determined as the unique weak solution of

$$\rho_t(t,\xi) + [\rho(t,\xi)(1 - \rho(t,\xi))]_\xi = 0 \tag{1}$$

with initial condition $\rho(0, \cdot) = \rho_0(\cdot)$, that satisfies the 'entropy condition'.

Remark 2.2. *The entropy condition can be stated in many equivalent forms. For example if $\rho(t, \cdot)$ is a smooth solution, then for any smooth function $h(r)$*

$$[h(\rho(t,\xi))]_t = h'(\rho(t,\xi))\rho_t(t,\xi) = -h'(\rho(t,\xi))(1 - 2\rho(t,\xi))\rho_\xi(t,\xi)$$

or

$$[h(\rho(t,\xi))]_t + [g(\rho(t,\xi))]_\xi = 0 \tag{2}$$

where g and h are related by

$$g'(r) = h'(r)(1-2r) \tag{3}$$

If $\rho(t,\cdot)$ is only a weak solution, then equation (2) may not hold even weakly. A weak solution of equation (1) is said to satisfy the entropy condition if for every convex function h and the corresponding g defined by equation (3),

$$[h(\rho(t,\xi))]_t + [g(\rho(t,\xi))]_\xi \le 0 \tag{4}$$

holds as a distribution on $[0,T]\times\mathbf{R}$ or $[0,T]\times\mathbf{S}$ as the case may be. Then for any initial value, the weak solution satisfying the entropy condition exists and is unique. The density profile of the TASEP converges to this unique solution.

We will not prove it here. For the special case when the sites are $\mathbf{Z}$ and $\eta_{x,N}(0) = 1$ for $x \le 0$ and 0 otherwise was carried out by Rost [4], who proved that in this case the solution $\rho(t,\xi)$ is the rarefaction wave,

$$\rho(t,\xi) = \begin{cases} 1 & if \quad \xi \le -t \\ \frac{t-\xi}{2t} & if \quad -t \le \xi \le t \\ 0 & if \quad \xi \ge t \end{cases}$$

and the density of the TASEP converges to it. Seppäläinen in [5] obtained a representation of the TASEP with arbitrary initial conditions in terms of a family of coupled processes with initial conditions of Rost type and was able to reduce the general case to the Rost case.

If we look at special solutions of the form

$$\rho(t,\xi) = \begin{cases} \rho & if \quad \xi \le 0 \\ 1-\rho & if \quad \xi \ge 0 \end{cases}$$

then this will be an entropic solution only when $\rho \le \frac{1}{2}$. In particular if $\rho = 1$, although the initial profile in the Rost case is a stationary weak solution it is not entropic. On the other hand if we hold the lead particle from jumping, then nothing can move. So with probability e^{-Nt}, the Rost initial profile can remain intact up to time t. This illustrates that non-entropic solutions are relevant for large deviations.

§3. Large Deviations. Some super exponential estimates

The validity of hydrodynamical scaling depends on some basic facts. We will state them in the periodic case. The needed modifications when we have the entire **Z** are obvious. The 'one block estimate' allows one to replace the microscopic flux by its expectations, given the densities over blocks of size $2k+1$. If

$$\mathcal{E}(N,k,t) = \frac{1}{N}\int_0^t \sum_x |e_{N,k,x}(s)| ds$$

where

$$e_{N,k,x}(s) = |\frac{1}{2k+1}\sum_{y:|y-x|\le k} \eta_y(s)(1-\eta_{y+1}(s)) - \bar{\eta}_x^k(s)(1-\bar{\eta}_x^k(s))|$$

and $\bar{\eta}_x^k = \frac{1}{2k+1}\sum_{y:|y-x|\le k}\eta_y$, then

$$\lim_{k\to\infty}\lim_{N\to\infty} E^{P_N}\Big[\mathcal{E}(N,k,t)\Big] = 0$$

The expectation is taken here with respect to the measure P_N that corresponds to some initial profile on the periodic lattice Z_N and evolves according to TASEP dynamics in the speeded up time scale. Then the two block estimate allows one to replace $\bar{\eta}_x^k$ with large k by $\bar{\eta}_x^{N\epsilon}$ with a small ϵ. One can exhibit this in many ways. For instance, if we define,

$$\mathcal{D}(N,\epsilon,k,t) = \int_0^t \big[\frac{1}{N}\sum_x [\bar{\eta}_{x,N}^k(s)]^2 - \frac{1}{N}\sum_x [\bar{\eta}_{x,N}^{N\epsilon}(s)]^2\big] ds$$

then, by proving

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\lim_{N\to\infty} E^{P_N}\Big[\mathcal{D}(N,\epsilon,k,t)\Big] = 0$$

one can establish that any limit of the empirical density is a weak solution of equation (1).

Remark 3.1. *Because of finite propagation speed, basically the effect of any change in a region is only felt over a finite macroscopic domain. This allows us to go back and forth between the periodic and the nonperiodic cases without much effort. If we take the domain large enough then the probability of any effect outside is superexponentially small. So even for large deviations, one can go back and forth.*

Theorem 3.2. *One has the super exponential 'one block' and 'two block estimates'. For any $\delta > 0$,*

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log P\left[\mathcal{E}(N,k,t)\geq\delta\right]=-\infty \tag{5}$$

$$\limsup_{\epsilon\to 0}\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log P\left[\mathcal{D}(N,\epsilon,k,t)\geq\delta\right]=-\infty \tag{6}$$

Sketch of proof: We look at the periodic case. The Dirichlet form

$$D(p)=\sum_{x,\eta}[\sqrt{p(\eta^{x,x+1})}-\sqrt{p(\eta)}]^2$$

can be used in conjunction with the Feynman-Kac formula to provide the first estimate. This is not any different from the symmetric case. The fact that the scaling factor is N and not N^2 does not affect the estimate. It only matters that it is large.

The second estimate on the other hand is a bit tricky. In the symmetric case the proof uses the full strength of the factor N^2, and does not work here. Instead the proof is carried out in several steps. First one proves that there is an exponential error bound, for large deviations from the hydrodynamical limit in the Rost case, by explicit computation. This is not hard and can be done by just following Rost's proof carefully. Then this is extended to arbitrary initial conditions by following through Seppäläinen's proof. One then notices that, by convexity, if $\mathcal{D}(N,\epsilon,k,t)$ does not go to zero, and the one block estimate holds, then the hydrodynamic limit cannot hold. Therefore the two block estimate holds with exponential error probability. Finally a bootstrap argument is used to improve the exponential error probability to a superexponential estimate. The space time region of size $N\times N$ is divided into ℓ^2 grids of size $\frac{N}{\ell}\times\frac{N}{\ell}$. The probability of a significant violation in the two block estimate is $e^{-c\frac{N}{\ell}}$ for one grid. The grids do not influence each other that much. Now the usual Bernoulli large deviation estimate yields a multiplication of the exponent by a factor ℓ^2, that equals the number of grids. If we pick ℓ large we are done.

Corollary 3.3. *Outside the set of weak solutions the probability measure P_N decays superexponentially fast.*

It is then natural to expect that the rate function for large deviations will be a measure of how 'nonentropic' the weak solution is.

§4. Macroscopic and Microscopic Entropies.

A microstate on the configurations on $\mathbf{Z}_N$ is a probability distribution $p_N(\eta)$ on the configurations $\eta \in \{0,1\}^{\mathbf{Z}_N}$. Its entropy (relative to the uniform distribution) is defined as

$$H_N(p_N) = N\log 2 + \sum_\eta p_N(\eta)\log[p_N(\eta)]$$

For a macroscopic density profile $\rho(\xi)$, the corresponding entropy function is defined by

$$\mathcal{H}(\rho(\cdot)) = \log 2 + \int_{\mathbf{S}} [\rho(\xi)\log\rho(\xi) + (1-\rho(\xi))\log(1-\rho(\xi))]d\xi$$

If p_N has asymptotic profile ρ, in the sense that

$$\lim_{N\to\infty}\frac{1}{N}\sum J(\frac{x}{N})\eta_x = \int J(\xi)\rho(\xi)d\xi$$

in probability with respect to p_N, then by Jensen's inequality

$$\liminf_{N\to\infty}\frac{1}{N}H_N(p_N) \geq \mathcal{H}(\rho(\cdot))$$

We need a result of Kosygina [3] that asserts that under certain additional conditions the equality holds, i.e.

$$\lim_{N\to\infty}\frac{1}{N}H_N(p_N) = \mathcal{H}(\rho(\cdot))$$

Two conditions are needed.

- The Dirichlet form is "small"

$$D_N(p_N) = \sum_{x,\eta}[\sqrt{p(\eta^{x,x+1})} - \sqrt{p(\eta)}]^2 = o(N)$$

- The two block estimate holds.

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\lim_{N\to\infty} E^{p_N}\Big[\mathcal{D}(N,\epsilon,k)\Big] = 0$$

where

$$\mathcal{D}(N,\epsilon,k) = \frac{1}{N}\sum_x[\bar{\eta}^k_{x,N}]^2 - \frac{1}{N}\sum_x[\bar{\eta}^{N\epsilon}_{x,N}]^2$$

The proof uses the fact that the control of Dirichlet form allows us to estimate $\frac{1}{N}H_N(p_N)$ by

$$\log 2 + E^{P_N}\left[\frac{1}{N}\sum_x [\bar{\eta}_x^k \log \bar{\eta}_x^k + (1-\bar{\eta}_x^k)\log(1-\bar{\eta}_x^k)]\right]$$

and the two block estimate allows k to be replaced by $N\epsilon$ and if the law of large numbers holds then we easily pass to $\mathcal{H}(\rho(\cdot))$, providing the upper bound. The lower bound as we mentioned is essentially Jensen's inequality.

With some additional work the following theorem due to Kosygina can be proved.

Theorem 4.1. *Consider the evolution according to TASEP in the periodic case with any initial conditions. Suppose the hydrodynamic limit holds with some profile $\rho(t,\xi)$. Then for any $\delta > 0$*

$$\limsup_{N\to\infty} \sup_{\delta \le s \le t} |\frac{1}{N}H_N(p_N(s)) - \mathcal{H}(\rho(s,\cdot))| = 0$$

Idea of proof: The discussion above will allow us to control it for most times s. But the entropy is monotone and cannot fluctuate wildly.

Remark 4.2. *Actually the theorem Kosygina will continue to hold even if we modify the dynamics by changing the rates, replacing in the speeded up scale N by $N\lambda_{x,x+1}(s,\eta)$, provided the relative entropy of the modified process with respect to the unperturbed process remains bounded by CN. This is because the estimates on the Dirichlet form, usually obtained by differentiating the entropy at time t, with respect to t can still be derived. Because the two block estimates has superexponential error estimates for the unperturbed process, they will continue to hold for the perturbed process which has relative entropy bounded by CN. Since the proof of Theorem 4.1 depends only on estimates on the Dirichlet form and two block estimates, the Theorem will continue to hold even when we perturb.*

Remark 4.3. *If for some p_N with profile ρ the entropy relation*

$$\limsup_{N\to\infty} |\frac{1}{N}H_N(p_N) - \mathcal{H}(\rho(\cdot))| = 0$$

holds, then from the super additivity of the entropy function over disjoint blocks, one has for the marginal $p_{N,B}$ of p_N on any block of size $N(b-a)$

say from $[Na, Nb]$,

$$\limsup_{N\to\infty} |\frac{1}{N} H_N(p_{N,B}) - \int_a^b h(\rho(\xi))d\xi| = 0$$

§5. Large Deviation. The rate function

The basic space on which we will carry out the large deviation is the space $\Omega = C[[0,T], \mathcal{M}]$ of continuous maps $\rho(t,d\xi)$ of $[0,T]$ into the space $\mathcal{M}$ of nonnegative measures on S. Although under P_N, $\rho(t,d\xi)$ consists of atoms with mass $\frac{1}{N}$, because of exclusion any conceivable limit will be supported on $\rho(t,d\xi)$ that have densities $\rho(t,\xi)d\xi$ that satisfy $0 \le \rho(t,\xi) \le 1$ for all $(t,\xi) \in [0,T] \times \mathbf{S}$ and are weakly continuous as mappings of $[0,T] \to \mathcal{M}$.

The rate function $\mathcal{I}(\rho(\cdot,\cdot))$ is defined as $+\infty$ if $\rho(\cdot,\cdot)$ is not a weak solution of

$$\rho_t + [\rho(1-\rho)]_\xi = 0$$

If it is a weak solution, then

$$\begin{aligned}
\mathcal{I}(\rho(\cdot,\cdot)) &= \int_{0+0}^{T-0} \int_{\mathbf{S}} \left[[h(\rho(\cdot,\cdot))]_t + [g(\rho(\cdot,\cdot))]_\xi\right]^+ dt d\xi \\
&= \sup_{J\in\mathcal{J}} \int_0^T \int_{\mathbf{S}} J(t,\xi)\left[[h(\rho(\cdot,\cdot))]_t + [g(\rho(\cdot,\cdot))]_\xi\right] dt d\xi \\
&= -\inf_{J\in\mathcal{J}} \int_0^T \int_{\mathbf{S}} \left[J_t(t,\xi)h(\rho(\cdot,\cdot)) + J_\xi(t,\xi)g(\rho(\cdot,\cdot))\right] dt d\xi
\end{aligned}$$

Here $h(r) = r\log r + (1-r)\log(1-r)$ and $g(r)$ as defined by equation (3) is

$$g(r) = r(1-r)\log\frac{r}{(1-r)} - r$$

and

$$\mathcal{J} = \{J(\cdot,\cdot) : 0 \le J(\cdot,\cdot) \le 1, J(0,\cdot) \equiv J(T,\cdot) \equiv 0\}$$

It is interesting to note that the set of weak solutions of nonlinear equations is in general not weakly closed. However a result on compensated compactness, that can be found in Tartar [6], tells us that the set C_ℓ of weak solutions for which $\mathcal{I}(\rho(\cdot,\cdot)) \le \ell$ is in fact compact in the strong topology, guaranteeing that the rate function is indeed lower semi continuous. It is easy to check uniform modulus of continuity in time in the weak topology. So the rate function in fact does have compact level sets.

§6. Upper Bounds

For upper bounds we will use the formulation of Ellis and Dupuis [1]. Suppose $\eta_{x,N}$ is a deterministic initial condition with a profile $\rho_0(\xi)$.

Theorem 6.1. *Suppose P_N is the measure on the configuration space $\{\eta_x(t)\}$ induced by the TASEP and Q_N is such that $Q_N << P_N$ and the measure $\widehat{Q}_N$ induced by Q_N on Ω converges to the degenerate distribution at $\rho(\cdot,\cdot) \in \Omega$. Then*

$$\liminf_{N\to\infty} \frac{1}{N} H(Q_N|P_N) \geq \mathcal{I}(\rho(\cdot,\cdot))$$

Remark 6.2. *This is easily seen to be equivalent to the standard upper bound LDP estimate.*

The proof is broken up into several lemmas.

Lemma 6.3. *Without loss of generality we can assume that Q_N is Markov with rates $N\widehat{\lambda}(t,x,\eta)$.*

Proof. Consider the probability distribution $q_N(t,\eta)$ of $\eta(t)$ at time t under Q_N. We have

$$\frac{1}{N}\sum_x J(\frac{x}{N})\eta_x \to \int J(\xi)\rho(t,\xi)d\xi$$

in probability with respect to $q_N(t,\eta)$. The process Q_N has some rates $N\lambda_N(t,x,\omega)$ of particles jumping from x to $x+1$, that may depend on the past history upto time t. This comes from general martingale theory. One can write the formal generator

$$(\mathcal{L}_{t,\omega}f)(\eta) = N\sum_x \lambda(t,x,\omega)\eta_x(1-\eta_{x+1})[f(\eta^{x,x+1}) - f(\eta)]$$

and with respect to Q_N,

$$f(\eta(t)) - f(\eta(0)) - \int_0^t (\mathcal{L}_{s,\omega}f)(\eta(s))ds$$

will be martingales. By Girsanov formula one can calculate on $[0,T]$,

$$\frac{1}{N}H(Q_N|P_N) = E^{Q_N}\left[\int_0^T \big[\sum_x \eta_x(t)(1-\eta_{x+1}(t))\theta(\lambda(t,x,\omega))\big]dt\right]$$

where $\theta(\lambda) = \lambda\log\lambda - \lambda + 1$. If we replace $\lambda(t,x,\omega)$ by its conditional expectation

$$\widehat{\lambda}(t,x,\eta) = E^{Q_N}\left[\lambda(t,x,\omega)|\eta(t)\right]$$

we see that

$$E^{Q_N}[f(\eta(t)) - f(\eta(0))] = E^{Q_N}\left[\int_0^T (\widehat{\mathcal{L}}_t f)(\eta(t))dt\right]$$

with

$$(\widehat{\mathcal{L}}_t f)(\eta) = N\sum_x \widehat{\lambda}(t,x,\eta)\eta_x(1-\eta_{x+1})[f(\eta^{x,x+1}) - f(\eta)]$$

In other words $q_N(t,\eta)$ is the solution of the forward equation corresponding to $\widehat{\mathcal{L}}$. On the other hand, since $\theta(\lambda)$ is a convex function of λ, by Jensen's inequality,

$$\begin{aligned} &E^{Q_N}\left[\eta_x(t)(1-\eta_{x+1}(t))\theta(\lambda(t,x,\omega))\right] \\ &\qquad\qquad \geq E^{Q_N}\left[\eta_x(t)(1-\eta_{x+1}(t))\theta(\widehat{\lambda}(t,x,\omega))\right] \end{aligned}$$

The Markov process with $\widehat{\mathcal{L}}_t$ as generator has the same marginals at time t as Q_N and will work as well. In other words for our theorem we can assume with out loss of generality that Q_N is Markov with rates $N\widehat{\lambda}(x,t,\eta)$. Q.E.D.

Consider the joint probability distribution $q_{N,x,k}(t,\eta)$ at the $2k+1$ sites $[x-k,\ldots,x+k]$ of $\{\eta_y\}$ under $q_N(t,\eta)$. We think of it as function of η that depends on the variables $\{\eta_y : |y-x| \leq k\}$.

We let

$$H(N,x,k,t) = \frac{1}{N}\sum_{\eta\in[0,1]^{2k+1}} q_{N,x,k}(t,\eta)\log q_{N,x,k}(t,\eta)$$

and compute

$$\begin{aligned} H_t(N,x,k,t) &= \frac{1}{N}\sum_{\eta\in[0,1]^{2k+1}} \dot{q}_{N,x,k}(t,\eta)[1+\log q_{N,x,k}(t,\eta)] \\ &= \frac{1}{N}\sum_{\eta\in[0,1]^{2k+1}} \dot{q}_{N,x,k}(t,\eta)\log q_{N,x,k}(t,\eta) \\ &= \frac{1}{N}\sum_{\eta\in[0,1]^{N}} \dot{q}_{N}(t,\eta)\log q_{N,x,k}(t,\eta) \end{aligned}$$

Using the forward equation $\dot{q}_N(t,\eta) = N(\mathcal{L}_t^* q_N)(t,\eta)$, we get

$$
\begin{aligned}
&H_t(N,x,k,t)\\
&= \sum_{\eta\in[0,1]^N} q_N(t,\eta)\mathcal{L}_t[\log q_{N,x,k}(t,\eta)]\\
&= \sum_{\substack{\eta\in[0,1]^N\\ x-k-1\le y\le x+k}} q_N(t,\eta)\widehat{\lambda}(t,y,\eta)e_{y,y+1}(\eta)\log\frac{q_{N,x,k}(t,\eta^{y,y+1})}{q_{N,x,k}(t,\eta)}
\end{aligned}
$$

where $e_{y,y+1}(\eta) = \eta_y(1-\eta_{y+1})$. We use the inequality

$$\lambda\log y \le \lambda\log\lambda - \lambda + 1 + e^a y - 1 - a\lambda$$

with the choice of $a = a_{N,x,y,k}(\eta)$ to be made later.

$$
\begin{aligned}
&H_t(N,x,k,t) \le\\
&\sum_{\substack{\eta\in[0,1]^N\\ x-k-1\le y\le x+k}} q_N(t,\eta)e_{y,y+1}(\eta)[\widehat{\lambda}(t,y,\eta)\log\widehat{\lambda}(t,y,\eta) - \widehat{\lambda}(t,y,\eta)+1]\\
&+ \sum_{\substack{\eta\in[0,1]^N\\ x-k-1\le y\le x+k}} q_N(t,\eta)e_{y,y+1}(\eta)\frac{e^{a_{N,x,y,k}}q_{N,x,k}(t,\eta^{y,y+1}) - q_{N,x,k}(t,\eta)}{q_{N,x,k}(t,\eta)}\\
&- \sum_{\substack{\eta\in[0,1]^N\\ x-k-1\le y\le x+k}} q_N(t,\eta)e_{y,y+1}(\eta)\widehat{\lambda}(t,y,\eta)a_{N,x,y,k}
\end{aligned}
$$

We rewrite this as

$$
\begin{aligned}
&\sum_{\substack{\eta\in[0,1]^N\\ x-k-1\le y\le x+k}} q_N(t,\eta)e_{y,y+1}(\eta)[\widehat{\lambda}(t,y,\eta)\log\widehat{\lambda}(t,y,\eta) - \widehat{\lambda}(t,y,\eta)+1]\\
&\qquad\qquad \ge H_t(N,x,k,t) - A_1(N,x,k,t) + A_2(N,x,k,t)
\end{aligned}
$$

where

$$
\begin{aligned}
&A_1(N,x,k,t) =\\
&\sum_{\substack{\eta\in[0,1]^N\\ x-k-1\le y\le x+k}} q_N(t,\eta)e_{y,y+1}(\eta)\frac{e^{a_{N,x,y,k}}q_{N,x,k}(t,\eta^{y,y+1}) - q_{N,x,k}(t,\eta)}{q_{N,x,k}(t,\eta)}
\end{aligned}
$$

and

$$A_2(N,x,k,t) = \sum_{\substack{\eta\in[0,1]^N \\ x-k-1\le y\le x+k}} q_N(t,\eta)e_{y,y+1}(\eta)\widehat{\lambda}(t,y,\eta)a_{N,x,y,k}$$

We now multiply by $J(t,\frac{x}{N})$, where $J\in\mathcal{J}$, sum over x, integrate with respect to t from 0 to T and finally multiply by $\frac{1}{(2k+2)N}$,

$$\begin{aligned}
&\frac{1}{N}H(Q_N|P_N) \\
&\ge \int_0^T \frac{1}{(2k+2)N}\sum_{x\in\mathbf{Z}_N} J(t,\frac{x}{N})d\Big[\sum_{\eta\in[0,1]^{2k+1}} q_{N,x,k}(t,\eta)\log q_{N,x,k}(t,\eta)\Big] \\
&\quad - E^{Q_N}\Bigg[\int_0^T\Bigg[\frac{1}{2k+2}\sum_{\substack{x,y \\ x-k-1\le y\le x+k}} J(t,\frac{x}{N})e_{y,y+1}(\eta)\times \\
&\qquad\qquad \frac{e^{a_{N,x,y,k}}q_{N,x,k}(t,\eta^{y,y+1})-q_{N,x,k}(t,\eta)}{q_{N,x,k}(t,\eta)}\Bigg]dt\Bigg] \\
&\quad + E^{Q_N}\Bigg[\int_0^T\Bigg[\frac{1}{2k+2}\sum_{\substack{x,y \\ x-k-1\le y\le x+k}} J(t,\frac{x}{N})e_{y,y+1}(\eta)\times \\
&\qquad\qquad \widehat{\lambda}(t,y,\eta)a_{N,x,y,k}\Bigg]dt\Bigg] \\
&= T_1(N,J(\cdot,\cdot),k) - T_2(N,J(\cdot,\cdot),k) + T_3(N,J(\cdot,\cdot),k)
\end{aligned}$$

Now we have to analyse the terms on the right. Let us look at

$$\begin{aligned}
&T_1(N,J(\cdot,\cdot),k) = \\
&\int_0^T \frac{1}{(2k+2)N}\sum_{x\in\mathbf{Z}_N} J(t,\frac{x}{N})d\Big[\sum_{\eta\in[0,1]^{2k+1}} q_{N,x,k}(t,\eta)\log q_{N,x,k}(t,\eta)\Big]
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&T_1(N,J(\cdot,\cdot),k) = \\
&-\int_0^T \sum_{x\in\mathbf{Z}_N} J_t(t,\frac{x}{N})\Big[\frac{1}{(2k+2)N}\sum_{\eta\in[0,1]^{2k+1}} q_{N,x,k}(t,\eta)\log q_{N,x,k}(t,\eta)\Big]dt
\end{aligned}$$

We pick $k=N\epsilon$ and let $\epsilon\to 0$ at the end.

Lemma 6.4. *If for $t \in [0,T]$, $q_N(t,\eta)$ leads to the profile $\rho(t,\cdot)$, then*

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}$$

$$\int_0^T \sum_{x\in \mathbf{Z}_N} J_t(t,\frac{x}{N})\left[\frac{1}{2N^2\epsilon}\sum_{\eta\in[0,1]^{2N\epsilon}} q_{N,x,N\epsilon}(t,\eta)\log q_{N,x,N\epsilon}(t,\eta)\right]dt$$

$$= \int_0^T \int_{\mathbf{S}} J_t(t,\xi)h(\rho(t,\xi))dtd\xi$$

Proof. Let us consider the quantity

$$H_N(t,x,\epsilon) = \log 2 + \frac{1}{2N\epsilon}\sum_{\eta\in[0,1]^{2N\epsilon+1}} q_{N,x,N\epsilon}(t,\eta)\log q_{N,x,N\epsilon}(t,\eta)$$

and the measure

$$\mu_N(t,\epsilon) = \frac{1}{N}\sum_x H_N(t,x,\epsilon)\delta_{\frac{x}{N}}$$

We need to prove the weak convergence of

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\mu_N(t,\epsilon)dt = h(\rho(t,\xi))dtd\xi$$

Since we are looking at relative entropy with respect to a product measure, i.e. uniform distribution on $[0,1]^{\mathbf{Z}_N}$, it is easy to see that

$$\liminf_{\epsilon\to 0}\liminf_{N\to\infty}\mu_N(t,\epsilon)dt \geq h(\rho(t,\xi))dtd\xi$$

in view of the remark at the end of the last section. On the other hand the total mass of $\mu_N(t,\epsilon)$ is dominated by the total entropy

$$\log 2 + \frac{1}{N}\sum_{\eta\in[0,1]^{\mathbf{Z}_N}} q_N(t,\eta)\log q_N(t,\eta)$$

and we are done. Q.E.D.

Now we try to control $T_2(N,J(\cdot,\cdot),N\epsilon) - T_2(N,J(\cdot,\cdot),N\epsilon)$ which is more difficult. The interior terms with $x-k-1<y<x+k$ are easy. We

choose $a_{N,x,y,k} = 0$.

$$E^{Q_N}\left[\frac{1}{2N\epsilon}\sum_{x-k-1<y<x+k} e_{y,y+1}(\eta)\frac{q_{N,x,N\epsilon}(t,\eta^{y,y+1}) - q_{N,x,N\epsilon}(t,\eta)}{q_{N,x,N\epsilon}(t,\eta)}\right]$$
$$= \sum_{\eta\in[0,1]^{\mathbf{Z}_{2N\epsilon}}}\frac{1}{2N\epsilon}\sum_{x-k-1<y<x+k} e_{y,y+1}(\eta)\times$$
$$[q_{N,x,N\epsilon}(t,\eta^{y,y+1}) - q_{N,x,N\epsilon}(t,\eta)]$$
$$= \sum_{\eta\in[0,1]^{\mathbf{Z}_{2N\epsilon}}}\frac{1}{2N\epsilon}\sum_{x-k-1<y<x+k}[\eta_{y+1}-\eta_y]q_{N,x,N\epsilon}(t,\eta)$$

If we carry out a summation by parts in x and integration over t, this leads in the limit to

$$-\int_0^T\int_{\mathbf{S}} J_\xi(t,\xi)\rho(t,\xi)dtd\xi$$

We look next at the boundary terms. Note that $k=[N\epsilon]$. The boundary terms equal $B = B_1 + B_2 + B_3 + B_4$

$$B_1 = -\,E^{Q_N}\left[\int_0^T\left[\frac{1}{2N\epsilon}\sum_x J(t,\frac{x}{N})\eta_{x-k-1}(1-\eta_{x-k})\times\right.\right.$$
$$\left.\left.\frac{e^{a_{N,x,-,k}}q_{N,x,k}(t,\eta^{x-k-1,x-k}) - q_{N,x,k}(t,\eta)}{q_{N,x,k}(t,\eta)}\right]dt\right]$$

$$B_2 = -\,E^{Q_N}\left[\int_0^T\left[\frac{1}{2N\epsilon}\sum_x J(t,\frac{x}{N})\eta_{x+k}(1-\eta_{x+k+1})\times\right.\right.$$
$$\left.\left.\frac{e^{a_{N,x,+,k}}q_{N,x,k}(t,\eta^{x+k,x+k+1}) - q_{N,x,k}(t,\eta)}{q_{N,x,k}(t,\eta)}\right]dt\right]$$

$$B_3 = +\,E^{Q_N}\left[\int_0^T\left[\frac{1}{2N\epsilon}\sum_x J(t,\frac{x}{N})\eta_{x-k-1}(1-\eta_{x-k})\times\right.\right.$$
$$\left.\left.\widehat{\lambda}(t,x-k-1,\eta)a_{N,x,-,k}\right]dt\right]$$

$$B_4 = + E^{Q_N}\left[\int_0^T \left[\frac{1}{2N\epsilon}\sum_x J(t,\frac{x}{N})\eta_{x+k}(1-\eta_{x+k+1}) \times \right.\right.$$
$$\left.\left. \widehat{\lambda}(t,x+k,\eta)a_{N,x,+,k}\right]dt\right]$$

We would like to make the choice of $a_{N,x,-,k} = -u(t,\frac{x-k-1}{N})$ and $a_{N,x,+,k} = u(t,\frac{x+k}{N})$ for some smooth u. We can combine B_3 and B_4 and write

$$\begin{aligned}
&B_3 + B_4\\
&= E^{Q_N}\left[\int_0^T \left[\frac{1}{2N\epsilon}\sum_x [J(t,\frac{x+k+1}{N}) - J(t,\frac{x-k}{N})] \times \right.\right.\\
&\qquad\left.\left. \eta_x(1-\eta_{x+1})\widehat{\lambda}(t,x,\eta)u(t,\frac{x}{N})\right]dt\right]\\
&= E^{Q_N}\left[\int_0^T \left[\frac{1}{2N\epsilon}\sum_x [J(t,\frac{x+k+1}{N}) - J(t,\frac{x-k}{N})] \times \right.\right.\\
&\qquad\left.\left. \eta_x(1-\eta_{x+1})u(t,\frac{x}{N})\right]dt\right]\\
&\quad + E^{Q_N}\left[\int_0^T \left[\frac{1}{2N\epsilon}\sum_x [J(t,\frac{x+k+1}{N}) - J(t,\frac{x-k}{N})] \times \right.\right.\\
&\qquad\left.\left. [\widehat{\lambda}(t,x,\eta)-1]u(t,\frac{x}{N})\right]dt\right]\\
&\simeq \frac{1}{2\epsilon}\int_0^T\int_{\mathbf{S}} [J(t,\xi+\epsilon) - J(t,x-\epsilon)]\rho(t,\xi)(1-\rho(t,\xi))u(t,\xi)dtd\xi\\
&+ Error
\end{aligned}$$

The error term is dominated by

$$CE^{Q_N}\left[\int_0^T \left[\frac{1}{N}\sum_x |\widehat{\lambda}(t,x,\eta(t))-1|\right]dt\right]$$

For any $\theta > 0$, there is a constant C_θ such that

$$|\lambda - 1| \le \theta + C_\theta[\lambda\log\lambda - \lambda + 1]$$

Therefore

$$Error \le C\theta + \frac{C_\theta}{N^2}H(Q_N|P_N)$$

We will get an estimate on B_1. The term B_2 is similar. B_1 is estimated by

$$\frac{1}{2N\epsilon}E^{Q_N}\left[\int_0^T \sum_x \eta_{x-1}(1-\eta_x)\left|e^{-u(t,\frac{x}{N})}\frac{q_{N,x,x+2k}(t,\eta^{x-1,x})}{q_{N,x,x+2k}(t,\eta)}-1\right|dt\right]$$

The quantity

$$R_N = \eta_{x-1}(1-\eta_x)\frac{q_{N,x,x+2k}(t,\eta^{x-1,x})}{q_{N,x,x+2k}(t,\eta)}$$

has to be looked at carefully. Take $x = 0$. If we denote $q_{N,x,x+2k}(t,\eta)$ by $f_N(\eta_0,\eta_1,\cdots,\eta_{2k})$ then

$$\begin{aligned} R_N &= \eta_1(1-\eta_0)\frac{f_N(1,\eta_1,\cdots,\eta_{2k})}{f_N(0,\eta_1,\cdots,\eta_{2k})} = \eta_1(1-\eta_0)\frac{p_N(1|\eta_1,\cdots,\eta_{2k})}{p_N(0|\eta_1,\cdots,\eta_{2k})} \\ &\simeq \eta_1(1-\eta_0)\frac{\rho(0)}{1-\rho(0)} \end{aligned}$$

Therefore it follows that

$$\limsup_{N\to\infty} B_1 \le \frac{1}{2\epsilon}\int_0^T\int_{\mathbf{S}} \rho(t,\xi)(1-\rho(t,\xi))\left|\frac{e^{-u(t,\xi)}\rho(t,\xi)}{1-\rho(t,\xi)}-1\right|dtd\xi$$

and similarly

$$\limsup_{N\to\infty} B_2 \le \frac{1}{2\epsilon}\int_0^T\int_{\mathbf{S}} \rho(t,\xi)(1-\rho(t,\xi))\left|\frac{e^{u(t,\xi)}(1-\rho(t,\xi))}{\rho(t,\xi)}-1\right|dtd\xi$$

If we let u approach $\log\frac{\rho(t,\xi)}{1-\rho(t,\xi)}$ both B_1 and B_2 tend to 0 for any positive ϵ. Finally we let $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}[B_3+B_4] = \int_0^T\int_{\mathbf{S}} J_\xi(t,\xi)\rho(t,\xi)(1-\rho(t,\xi))\log\frac{\rho(t,\xi)}{1-\rho(t,\xi)}dtd\xi$$

This proves

$$\begin{aligned} &\liminf_{N\to\infty}\frac{1}{N}H(Q_N|P_N) \ge \\ &\quad -\int_0^T\int_{\mathbf{S}} J_t(t,\xi)h(\rho(t,\xi))dtd\xi + \int_0^T\int_{\mathbf{S}} J_\xi(t,\xi)\rho(t,\xi)dtd\xi \\ &\quad -\int_0^T\int_{\mathbf{S}} J_\xi(t,\xi)\rho(t,\xi)(1-\rho(t,\xi))\log\frac{\rho(t,\xi)}{1-\rho(t,\xi)}dtd\xi \\ &= -\int_0^T\int_{\mathbf{S}}[J_t(t,\xi)h(\rho(t,\xi)) + J_\xi(t,\xi)g(\rho(t,\xi))]dtd\xi \end{aligned}$$

Since J is arbitrary, we are done.

§7. Lower Bounds

The situation with the lower bounds is not totally satisfactory. Ideally one should construct an explicit perturbation of the rates that produces a particular profile, the entropy cost of such a perturbation being approximately equal to the rate function for such a large deviation. This one is not able to do at this time. The best we can do is to prove the existence of such perturbations and construct them implicitly. Even this we can do only to produce a single non-entropic shock traveling at a constant speed. By patching together, one can possibly handle a finite number of shocks of varying speeds, even crossing each other forming caustics. However one does not see at the moment how to produce a 'general' non-entropic weak solution, partly because one does not know what it is. Ideally there would be an approximation theorem allowing us to pass from a solution with a finite number of shocks to a general weak solution with a finite large deviation rate.

We will sketch the proof for the simple case of a stationary non-entropic shock at 0 starting from a special initial configuration. Suppose we are given on $\mathbf{Z}$ an initial configuration of particles where every site $x \leq 0$ is filled and every site $x > 0$ is empty. We wish to perturb the standard speeded up TASEP dynamics with new rates $N\lambda_N(t, x, \eta)$, such that for the modified process Q_N, for every test function J with compact support and every $t \in [0, T]$, we have in probability,

$$\lim_{N\to\infty} \sum_x J(\frac{x}{N})\eta_x(t) = \int_{-\infty}^{\infty} J(\xi)\rho(t,\xi)d\xi$$

where $\rho(t, \xi)$ is the following special weak solution.

$$\rho(t,\xi) = \begin{cases} 1 & if \quad x \leq -t \\ \frac{t-x}{2t} & if \quad -t \leq x \leq -t(2\rho-1) \\ \rho & if \quad -t(2\rho-1) \leq x \leq 0 \\ 1-\rho & if \quad 0 \leq x \leq t(2\rho-1) \\ \frac{t-x}{2t} & if \quad t(2\rho-1) \leq x \leq t \\ 0 & if \quad x \geq t \end{cases} \tag{7}$$

Here $\rho > \frac{1}{2}$ and there is a non-entropic shock at 0 where the density jumps from a higher value of $\rho > \frac{1}{2}$ to the lower value of $1 - \rho < \frac{1}{2}$. The rate function for this profile in the interval $[0,T]$ is proportional to T, i.e. equals $c(\rho)T$, where

$$c(\rho) = 2\rho - 1 - 2\rho(1-\rho)\log\frac{\rho}{1-\rho} \tag{8}$$

The problem is to find rates $N\lambda_N(x,\eta)$ such that the process with these new rates has a law of large numbers with the profile $\rho(t,\xi)$ given by (7) and achieve this with an entropy cost that is roughly $c(\rho)N$. We know from the upper bound that we cannot do better. Since we want to slow down particles at or near 0, ideally cutting down the rate at 0 should do it. If we slow down the rate at 0 to $N\lambda$ with some fixed $\lambda < 1$, holding all other rates at N, we will produce a profile of the type we want with some $\rho = \rho(\lambda)$ that is hard to determine, except in the trivial case of $\lambda = 0, \rho = 1, c(0) = 1$. The cost is surely not going to be optimal. We can however lower the rate at several points around 0, depending on the current configuration of particles. The new rates $N\lambda_N(x,\eta)$ will do the trick. We will implicitly construct them. We will then have to see how this will work for any initial condition. After that we need to modify the construction for shocks moving with constant velocity. Then patch things together for one or more shocks with non constant jumps and non constant velocities that may cross each other.

The idea for a single shock is simple enough. A non-entropic shock is entropic if time is reversed. Let us begin with a generator of a TASEP with jumps to the left rather than to the right. The generator is given by

$$(\widehat{\mathcal{L}}f)(\eta) = N\sum_x \eta_{x+1}(1-\eta_x)[f(\eta^{x,x+1}) - f(\eta)]$$

We start with some initial configuration at $t = 0$, that produces the density profile of $\rho(T,\xi)$ specified by equation (7). The hydrodynamical scaling limit will be an entropic solution of

$$\widehat{\rho}_t - [\widehat{\rho}(1-\widehat{\rho})]_x = 0$$

with $\widehat{\rho}_T(0,\xi) = \rho(T,\xi)$. This is seen to be the time reversal of the profile given in (7).

$$\widehat{\rho}_T(t,\xi) = \rho(T-t,\xi)$$

for $0 \le t \le T$ and $\xi \in R$. If we now take the process Q_N corresponding to $\widehat{\mathcal{L}}$ and reverse time to get trajectories $\eta(T-t)$, the new process will have some generator $\mathcal{L}_{N,T,t}$ that is time inhomogeneous and nearly impossible to compute. However it does have the advantage that it has a hydrodynamical limit with a profile that is the time reversed version of $\widehat{\rho}_T(t,\xi)$ which is of course $\rho(t,\xi)$. The entropy will match, because while the forward motion is losing entropy at the shock, the time reversed motion will put it back and this is done by the new rates for the reversed process. If we do not waste entropy at microscopic scale, then the book keeping at micro and macro levels match and will give us the lower bound for large deviations. However the rates for $\widehat{\mathcal{L}}_{N,T,t}$ are too messy and one has to make it independent of T,N and t, and localize it, so that it is transportable and can be used as a module that we can use at any place and time to slow the flow, which is all that any non-entropic solution is expected to do. We start with a fairly general simple calculation.

Let P be a time homogeneous Markov process with trajectories $x(t)$ in a finite time interval $[0,T]$, on a finite state space with generator

$$(Af)(x) = \sum_y c(x,y)f(y).$$

Let $\pi(t,x)$ be the marginal distributions in the time interval $[0,T]$. We denote by $C(x) = -c(x,x) = \sum_{y\neq x} c(x,y)$. Let $\widehat{P}_T$ be the process that corresponds to the time reversed trajcetories $y(t) = x(T-t)$. Although $\widehat{P}_T$ is a Markov process, it is in general time inhomogeneous and will have a generator that depends on the marginals $\pi(\cdot,\cdot)$. We denote its time dependent generator by

$$(\widehat{\mathcal{A}}_{T,t}f)(x) = \sum_{y\neq x} \widehat{c}_T(t,x,y)f(y)$$

and

$$\widehat{C}_T(x) = -\widehat{c}_T(t,x,x) = \sum_y \widehat{c}_T(t,x,y)$$

We can also reverse the generator and define $\widehat{\mathcal{A}}$ as

$$(\widehat{\mathcal{A}}f)(x) = \sum_y \widehat{c}(x,y)f(y)$$

with $\widehat{c}(x,y) = c(y,x)$ for $x \neq y$ and

$$\widehat{C}(x) = -\widehat{c}(x,x) = \sum_{y\neq x} \widehat{c}(x,y) = \sum_{y\neq x} c(y,x)$$

We denote by $\widehat{Q}_T$, the process with generator $\widehat{\mathcal{A}}$ with initial distribution $\pi(T,\cdot)$.

Theorem 7.1. *We have the following simple formula connecting the function*

$$H(t) = \sum_x \pi(t,x)\log \pi(t,x)$$

and the relative entropy $H(\widehat{P}_T, \widehat{Q}_T)$.

$$H(\widehat{P}_T|\widehat{Q}_T) = H(0) - H(T) + E^{\widehat{Q}_T}\left[\int_0^T [\widehat{C}(x(t)) - C(x(t))]dt\right] \tag{9}$$

Proof. The probabilities $\pi(t,x)$ satisfy the forward equation

$$\frac{d\pi(t,y)}{dt} = \sum_{x\neq y} c(x,y)\pi(t,x) - C(y)\pi(t,y)$$

The time reversed process $\widehat{P}_T$ defined by $y(t) = x(T-t)$ will have marginals $\pi(T-t,y)$ and some generator

$$(\widehat{\mathcal{A}}_{T,t}f)(x) = \sum_y \widehat{c}_T(t,x,y)f(y)$$

Of course

$$\begin{aligned}\frac{d\pi(T-t,y)}{dt} &= -\sum_{x\neq y} c(x,y)\pi(T-t,x) + C(y)\pi(t,y)\\ &= \sum_{x\neq y} \widehat{c}_T(t,x,y)\pi(T-t,x) - \widehat{C}_T(t,y)\pi(T-t,y)\end{aligned}$$

Actually it is not hard to see that for $x \neq y$

$$\pi(T-t,x)\widehat{c}_T(t,x,y) = \widehat{c}(x,y)\pi(T-t,y)$$

and

$$\widehat{C}_T(t,x) = \frac{1}{\pi(T-t,x)}\sum_{y\neq x}\widehat{c}(x,y)\pi(T-t,y)$$

Our goal is to compute the relative entropy

$$\begin{aligned}H(\widehat{P}_T|\widehat{Q}_T) = \int_0^T \sum_x \pi(T-t,x) \sum_{y:y\neq x} [c_T(t,x,y)\log\frac{c_T(t,x,y)}{\widehat{c}(x,y)}\\ - c_T(t,x,y) + \widehat{c}(x,y)]dt\end{aligned}$$

$$
\begin{aligned}
&H(\widehat{P}_T|\widehat{Q}_T)\\
&\quad = \int_0^T \Bigg[\sum_x \pi(T-t,x)\Bigg[\Bigg[\sum_{y:y\neq x} \frac{c(y,x)\pi(T-t,y)}{\pi(T-t,x)} \log \frac{\pi(T-t,y)}{\pi(t-t,x)}\Bigg]\\
&\qquad\qquad\qquad\qquad\qquad\qquad - C_T(t,x) + \widehat{C}(x)\Bigg]\Bigg] dt\\
&\quad = \int_0^T \sum_{x\neq y} c(y,x)\Bigg[\pi(T-t,y)\log\frac{\pi(T-t,y)}{\pi(T-t,x)}\\
&\qquad\qquad\qquad\qquad\qquad - \pi(T-t,y) + \pi(T-t,x)\Bigg] dt\\
&\quad = \int_0^T \sum_{x\neq y} c(y,x)\Bigg[\pi(t,y)\log\frac{\pi(t,y)}{\pi(t,x)} - \pi(t,y) + \pi(t,x)\Bigg] dt
\end{aligned}
$$

We begin by differentiating $H(t) = \sum_y \pi(t,y)\log\pi(t,y)$.

$$
\begin{aligned}
H'(t) &= \frac{d}{dt}\sum_y \pi(t,y)\log\pi(t,y) = \sum_y \pi(t,y)(\mathcal{A}\log\pi(t,\cdot))(y)\\
&= \sum_y \pi(t,y)\Bigg[[\sum_{x\neq y} c(y,x)\log\pi(t,x)] - C(y)\log\pi(t,y)\Bigg]\\
&= \sum_{x\neq y} c(y,x)[\pi(t,y)\log\pi(t,x) - \pi(t,y)\log\pi(t,y)]\\
&= -\sum_{x\neq y} c(y,x)\pi(t,y)\log\frac{\pi(t,y)}{\pi(t,x)}
\end{aligned}
$$

This proves (9). Q.E.D.

Let us start the backward TASEP $\widehat{\mathcal{L}}$, with an initial distribution μ_N concentrated on the finite set

$$
\Omega_{N,L} = \Big\{\eta : \eta_x = 1 \text{ for } x < -NL \quad \text{and} \quad \eta_x = 0 \text{ for } x \geq NL\Big\}
$$

for some $L \geq T$. Our initial distribution μ_N will be a Bernoulli with $\mu_N[\eta_x = 1] = \rho(T, \frac{x}{N})$ given in equation (7). Assume $L \geq T$. Then the

TASEP will have profile $\rho(T-t,\xi)$ and at time $t=T$ end up at $\rho(0,\xi)$. The time reversed process will now be a perturbation of the TASEP going in the right direction with the profile we need. (7.1). We note that

$$\begin{aligned}\widehat{C}(\eta)-C(\eta) &= N\sum_x \eta_x(1-\eta_{x+1}) - N\sum_x \eta_{x+1}(1-\eta_x)\\ &= N\sum_x[\eta_x-\eta_{x+1}] \equiv N.\end{aligned}$$

Moreover $H(T)=0$ and

$$\begin{aligned}\frac{1}{NT}H(0) &\simeq \frac{1}{T}\int_{-T}^{T}[\rho(T,\xi)\log\rho(T,\xi)+(1-\rho(T,\xi))\log(1-\rho(T,\xi))]d\xi\\ &= 2(2\rho-1)[\rho\log\rho+(1-\rho)\log(1-\rho)]\\ &\quad +2\int_{2\rho-1}^{1}[\frac{1-\xi}{2}\log\frac{1-\xi}{2}+\frac{1+\xi}{2}\log\frac{1+\xi}{2}]d\xi\\ &= 2\rho-2-2\rho(1-\rho)\log\frac{\rho}{1-\rho}\end{aligned}$$

The relative entropy can now be computed using formula (9) and is seen to be asymptotic to CTN with

$$\begin{aligned}C=c(\rho) &= 2\rho-2-2\rho(1-\rho)\log\frac{\rho}{1-\rho}+1\\ &= 2\rho-1-2\rho(1-\rho)\log\frac{\rho}{1-\rho}\end{aligned}$$

agreeing with (8).

This perturbation is neither stationary in time nor local in nature. We need to modify it.

The special initial configuration of particles at every site $x\le 0$ and no particles at any site $x>0$ will be denoted by $\bar\eta$. Let $N(T)$ be the number of transitions from 0 to 1 during $[0,T]$ for the TASEP. Let P be the unpertubed TASEP from this special configurartion. Our initial goal is to construct, for each given ρ a perturbed measure Q_ρ such that $Q_\rho << P$, with

$$E^{Q_\rho}[N(T)] \simeq \rho(1-\rho)T$$

and

$$H(Q_\rho|P) \simeq Tc(\rho)$$

with $c(\rho)$ given by (8). We wish to do this with a local, time independent perturbation at least approximately. We can work out the algebra and restate it as trying to make for $a < \frac{1}{4}$,

$$E^{Q_\rho}[N(T)] \simeq aT$$

with an entropy cost not exceeding

$$I(a) = \sqrt{1-4a} - 2a \log \frac{1+\sqrt{1-4a}}{1-\sqrt{1-4a}} \tag{10}$$

We consider for $\sigma > 0$,

$$U(\sigma, t, \eta) = E^\eta[e^{-\sigma N(t)}]$$

where η is the initial configurartion. First we note that by a simple coupling argument

$$U(\sigma, t, \bar{\eta}) \le U(\sigma, t, \eta) \le U(\sigma, t, \bar{\eta}) e^{\sigma g(\eta)}$$

with

$$g(\eta) = \sum_{x \le 0} (1 - \eta_x) + \sum_{x > 0} \eta_x$$

for all configurations η with only a finite number of ocupied sites x with $x > 0$ and finite number of empty sites x with $x \le 0$. By Markov property, if

$$A(\sigma, t) = \inf_\eta U(\sigma, t, \eta) = U(\sigma, t, \bar{\eta})$$

then $A(\sigma, t+s) \ge A(\sigma, t) A(\sigma, s)$ and $-\log A(t)$ is subadditive and

$$\lim_{t \to \infty} \frac{\log A(\sigma, t)}{t} = \sup_t \frac{\log A(\sigma, t)}{t} = -\lambda(\sigma) \tag{11}$$

exists. Moreover

$$e^{-t\lambda(\sigma) - \theta(t)} \le U(\sigma, t, \bar{\eta}) \le U(\sigma, t, \eta) \le e^{-t\lambda(\sigma) + \sigma g(\eta)}$$

where $\theta(t) = o(t)$ as $t \to \infty$. We can write down a differential equation satisfied by $U(\sigma, t, \eta)$

$$\frac{\partial U(\sigma, t, \eta)}{\partial t} = (\mathcal{L}_\sigma U)(\sigma, t, \eta)$$

with

$$U(\sigma, 0, \eta) = 1$$

The genertaor $\mathcal{L}_\sigma$ is obtained by a combination of Girsanov formula and Feynman-Kac formula. It takes the form

$$(\mathcal{L}_\sigma U)(\eta) = \sum_x c_{x,x+1}\eta_x(1-\eta_{x+1})[U(\eta^{x,x+1}) - U(\eta)] - (1-e^{-\sigma})U(\eta)$$

where $c_{x,x+1} = 1$ for $x \neq 0$ and $c_{0,1} = e^{-\sigma}$.

Theorem 7.2. *Let $\sigma > 0$ be given. For each $\epsilon > 0$, there exists a positive local function $V = V_{\sigma,\epsilon}(\eta)$ that satisfies*

$$(\mathcal{L}_\sigma V)(\eta) + (\lambda(\sigma)+\epsilon)V(\eta) \geq 0$$

for all η.

Proof. As a first step we produce a function that is continuous, i.e. depends weakly on far away sites and then approximate it to get a local function. Our first choice is

$$W(\eta) = \frac{1}{t_0}\int_0^{t_0} \exp[(\lambda(\sigma)+\frac{\epsilon}{2})t]U(\sigma,t,\eta)dt$$

Because of the lower bound on U we can assume that t_0 is large enough so that for all η,

$$e^{[\lambda(\sigma)+\frac{\epsilon}{2}]t_0}U(\sigma,t_0,\eta) \geq 1$$

Let us compute $\mathcal{L}_\sigma W$.

$$\begin{aligned}(\mathcal{L}_\sigma W)(\eta) &= \frac{1}{t_0}\int_0^{t_0} \exp[(\lambda(\sigma)+\frac{\epsilon}{2})t](\mathcal{L}_\sigma U)(\sigma,t,\eta)dt\\ &= \frac{1}{t_0}\int_0^{t_0} \exp[(\lambda(\sigma)+\frac{\epsilon}{2})t]U_t(\sigma,t,\eta)dt\\ &= \frac{1}{t_0}[e^{[\lambda(\sigma)+\frac{\epsilon}{2}]t_0}U(\sigma,t_0,\eta)-1] - (\lambda(\sigma)+\frac{\epsilon}{2})W(\eta)\\ &\geq -(\lambda(\sigma)+\frac{\epsilon}{2})W(\eta)\end{aligned}$$

Since t_0 is finite, W depends weakly on far away sites, and can be nicely approximated by a V that is local. Q.E.D.

The next step is to use $V = V_{\sigma,\epsilon}$ to construct our perturbations. These perturbations cost entropy but will limit the flow between 0 and 1. There is a trade off and σ is the parameter that will control this trade off. Opitmality in the trade off is reached as $\epsilon \to 0$. We begin by defining the rates

$$c_{x,x+1}(\sigma,\epsilon,\eta) = c_{x,x+1}(\sigma)\frac{V(\eta^{x,x+1})}{V(\eta)}$$

Note that $c_{x,x+1}(\sigma) = 1$ for $x \neq 0$ and $c_{0,1} = e^{-\sigma}$. The corresponding perturbed evolution

$$(\mathcal{L}_{\sigma,\epsilon} f)(\eta) = \sum_x c_{x,x+1}(\sigma, \epsilon, \eta)\eta_x(1-\eta_{x+1})[f(\eta^{x,x+1}) - f(\eta)]$$

is local. Let us speed up by N, and do the hydrodynamic scaling for this perturbation. Let Q_N be the perturbed process and P_N be the original process both rescaled and with the special initial configuration $\bar{\eta}$.

Theorem 7.3. *For every N, we have on the interval $[0,T]$,*

$$H(Q_N|P_N) \leq E^{Q_N}\Big[\log V(\eta(T)) - \log V(\eta(0)) - \sigma N(T)\Big] + NT[\lambda(\sigma)+\epsilon]$$

Proof.

$$\frac{1}{N} H(Q_N|P_N) = E^{Q_N}\left[\int_0^T \phi_{\sigma,\epsilon}(\eta(t))dt\right]$$

where

$$\phi_{\sigma,\epsilon}(\eta) = e_{x,x+1}(\eta)[c_{x,x+1}(\sigma,\epsilon,\eta)\log c_{x,x+1}(\sigma,\epsilon,\eta) - c_{x,x+1}(\sigma,\epsilon,\eta) + 1]$$

We use our definition of $c_{x,x+1}(\sigma,\epsilon,\eta)$ and an easy calculation to get

$$\phi_{\sigma,\epsilon}(\eta) = (\mathcal{L}_{\sigma,\epsilon}\log V)(\eta) - \sigma\, e_{0,1}(\eta)c_{0,1}(\sigma,\epsilon,\eta) - \frac{(\mathcal{L}_\sigma V)(\eta)}{V(\eta)}$$

The proof is completed by integrating with respect to t and taking expectations, noting

$$NE^{Q_N}\left[\int_0^T (\mathcal{L}_{\sigma,\epsilon}\log V)(\eta(t))dt\right]$$
$$= E^{Q_N}\left[\int_0^T [\log V(\eta(T)) - \log V(\eta(0))]\right]$$

Q.E.D.

Now the rest of the argument is relatively straight forward. First, we need a lemma.

Lemma 7.4. *The limit $\lambda(\sigma)$ defined in (11) satisfies*

$$\lambda(\sigma) = \inf_{\rho\geq\frac{1}{2}}[\sigma\rho(1-\rho) + c(\rho)] = \inf_{a\leq\frac{1}{4}}[a\sigma + I(a)]$$

where $I(a)$ is as in (10).

Proof. *Upper bound*: If $F(\rho(\cdot,\cdot))$ is the flow through the origin during $[0,T]$ for a weak solution $\rho(\cdot,\cdot)$, then from the upper bound already established

$$T\,\lambda(\sigma) \geq -\inf_{\rho(\cdot,\cdot)}\Big[\sigma\, F(\rho(\cdot,\cdot)) + I(\rho(\cdot,\cdot))\Big]$$

If one fixes $F(\rho(\cdot,\cdot)) = aT = \rho(1-\rho)T$, the infimum of $I(\rho(\cdot,\cdot))$ is shown by a variational argument to equal $Tc(\rho) = TI(a)$.

Lower bound: By a simple calculation using Jensen's inequality

$$\begin{aligned}
\log E^{P_N}\left[e^{-\sigma N(T)}\right] &= \log E^{Q_N}\left[e^{-\sigma N(T)}\frac{dP_N}{dQ_N}\right] \\
&= \log E^{Q_N}\left[e^{-\sigma N(T)+\log[\frac{dP_N}{dQ_N}]}\right] \\
&= \log E^{Q_N}\left[e^{-\sigma N(T)-\log[\frac{dQ_N}{dP_N}]}\right] \\
&\geq -E^{Q_N}\left[\sigma N(T) + \log[\frac{dQ_N}{dP_N}]\right] \\
&= -E^{Q_N}\left[\sigma N(T)\right] - H(Q_N|P_N)
\end{aligned}$$

We can take any Q_N and we pick it as the time reversal of the backward TASEP. Our earlier calculations establish the lower bound of $Tc(\rho)$ for the relative entropy and $T\rho(1-\rho)$ for the flow. One checks that $I(a)$ is a strictly convex function of a. Q.E.D.

This proves that if we perturb by $\mathcal{L}_{\sigma,\epsilon}$ and take the limit as $\epsilon \to 0$, the profiles we get will satisfy the entropy condition, will have flow at 0 limited by Ta and the realtive entropy will be bounded by $TI(a)$, where a is dual to σ.

The final step in the proof is to prove that the only profile, that satisfies the entropy condition away from 0, has the rate function bounded by $TI(a)$ and the flow through the origin bounded by Ta, is given by (7) with $\rho > \frac{1}{2}$ chosen so that $a = \rho(1-\rho)$.

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