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Relative positions of four subspaces in a Hilbert space and subfactors

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Abstract.

We study relative positions of four subspaces in a Hilbert space. Gelfand-Ponomarev gave a complete classification of indecomposable systems of four subspaces in a finite-dimensional space. In this note we show that there exist uncountably many indecomposable systems of four subspaces in an infinite-dimesional Hilbert space. We extend a numerical invariant, called defect, for a certain class of systems of four subspaces using Fredholm index. We show that the set of possible values of the defect is $\{\frac{n}{3};\ n \in \mathbf{Z}\}$.

§1. Introduction

This is an announcement of the joint work [EW] with M. Enomoto.

The relative position of one subfactor in a factor has been proved quite rich after the work [J] of Jones. But the relative position of one subspace of a Hilbert space is extremely poor and simply determined by its dimension and co-dimension. The aim of the paper is to cover up the poorness by considering the relative position of several subspaces.

It is a well-known fact that the relative position of two subspaces E and F in a Hilbert space H can be described completely up to unitary equivalence as in Dixmier [D] and Halmos [H]. The Hilbert space is the direct sum of five subspaces:

$$H = (E \cap F) \oplus (\text{the rest}) \oplus (E \cap F^{\perp}) \oplus (E^{\perp} \cap F) \oplus (E^{\perp} \cap F^{\perp}).$$

In the "rest part", E and F are in generic position and the relative position is described only by the "angles" between them. In fact, let e and f be the projections onto E and F respectively. Then e and f look like

$$e = I_{(e \wedge f)H} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus I_{(e \wedge f^{\perp})H} \oplus 0 \oplus 0,$$

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$$f = I_{(e\wedge f)H} \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus I_{(e^{\perp}\wedge f)H} \oplus 0,$$

where c and s are two positive operators with null kernels and $c^2+s^2=1$. By the functional calculus, there exists a unique positive operator θ, called the angle operator, such that $c = \cos\theta$ and $s = \sin\theta$ with $0 \leq \theta \leq \frac{\pi}{2}$.

Consider two self-adjoint unitaries $u = 2e-1$ and $v = 2f-1$. It is obvious that there is a bijective correspondence between the set of two subspaces in a Hilbert space H and the set of unitary representations π of the free product $G_2 = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} = \langle a, b\rangle$ of the cyclic groups of order two on H through $\pi(a) = u$ and $\pi(b) = v$. Similarly there is a bijective correspondence between the set of n subspaces in a Hilbert space H and the set of unitary representations of the free product $G_n = \mathbf{Z}/2\mathbf{Z} * \cdots * \mathbf{Z}/2\mathbf{Z}$ (n-times) of the cyclic groups of order two. It is well-known that for $n \geq 3$ the group G_n is non-type I and non-amenable. Therefore it seems brave and stupid to study the relative positions of n subspaces for $n \geq 3$ up to unitary equivalence. To avoid the difficulty, we forget the angles and consider a weaker equivalent relation for the systems of n-subspaces as topological vector spaces.

We say that two systems $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E_1', \cdots, E_n')$ of n subspaces in Hilbert spaces H and H' are similar if their exists a bounded invertible operator $T : H \to H'$ satisfying $TE_i = E_i'$ for $i = 1, \cdots, n$.

We should study an *indecomposable* system $\mathcal{S} = (H; E_1, \cdots, E_n)$ of n-subspaces in the sense that the system $\mathcal{S}$ can not be similar to a direct sum of two non-zero systems. Consider the case that the Hilbert space H is finite-dimensional. Then we have four indecomposable systems of two subspaces. We have nine indecomposable systems of three subspaces. They are trivial ones, that is, H is one dimensional, except one system. But, in the old paper [GP], Gelfand and Ponomarev showed that there exist infinitely many indecomposable systems of four subspaces with higher finite dimensions and surprisingly they completely classified them.

We shall show that there exist infinitely many indecomposable systems of four subspaces in an infinite-dimensional Hilbert space H.

The most important numerical invariant of a subfactor $N \subset M$ is the Jones index $[M : N]$ introduced in [J]. Similarly the most important numerical invariant of a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a finite-dimensional Hilbert space H is the defect

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} dim\ E_i - 2dim\ H$$

introduced by Gelfand-Ponomarev in [GP]. We shall extend their notion of defect $\rho(\mathcal{S})$ for a certain class of systems $\mathcal{S}$ of four subspaces in an infinite-dimensional Hilbert space H using Fredholm index. If a pair $N \subset M$ of factor-subfactor is finite-dimensional, then Jones index $[M : N]$ is an integer. But if $N \subset M$ is infinite-dimensional, then Jones index $[M : N]$ is a non-integer in general. One of the amazing facts was that the possible value of Jones index is in $\{4cos^2 \frac{\pi}{n} \mid n = 3, 4, \cdots\} \cup [4, \infty]$. Similarly if a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is finite-dimensional, then the defect $\rho(\mathcal{S})$ is an integer. Gelfand-Ponpmarev showed that if a system $\mathcal{S}$ is indecomposable and finite-dimensional, then the possible value of defect $\rho(\mathcal{S})$ is exactly in $\{-2, -1, 0, 1, 2\}$. We show that the set of values of defect for indecomposable systems of four subspaces in an infinite-dimesional Hilbert space is $\{\frac{n}{3}; n \in \mathbf{Z}\}$.

Sunder also considered n subspaces in [S]. But his interest is opposite to ours. In fact he studied the case that the Hilbert space H is the algebraic sum of the n subspaces and solved the statistical problem of computing the canonical partial correlation coefficients between three sets of random variables.

§2. Systems of n subspaces

Our purpose is to study relative positions of n subspaces in a Hilbert space. Let H be a (separable) Hilbert space and $E_1, \cdots, E_n$ be a finite family of subspaces of H. We shall write $\mathcal{S} = (H; E_1, \cdots, E_n)$ for such a system of n subspaces. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ be systems of n subspaces. We say that $\mathcal{S}$ and $\mathcal{S}'$ are *similar*, denoted by $\mathcal{S} \sim \mathcal{S}'$, if there exists a bounded linear operator $T : H \to H'$ such that $E'_i = TE_i$ for $i = 1, \cdots, n$. We say that $\mathcal{S}$ and $\mathcal{S}'$ are *unitary equivalent* if there exists a unitary operator $u : H \to H'$ such that $E'_i = uE_i$ for $i = 1, \cdots, n$. We study relative positions of n subspaces up to similarity to ignore angles between subspaces in a certain sense. We denote by $\mathcal{S} \oplus \mathcal{S}'$ the direct sum $(H \oplus H'; E_1 \oplus E'_1, \cdots, E_n \oplus E'_n)$ of two systems $\mathcal{S}$ and $\mathcal{S}'$. We write $\mathcal{S} = 0$ if $H = 0$.

Lemma 1. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, E_2)$ a system of two subspaces. Then the following are equivalent:*

1. *There exists a closed subspace $M \subset H$ such that*
$$(H; E_1, E_2) \sim (H; M, M^{\perp})$$
2. $H = E_1 + E_2$ *and* $E_1 \cap E_2 = 0$.

Definition. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in a Hilbert space H. We say that $\mathcal{S}$ is *decomposable* if there exists non-zero systems $\mathcal{S}_1$ and $\mathcal{S}_2$ of n subspaces such that $\mathcal{S} \sim \mathcal{S}_1 \oplus \mathcal{S}_2$. It is useful to note that $\mathcal{S}$ is decomposable if and only if there exist non-zero closed subspaces H_1 and H_2 of H such that $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$ for $i = 1, \cdots, n$. We say $\mathcal{S}$ is *indecomposable* if $\mathcal{S}$ is not decomposable.

Example 1. Let $H = \mathbf{C}^2$. Fix an angle θ with $0 < \theta < \pi/2$. Put $E_1 = \mathbf{C}(1,0)$ and $E_2 = \mathbf{C}(cos\theta, sin\theta)$. Then

$$(H; E_1, E_2) \sim (\mathbf{C}; \mathbf{C}, 0) \oplus (\mathbf{C}; 0, \mathbf{C})$$

Hence $(H; E_1, E_2)$ is decomposable. Let e_1 and e_2 be the projections onto E_1 and E_2. Then the C^*-algebra $C^*(\{e_1, e_2\})$ generated by e_1 and e_2 is exactly $B(H) \cong M_2(\mathbf{C})$. Thus the irreducibility of $C^*(\{e_1, e_2\})$ does not imply the indecomposability of $\mathcal{S} = (H; E_1, E_2)$.

Remark. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in a Hilbert space H. Let e_i be the projection of H onto E_i for $i = 1, \cdots, n$. If $\mathcal{S} = (H; E_1, \cdots, E_n)$ is indecomposable, then the $C^*(\{e_1, \cdots, e_n\})$ generated by $e_1, \cdots, e_n$ is irreducible. But the converse is not true as in Example 1.

Example 2. Let $H = \mathbf{C}^2$. Put $E_1 = \mathbf{C}(1,0)$, $E_2 = \mathbf{C}(0,1)$ and $E_3 = \mathbf{C}(1,1)$. Then $\mathcal{S} = (H; E_1, E_2, E_3)$ is indecomposable.

Example 3. Let $H = \mathbf{C}^3$ and $\{a_1, a_2, a_3\}$ be a linearly independent subset of H. Put $E_1 = \mathbf{C}a_1$, $E_2 = \mathbf{C}a_2$ and $E_3 = \mathbf{C}a_3$. Then $\mathcal{S} = (H; E_1, E_2, E_3)$ is decomposable. In fact, let $H_1 = E_1 \vee E_2 \neq 0$ and $H_2 = E_3 \neq 0$. Then $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$, for $i = 1, 2, 3$.

Example 4. Let $H = \mathbf{C}^3$ and $\{b_1, b_2, b_3, b_4\}$ be a subset of H. Put $E_i = \mathbf{C}b_i$ for $i = 1, \cdots, 4$. Consider a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces. Then the following are equivalent:

1. $\mathcal{S}$ is indecomposable.
2. Any three vectors of $\{b_1, b_2, b_3, b_4\}$ is linearly independent.
3. The set $\{b_1, b_2, b_3\}$ is linearly independent and $b_4 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$ for some scalars $\lambda_i \neq 0$ $(i = 1, 2, 3)$.

Assume that $\{u_1, u_2, u_3, u_4\} \subset H$ and $\{v_1, v_2, v_3, v_4\} \subset H$ satisfy the above condition (2). Then $\mathcal{S} = (H; \mathbf{C}u_1, \mathbf{C}u_2, \mathbf{C}u_3, \mathbf{C}u_4)$ and $\mathcal{T} = (H; \mathbf{C}v_1, \mathbf{C}v_2, \mathbf{C}v_3, \mathbf{C}v_4)$ are similar.

Example 5. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(1,1,1)$ and $E_3 = \mathbf{C}(1,2,3)$. Then a system $\mathcal{S} = (H; E_1, E_2, E_3)$ is decomposable. In fact, let $E_1' = (E_2 \vee E_3) \cap E_1$ and $H_1 = E_1 \cap (E_1')^{\perp} \neq 0$. Let

$H_2 = E_2 \vee E_3 \neq 0$. Then $H_1 + H_2 = H$, $H_1 \cap H_2 = H$ and $E_i = E_i \cap H_1 + E_i \cap H_2$, for $i = 1, 2, 3$.

Example 6. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(1,0,1)$. Then a system $\mathcal{S}_1 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 7. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(1,0,1)$. Then a system $\mathcal{S}_2 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 8. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(1,0,1) \oplus \mathbf{C}(0,1,0)$. Then a system $\mathcal{S}_3 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 9. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,0)$, $E_2 = \mathbf{C}(0,1,0) \oplus \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(0,0,1) \oplus \mathbf{C}(1,1,0)$. Then a system $\mathcal{S}_4 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Remark Any two of the above indecomposable systems $\mathcal{S}_1, \cdots, \mathcal{S}_4$ of four subspaces are not similar.

Example 10. Let $K = \ell^2(\mathbf{N})$ and $H = K \oplus K$. Consider a unilateral shift $S : K \to K$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Sx) \in H | x \in K\}$ and $E_4 = \{(x, x) \in H | x \in K\}$. Then a system $\mathcal{S}_4 = (H; E_1, E_2, E_3, E_4)$ of four subspaces in H is indecomposable.

Example 11.(Harrison-Radjavi-Rosental [HRR]) Let $K = \ell^2(\mathbf{Z})$ and $H = K \oplus K$. Consider a sequence $(\alpha_n)_n$ given by $\alpha_n = 1$ for $n \leq 0$ and $\alpha_n = exp((-1)^n n!)$ for $n > 1$. Consider a bilateral weighted shift $S : \mathcal{D}_T \to K$ such that $T(x_n)_n = (\alpha_{n-1} x_{n-1})_n$ with the domain $\mathcal{D}_T = \{(x_n)_n \in \ell^2(\mathbf{Z}) | \sum_n |\alpha_n x_n|^2 < \infty\}$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Tx) \in H | x \in \mathcal{D}_T\}$ and $E_4 = \{(x, x) \in H | x \in K\}$. Since $\{0, H, E_1, E_2, E_3, E_4\}$ is a transitive lattice, a system $\mathcal{S}_4 = (H; E_1, E_2, E_3, E_4)$ of four subspaces in H is indecomposable.

Definition. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in a Hilbert space H. Then the orthogonal complement of $\mathcal{S}$, denoted by $\mathcal{S}^\perp$, is defined by $\mathcal{S}^\perp = (H; E_1^\perp, \cdots, E_n^\perp)$.

Proposition 2. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, \cdots, E_n)$ a system of four subspaces in H. Then $\mathcal{S}$ is indecomposable if and only if $\mathcal{S}^\perp$ is indecomposable.*

§3. Classification of two subspaces

Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in H. We say that $\mathcal{S}$ is *trivial* if *dim* $H = 1$.

Gelfand-Ponomarev [GP] claim that if H is finite-dimensional, then every indecomposable system of $\mathcal{S} = (H; E_1, E_2)$ of two subspaces is trivial and similar to one of the following four systems:

$$\mathcal{S}_1 = (\mathbf{C}; \mathbf{C}, 0), \quad \mathcal{S}_2 = (\mathbf{C}; 0, \mathbf{C}), \quad \mathcal{S}_3 = (\mathbf{C}; \mathbf{C}, \mathbf{C}), \quad \mathcal{S}_4 = (\mathbf{C}; 0, 0).$$

Any system of two subspaces is similar to a direct sum of a finite number of indecomposable systems above.

We consider the case that H is infinite-dimensional.

Proposition 3. *Let H be a separable infinite-dimesional Hilbert space and $\mathcal{S} = (H; E_1, E_2)$ a system of two subspaces in H. If $\mathcal{S}$ is indecomposable, then $\mathcal{S}$ is similar to one of the following four systems:*

$$\mathcal{S}_1 = (\mathbf{C}; \mathbf{C}, 0), \quad \mathcal{S}_2 = (\mathbf{C}; 0, \mathbf{C}), \quad \mathcal{S}_3 = (\mathbf{C}; \mathbf{C}, \mathbf{C}), \quad \mathcal{S}_4 = (\mathbf{C}; 0, 0).$$

§4. Classification of three subspaces

Gelfand-Ponomarev ([GP]) also claim that if H is finite-dimensional, then there exist nine different indecomposable system $\mathcal{S} = (H; E_1, E_2, E_3)$ of three subspaces in H. The eight of them are trivial and similar to one of the following systems:

$$\mathcal{S}_1 = (\mathbf{C}; 0, 0, 0), \quad \mathcal{S}_2 = (\mathbf{C}; \mathbf{C}, 0, 0), \quad \mathcal{S}_3 = (\mathbf{C}; 0, \mathbf{C}, 0),$$

$$\mathcal{S}_4 = (\mathbf{C}; 0, 0, \mathbf{C}), \quad \mathcal{S}_5 = (\mathbf{C}; \mathbf{C}, \mathbf{C}, 0), \quad \mathcal{S}_6 = (\mathbf{C}; \mathbf{C}, 0, \mathbf{C}),$$

$$\mathcal{S}_7 = (\mathbf{C}; 0, \mathbf{C}, \mathbf{C}), \quad \mathcal{S}_8 = (\mathbf{C}; \mathbf{C}, \mathbf{C}, \mathbf{C})$$

The only non-trivial indecomposable system of three subspaces is

$$\mathcal{S} = (\mathbf{C}^2; \mathbf{C}(1,0), \mathbf{C}(0,1), \mathbf{C}(1,1))$$

up to similarity.

§5. Classification of four subspaces

The classification of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a Hilbert space H is a central problem. If H is finite-dimensional, Gelfand-Ponomarev [GP] completely classified them and gave a complete list of their canonical forms. Their important numerical invariants are *dim* H and the defect

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} dim\ E_i - 2dim\ H.$$

Proposition 4 (Gelfand-Ponomarev [GP]). *If a system $\mathcal{S}$ of four subspaces in a finite-dimensional H is indecomposable, then a possible value of the defect $\rho(\mathcal{S})$ is exactly in the set $\{-2, -1, 0, 1, 2\}$.*

The defect characterizes an essential feature of the system. If $\rho(\mathcal{S}) = 0$, then there exists a pair of linear operators $A : E \to F$ and $B : F \to E$ and the system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is described up to permutation by $H = E \oplus F$, $E_1 = E \oplus 0$, $E_2 = 0 \oplus F$, $E_3 = \{(x, Ax) \in H | x \in E\}$ and $E_4 = \{(By, y) \in H | y \in F\}$. If $\rho(\mathcal{S}) = \pm 1$, then $\mathcal{S}$ is given up to permutation by $H = E \oplus F$, $E_1 = E \oplus 0$, $E_2 = 0 \oplus F$, E_3 and E_4 are subspaces of H that do not reduced to the graphs of the operators as in the case that $\rho(\mathcal{S}) = 0$. A system with $\rho(\mathcal{S}) = \pm 2$ cannot be described in the above forms.

Following [GP], we write down the canonical forms of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in an finite-dimensional space H up to permutation. We first consider the case when *dim* H is even and $2k$ for some positive integer k. There exist no indecomposable systems $\mathcal{S}$ with $\rho(\mathcal{S}) = \pm 2$. Let H be a space with a basis $\{e_1, \cdots, e_k, f_1, \cdots, f_k\}$.

The system $\mathcal{S}_3(2k, -1) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = -1$ and given by

$$E_1 = [e_1, \cdots, e_k],\ E_2 = [f_1, \cdots, f_k],$$

$$E_3 = [(e_2 + f_1), \cdots, (e_k + f_{k-1})],\ E_4 = [(e_1 + f_1), \cdots, (e_k + f_k)].$$

The system $\mathcal{S}_3(2k, 1) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = 1$ and given by

$$E_1 = [e_1, \cdots, e_k],\ E_2 = [f_1, \cdots, f_k],$$

$$E_3 = [e_1, (e_2 + f_1), \cdots, (e_k + f_{k-1}), f_k],\ E_4 = [(e_1 + f_1), \cdots, (e_k + f_k)].$$

The system $\mathcal{S}_{1,3}(2k, 0) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = 0$ and given by

$$E_1 = [e_1, \cdots, e_k],\ E_2 = [f_1, \cdots, f_k],$$

$$E_3 = [e_1, (e_2 + f_1), \cdots, (e_k + f_{k-1})],\ E_4 = [(e_1 + f_1), \cdots, (e_k + f_k)].$$

The system $\mathcal{S}(2k,0;\lambda) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = 0$ and given by

$$\begin{aligned} E_1 &= [e_1,\cdots,e_k],\ E_2 = [f_1,\cdots,f_k], \\ E_3 &= [(e_1+\lambda f_1),(e_2+f_1+\lambda f_2),\cdots,(e_k+f_{k-1}+\lambda f_k)], \\ E_4 &= [(e_1+f_1),\cdots,(e_k+f_k)]. \end{aligned}$$

Every other system $\mathcal{S}_i(2k,\rho)$, $\mathcal{S}_{i,j}(2k,0)$ can be obtained from the systems $\mathcal{S}_3(2k,\rho)$, $\mathcal{S}_{i,3}(2k,0)$ by a suitable permutation of the subspaces. Let σ be a permutation on the set $\{1,2,3,4\}$ and $\mathcal{S} = (H;E_1,E_2,E_3,E_4)$ a system of four subspaces. We define

$$\sigma\mathcal{S} = (H;E_{\sigma^{-1}(1)},E_{\sigma^{-1}(2)},E_{\sigma^{-1}(3)},E_{\sigma^{-1}(4)}).$$

Let $\sigma_{i,j}$ be the transposition (i,j). We put $\mathcal{S}_i(2k,\rho) = \sigma_{3,i}\mathcal{S}_3(2k,\rho)$ for $\rho = -1,1$. We also define $\mathcal{S}_{i,j}(2k,0) = \sigma_{1,i}\sigma_{3,j}\mathcal{S}_{1,3}(2k,0)$ for $i,j \in \{1,2,3,4\}$.

We next consider the case $dim\ H = 2k+1$, odd (for some positive integer k). Let H be a space with a basis $\{e_1,\cdots,e_k,e_{k+1},f_1,\cdots,f_k\}$.

The system $\mathcal{S}_1(2k+1,-1) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = -1$ and given by

$$E_1 = [e_1,\cdots,e_k,e_{k+1}],\ E_2 = [f_1,\cdots,f_k],$$

$$E_3 = [(e_2+f_1),\cdots,(e_{k+1}+f_k)],\ E_4 = [(e_1+f_1),\cdots,(e_k+f_k)].$$

The system $\mathcal{S}_2(2k+1,1) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = 1$ and given by

$$E_1 = [e_1,\cdots,e_k,e_{k+1}],\ E_2 = [f_1,\cdots,f_k],$$

$$E_3 = [e_1,(e_2+f_1),\cdots,(e_{k+1}+f_k)],\ E_4 = [(e_1+f_1),\cdots,(e_k+f_k),e_{k+1}].$$

The system $\mathcal{S}_{1,3}(2k+1,0) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = 0$ and given by

$$E_1 = [e_1,\cdots,e_k,e_{k+1}],\ E_2 = [f_1,\cdots,f_k],$$

$$E_3 = [e_1,(e_2+f_1),\cdots,(e_{k+1}+f_k)],\ E_4 = [(e_1+f_1),\cdots,(e_k+f_k)].$$

The system $\mathcal{S}(2k+1,-2) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = -2$ and given by

$$\begin{aligned} E_1 &= [e_1,\cdots,e_k,e_{k+1}],\quad E_2 = [f_1,\cdots,f_k], \\ E_3 &= [(e_2+f_1),\cdots,(e_{k+1}+f_k)], \\ E_4 &= [(e_1+f_2),\cdots,(e_{k-1}+f_k),(e_k+e_{k+1})]. \end{aligned}$$

The system $\mathcal{S}(2k+1,2) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = 2$ and given by

$$\begin{aligned} E_1 &= [e_1, \cdots, e_k, e_{k+1}], \quad E_2 = [f_1, \cdots, f_k], \\ E_3 &= [e_1, (e_2 + f_1), \cdots, (e_{k+1} + f_k)], \\ E_4 &= [f_1, (e_1 + f_2), \cdots, (e_{k-1} + f_k), (e_k + e_{k+1})]. \end{aligned}$$

We put $\mathcal{S}_i(2k+1,-1) = \sigma_{1,i}\mathcal{S}_1(2k+1,-1)$, $\mathcal{S}_i(2k+1,+1) = \sigma_{2,i}\mathcal{S}_2(2k+1,1)$, $\mathcal{S}_{i,j}(2k+1,0) = \sigma_{1,i}\sigma_{3,j}\mathcal{S}_{1,3}(2k+1,0)$ for $i,j \in \{1,2,3,4\}$.

Theorem 5 (Gelfand-Ponomarev [GP]). *If a system $\mathcal{S}$ of four subspaces in a finite-dimensional H is indecomposable, then $\mathcal{S}$ is similar to one of the systems $\mathcal{S}_{i,j}(m,0)$, $(i < j, i,j \in \{1,2,3,4\}, m = 1,2,\cdots)$; $\mathcal{S}(2k,0,\lambda)$, $(\lambda \in \mathbf{C}, \lambda \neq 0, \lambda \neq 1, k = 1,2,\cdots)$; $\mathcal{S}_i(m,-1)$, $\mathcal{S}_i(m,1)$, $(i \in \{1,2,3,4\}, m = 1,2,\cdots)$; $\mathcal{S}(2k+1,-2)$, $\mathcal{S}(2k+1,+2), k = 0,1,\cdots)$*

We would like to investigate the case when H is infinite-dimensional. The complete classification is at present far from being solved. But we can show the existence of plenty of examples.

Theorem 6 ([EW]). *There exist uncountably many indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in an infinite-dimensional Hilbert space H.*

We shall extend the notion of the defect for a certain class of systems using Fredholm index.

Definition. Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces in a Hilbert space H . For any $i \neq j \in \{1,2,3,4\}$, define a bounded linear operator $T_{ij} = E_i \oplus E_j \to H$ by $T_{ij}(x,y) = x + y$. If T_{ij} is a Fredholm operator, then $ind\ T_{ij} = dim\ (E_i \cap E_j) - dim\ (E_i + E_j)^{\perp}$. We say that $\mathcal{S}$ is a *Fredholm system* if T_{ij} is a Fredholm operator for any $i \neq j \in \{1,2,3,4\}$. We also say that $\mathcal{S}$ is a *weak Fredholm system* if $ker\ T_{ij}$ and $ker\ T_{ij}^*$ is finite-dimensional for any $i \neq j \in \{1,2,3,4\}$. It is clear that if $\mathcal{S}$ is a Fredholm system, then $\mathcal{S}$ is a weak Fredholm system. For any weak Fredholm system $\mathcal{S}$ we define the defect of $\mathcal{S}$, denoted by $\rho(\mathcal{S})$, by

$$\rho(\mathcal{S}) = \frac{1}{3} \sum_{1 \leq i < j \leq 4} Ind\ T_{i,j}.$$

The new definition of the defect agrees with the original one when H is finite-dimensional. In that case the value of the defect is an integer.

Proposition 7 ([EW]). *If $\mathcal{S}$ is a weak Fredholm system, then the orthogonal complement $\mathcal{S}^{\perp}$ is also a weak Fredholm system and $\rho(\mathcal{S}^{\perp}) = -\rho(\mathcal{S})$.*

Recall that one of the amazing fact in subfactor theory was that the possible value of the Jones index for a subfactor is in $\{4cos^2\frac{\pi}{n} \mid n = 3, 4, \cdots\} \cup [4, \infty]$. We shall determine the possible value of the defect for an indecomposable system $\mathcal{S}$ of four subspaces in an infinite-dimesional Hilbert space.

Theorem 8 ([EW]). *The set of possible values of the defect for indecomposable systems of four subspaces in an infinite-dimesional Hilbert space is* $\{\frac{n}{3}; n \in \mathbf{Z}\}$.

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