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Quantum spin chain and Popescu systems

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§1. Introduction

In this article, we explain how Popescu systems and their dilation to representations of the Cuntz algebra are related to some problems of quantum statistical mechanics. The physics we discuss here is the quasi one-dimensional material, closely related to an unsolved problem of anti-ferromagnetic Heisenberg models. First we begin by stating our notation and the mathematical problem precisely. Our quantum spin models with an infinite degree of freedom are described as a C^*-dynamical system on a UHF C^*-algebra. The standard references of this mathematical approach are [9] and [10]. The algebra of local observables is the infinite tensor product $\mathfrak{A}_{loc}$ of the full matrix algebras. For the usual quantum system with spin s ($s = 1/2, 1, 3/2, \cdots$), the one site algebra is $M_{2s+1}(\mathbf{C})$, the set of $2s+1$ by $2s+1$ matrices, and in this case

$$\mathfrak{A}_{loc} = \bigotimes_{\mathbf{Z}} M_{2s+1}(\mathbf{C}).$$

Each component of the tensor product above is specified with a lattice site $j \in \mathbf{Z}$. The C^*-completion of $\mathfrak{A}_{loc}$ is denoted by $\mathfrak{A}$.

For any integer j and any matrix Q in $M_{2s+1}(\mathbf{C})$, $Q^{(j)}$ will be an observable Q located at the lattice site j. Thus, by $Q^{(j)}$ we denote the following element of $\mathfrak{A}$:

$$\cdots \otimes 1 \otimes 1 \otimes \underbrace{Q}_{\text{the j-th component}} \otimes 1 \otimes 1 \otimes \cdots \in \mathfrak{A}.$$

Given a subset Λ of $\mathbf{Z}$, $\mathfrak{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathfrak{A}$ generated by all $Q^{(j)}$ with $Q \in M_n(\mathbf{C})$, $j \in \Lambda$. If φ is a state of $\mathfrak{A}$ the restriction of φ to $\mathfrak{A}_\Lambda$ will be denoted by φ_Λ:

$$\varphi_\Lambda = \varphi|_{\mathfrak{A}_\Lambda}.$$

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The translation τ_k (shift on the integer lattice $\mathbf{Z}$) is an automorphism of $\mathfrak{A}$ determined by

$$\tau_k(Q^{(j)}) = Q^{(j+k)}.$$

As the Lie group $SU(2)$ acts on the 2s+1 dimensional vector space irreducibly, each one site algebra $M_{2s+1}(\mathbf{C})$ has the adjoint action of $SU(2)$. From this action we obtain the product type action β_g of $SU(2)$ on $\mathfrak{A}$ which commutes with the lattice translation τ_k.

The time evolution of the system is governed by the one parameter group of automorphisms α_t on $\mathfrak{A}$. The generator δ of α_t is an approximate inner derivation obtained by the infinite volume limit of local Hamiltonians H_Λ on the finite subset Λ of $\mathbf{Z}$:

$$\frac{d}{dt}\alpha_t(Q)|_{t=0} = \delta(Q) = \lim_{\Lambda\to\mathbf{Z}} \;[H_\Lambda, Q]$$

for Q in $\mathfrak{A}_{loc}$. A standard Hamiltonian of the spin s antiferromagnetic chain is the Heisenberg Hamiltonian H_Λ

$$H_\Lambda = \sum_{j\in\Lambda} (S^{(j)}, S^{(j+1)}), \tag{1.1}$$

where $S^{(j)}_\alpha$ is the spin operator at the site j and

$$(S^{(j)}, S^{(j+1)}) = \sum_{\alpha=x,y,z} S^{(j)}_\alpha S^{(j+1)}_\alpha. \tag{1.2}$$

Another Hamiltonian frequently used in Sold State Physics is the following spin 1 Hamiltonian ($s = 1$):

$$H_\Lambda = \sum_{j\in\Lambda}\left\{ J_1(S^{(j)}, S^{(j+1)}) + J_2(S^{(j)}, S^{(j+1)})^2\right\}. \tag{1.3}$$

The above Hamiltonians of (1.1) and (1.3) and the time evolution α_t associated with them are obviously $SU(2)$ invariant:

$$\beta_g \circ \alpha_t = \alpha_t \circ \beta_g.$$

Mathematically we may consider more general Hamiltonians as well. Although these two Hamiltonians are approximation of more complicated interactions, it is believed that the qualitative feature of these models represent "universal property" of $SU(2)$ invariant antiferromagnetic systems.

Decay of correlation and the spectrum property of these C^*-dynamical system $\{\mathfrak{A}, \alpha_t\}$ are of prime interest in mathematical physics. Next we formulate the problem more precisely.

States considered for the above models are ground states. As far as the spectrum and decay of correlation are concerned, KMS states are of no interest as they always have exponentially decay of correlation and each model does not exhibit any individual character.

By definition, a state φ is a ground state of the C^*-dynamical system $\{\mathfrak{A}, \alpha_t\}$ if

$$\frac{1}{i}\frac{d}{dt}\varphi(Q^*\alpha_t(Q)) \geq 0$$

for any Q of $\mathfrak{A}_{loc}$. Let $\{\pi_\varphi, \mathfrak{H}_\varphi, \Omega_\varphi\}$ be the GNS representation of a ground state φ where π_φ is the representation, and $\mathfrak{H}_\varphi$ is the Hilbert space and Ω_φ is the GNS cyclic vector. When the state φ is a ground state, there exists a positive (unbounded) selfadjoint operator H_φ on $\mathfrak{H}_\varphi$ such that

$$(1.4) \qquad e^{itH_\varphi}\pi_\varphi(Q)e^{-itH_\varphi} = \pi_\varphi\alpha_t((Q)), \qquad H_\varphi\Omega_\varphi = 0.$$

By spectrum of the infinite volume Heisenberg model, we mean the spectrum of H_φ.

When the spin s is 1/2, the Heisenberg model of (1.1) is exactly solved and eigenvectors and their eigenvalues of (1.1) have been found by Bethe ansatz. Even though the completeness of eigenvectors is not yet proved, a lot of heuristic argument has been published. On the other hand, for higher spin, nothing is rigorously proved for the Hamiltonian (1.1) so far. Nevertheless due to the heuristic arguments and numerical simulations, the following are now believed.

Conjecture 1.1. *(i) The ground state of the antiferromagnetic Heisenberg model obtained by the infinite volume of (1.1) is unique.*
(iia) If the spin s is a half odd integer ($s = (2n-1)/2$) the spectrum of H_φ for the unique ground state φ has no gap. Namely for any positive number δ $Spec\,H_\varphi \cap (0,\delta) \neq \emptyset$. The decay of correlation has the power law, so there exists Q and Q' in $\mathfrak{A}_{loc}$ such that the limit $\lim_{n\to\infty} |\varphi(Q\tau_n(Q')) - \varphi(Q)\varphi(Q')| = 0$ decays in a negative power of n.
(iib) If the spin s is an integer, the spectrum of H_φ has the gap in the sense that $Spec\,H_\varphi \cap (0,\delta) = \emptyset$ for a positive number δ.

It is possible to show that the decay of correlation is exponential if the spectral gap of H_φ is open. In the case of integer spin Heisenberg models, the conjecture suggests the exponential decay of correlation of $|\varphi(Q\tau_n(Q')) - \varphi(Q)\varphi(Q')|$ for any Q and Q' in $\mathfrak{A}_{loc}$ (c.f. Theorem 1.4 below).

Note that when the ground state is unique, it is pure and SU(2) invariant (invariant under the product type action β_g). The conjecture (iia) and (iib) was proposed by Haldane in the beginning of 1980's. The spin 1/2 case of Haldane's conjecture is supported by the exact (non-rigorous) solution. For the half odd integer spin I.Affleck and E.Lieb have shown that the gap of the spectrum of the finite volume Hamiltonian H_Λ vanishes in the infinite volume limit (see [2]). When the spin is integer nothing is known but in [3] I.Affleck, T.Kennedy, E.Lieb and H.Tasaki found an example of Hamiltonian with the similar property to the property Haldane's conjecture claims. In fact, they have shown that when the spin is one and $J_1 = 1$, $J_2 = 1/3$ the Hamiltonian (1.3) has the unique infinite volume ground state which has the exponential decay correlation and the spectral gap. Their Hamiltonian H_{AKLT} is referred to as the AKLT model or the AKLT Hamiltonian:

$$H_{AKLT} = \sum_j \left\{ (S^{(j)}, S^{(j+1)}) + \frac{1}{3}(S^{(j)}, S^{(j+1)})^2 \right\}. \tag{1.5}$$

To prove their results, I.Affleck, T.Kennedy, E.Lieb and H.Tasaki constructed the ground state of the AKLT Hamiltonian with a finite algebraic manipulation (the iteration of complete positive maps on finite matrix algebras). The state they obtained in [3] is named Valence Bond Solid state. Since the paper [3] was published, Valence Bond Solid states was studied extensively. We will return "VBS" states later.

Unlike the integer spin case, we do not have any definite result on the decay of correlation for the half odd integer spin case. However, M.Aizenman and B.Nachtergaele obtained a measure theoretic representation of ground states for a class of Hamiltonians. Within their class of Hamiltonians, M.Aizenman and B.Nachtergaele have found that translation symmetry breaking occurs if the decay of correlation is fast. It is interesting to ask if the result which M.Aizenman and B.Nachtergaele proved is a universal phenomenon or due to special choice of Hamiltonians. The theorem below is related to this question.

Theorem 1.2. *Suppose that the spin s is a half odd integer and consider a translationally invariant pure state φ of $\mathfrak{A}$. If φ is $SU(2)$ invariant, φ cannot have the following uniform cluster property:*

$$\lim_{k\to\infty} \sup_{\|Q\|\leq 1} \left| \sum_i (\varphi(Q_i \tau_k(R_i)) - \varphi(Q_i)\varphi(R_i)) \right| = 0, \tag{1.6}$$

where Q is any local observable in $\mathfrak{A}_{loc}$ written in the finite sum

$$Q = \sum_i \gamma_i Q_i R_i, \qquad Q_j \in \mathfrak{A}_{(-\infty,-1]} \quad R_i \in \mathfrak{A}_{[0,\infty)}.$$

We can formulate the same kind of results as above for an arbitrary compact semisimple Lie group.

Purity and translation invariance of φ implies the cluster property

$$\lim_{k\to\infty}\left|\sum_{i,j}(\varphi(Q_j\tau_k(R_i))-\varphi(Q_j)\varphi(R_i))\right|=0.$$

Let $\{\pi_\varphi, \mathfrak{H}_\varphi\Omega_\varphi\}$ be the GNS representation of a translation invariant factor state φ. Consider the von Neumann algebras $M_1=\pi_\varphi(\mathfrak{A}_{(-\infty,-1]})''$ and $M_2=\pi_\varphi(\mathfrak{A}_{[0,\infty)})'$. The uniformity of the cluster property in (1.6) is equivalent to the condition that the inclusion $M_1\subset M_2$ contains an intermediate type I factor $\mathfrak{N}$. Following R.Longo in [18] we call such an inclusion $M_1\subset\mathfrak{N}\subset M_2$ split.

Definition 1.3. *Let φ be a translation invariant factor state of $\mathfrak{A}$. We say that the state φ is split if the uniform cluster condition (1.6) is satisfied.*

Any Gibbs state for a finite range interaction is split and so is any VBS state. The construction of non-split pure states is a non trivial mathematical problem. We believe that the exponential decay correlation implies the split property of state, as we are not aware of any counter-example to this claim. Moreover Theorem 1.4 tells us that the spectral gap implies the exponential decay of correlation. If the exponential decay of correlation implies the split property, Theorem 1.1 is a solution to a part of Haldane's conjecture.

Theorem 1.4. *Consider a finite range translationally invariant Hamiltonian H_Λ. Suppose that the state φ is a pure ground state, and that the ground state energy of H_φ is non-degenerate and the spectral gap opens in the sense specified in Conjecture 1.1. For any Q and Q' in $\mathfrak{A}_{loc}$,*

$$|\varphi(Q\tau_n(Q'))-\varphi(Q)\varphi(Q')|\le C(Q,Q')e^{-Mn},$$

where $C(Q,Q')$, M are positive constants dependent on Q and Q'

This theorem is a lattice version of Cluster Theorem of K.Fredenhagen (c.f. [15]). A crucial assumption for Cluster Theorem of K.Fredenhagen is strict locality for the time evolution, which is not valid in our case. However we have entire analyticity and quasi-locality of our time evolution, with which we obtain Theorem 1.4. Unfortunately we do not have good estimates for constants $C(Q,Q')$ and M.

About the question whether there exist any pure state without split property, we have the following answer.

Theorem 1.5. *Consider the spin $\frac{1}{2}$ chain. Let φ be a translationally invariant pure state. Suppose that it is invariant under the torus $U(1)$ in $SU(2)$: $\varphi \circ \beta_z = \varphi$ for any $z \in U(1)$. Then φ is either a product state or a non-split state.*

Due to this result, the ground state of the one-dimensional XY model is non-split. We will also discuss this model later.

Next we sketch the key point of our proof for Theorem 1.2, 1.4. The proof is based on notions of split property, the shift of $\mathfrak{B}(\mathfrak{H})$ and Popescu systems.

A standard argument of quasi-local algebra implies that a state φ is split if and only if φ is quasi-equivalent to the state $\varphi_{(-\infty,-1]} \otimes \varphi_{[0,\infty)}$ where $\varphi_{(-\infty,-1]}$ and $\varphi_{[0,\infty)}$ are the restriction of φ to $\mathfrak{A}_{(-\infty,-1]}$ and $\mathfrak{A}_{[0,\infty)}$. The split property of states is one of basic concepts in the local quantum field theory (see [16], [11] and [12] and the references therein).

When φ is a translation invariant pure split state, the restriction $\varphi_{[0,\infty)}$ is a type I factor state. Passing to the GNS representation and the restriction of τ_j $(j \geq 0)$ to $\mathfrak{A}_R$ we obtain the shift of $\mathfrak{B}(\mathfrak{H})$ in the sense of R.Powers ([21]). Any shift of $\mathfrak{B}(\mathfrak{H})$ with Powers index d is implemented by the generator of the Cuntz algebra O_d. Thus the translation invariant state φ is extendible to a state of the Cuntz algebra O_d. The Popescu system describes the connection of symmetry of φ and the representation of the Cuntz algebra O_d. We will explain this connection more explicitly in the next section.

§2. Popescu Systems

Here we begin with the definition of the Popescu system. This notion naturally appears when a translationally invariant state $\varphi_{[0,\infty)}$ of $\mathfrak{A}_{[0,\infty)}$ is extended to a state on the Cuntz algebra O_d $(d = 2s+1)$.

Definition 2.1. *Let $\mathfrak{K}$ be a separable Hilbert space. By Popescu system on $\mathfrak{K}$ we mean the triple $\{\mathfrak{M}, V, \psi\}$ satisfying the following conditions.*

$\mathfrak{M}$ is a von Neumann algebra acting on $\mathfrak{K}$ non-degenerately. V is an isometry from $\mathfrak{K}$ to $\mathbf{C}^d \otimes \mathfrak{K}$. ψ is a normal faithful state of $\mathfrak{M}$ satisfying the invariance

$$\psi(R) = \psi(E(1_{\mathbf{C^d}} \otimes R)), \tag{2.1}$$

where R is any element of $\mathfrak{M}$ and $E(Q)$ is the unital completely positive map from $M_d(\mathbf{C}) \otimes \mathfrak{B}(\mathfrak{K})$ to $\mathfrak{B}(\mathfrak{K})$ determined by

$$E(Q) = V^* Q V \qquad \text{for any } Q \text{ in } M_d(\mathbf{C}) \otimes \mathfrak{B}(\mathfrak{K}). \tag{2.2}$$

Given a Popescu system $\{\mathfrak{M}, V, \psi\}$ on $\mathfrak{K}$, we can construct a translationally invariant state of the UHF algebra $\mathfrak{A}$ (d=2s+1) by the formula

$$\begin{aligned}(2.3)\qquad & \varphi(Q_0^{(j)}Q_1^{(j+1)}Q_2^{(j+2)}\cdots Q_l^{(j+l)}) \\ &= \psi(E(Q_0\otimes E(Q_1\otimes\cdots E(Q_l\otimes 1_{\mathfrak{K}}))\cdots)).\end{aligned}$$

When the dimension of $\mathfrak{K}$ is finite, the state φ determined by (2.3) is called the finitely correlated state or quantum Markov state or Matrix product state in mathematical physics. L.Accardi introduced quantum Markov states as a non-commutative extension of Markov measures in [1]. After the discovery of the AKLT Hamiltonian, M.Fannes, B.Nachtergaele and R,Werner found the relationship between VBS states and quantum Markov states in [13]. (See also [14].) The ground state of the AKLT Hamiltonian is described by the Popescu system where $\mathfrak{K}$ is two dimensional and V is the intertwiner of the representations of SU(2).

Any translationally invariant state is described by the Popescu system if we allow the dimension of $\mathfrak{K}$ to be infinite.

Lemma 2.2. *Let φ be a translationally invariant state of $\mathfrak{A}$. There exists a Popescu system on $\mathfrak{K}$, $\{\mathfrak{M}, V, \psi\}$ such that the state φ is described by (2.3). Furthermore when φ is a factor state, the complete positive map E of (2.2) has the following ergodicity:*
If $E(1_{\mathbf{C}^d}\otimes Q) = Q$ for Q in $\mathfrak{M}$, Q is a scalar $Q = c1$.

The lemma is based on the following observation. Let $\{\pi_\varphi, \mathfrak{H}_\varphi, \Omega_\varphi\}$ be the GNS triple of φ. Consider the von Neumann algebra $\pi(\mathfrak{A}_{[0,\infty)})''$ and the shift τ_1 restricted to $\mathfrak{A}_{[0,\infty)}$. τ_1 is extendible to the endomorphism Θ on the von Neumann algebra $\pi(\mathfrak{A}_{[0,\infty)})''$. Θ is implemented by a representation of the Cuntz algebra. Namely, on $\mathfrak{H}_\varphi$ there exists isometries S_k $(k = 1, 2, ..d)$ such that

$$S_k^* S_j = \delta_{kj}1, \quad \sum_{k=1}^{d} S_k S_k^* = 1$$

and

$$\Theta(Q) = \sum_{k=1}^{d} S_k Q S_k^* \qquad Q \in \pi(\mathfrak{A}_{[0,\infty)})''.$$

Let P be the support projection of φ as the state of $\pi(\mathfrak{A}_{[0,\infty)})''$. Let $\mathfrak{K}$ be the range of P. Then we set $\mathfrak{M} = P\pi(\mathfrak{A}_{[0,\infty)})''P$, $V = (V_1, V_2, ...V_d) = (PS_1^*P, PS_2^*P, ...PS_d^*P)$ and we obtain the claim of Lemma.

In [8] O.Bratteli, P.Jorgensen, A.Kishimoto and R.Werner studied the connection of the Popescu system and the shift of the von Neumann algebra $\mathfrak{M} = P\pi(\mathfrak{A}_{[0,\infty)})''P$ when it is of type I. We also consider the same situation.

Lemma 2.3. *Let φ be a translationally invariant pure state of $\mathfrak{A}$. The following conditions are equivalent:*
(i) The state φ is split;
(ii) The GNS representation of $\mathfrak{A}_{[0,\infty)}$ associated with $\varphi_{[0,\infty)}$ is type I factor.

An endomorphism Θ on a von Neumann algebra $\mathfrak{M}$ is shift if it satisfies the following condition:

$$\cap_{n=01,2,..}\Theta^n(\mathfrak{M}) = \mathbf{C1}.$$

It is known that the shift of a type I factor is inner in the sense that Θ is implemented by the canonical endomorphism of the Cuntz algebra O_d in $\mathfrak{M}$.

Let E be the completely positive map of (2.2) and set

$$E_Q(R) = E(Q \otimes R)$$

for $Q \in M_d(\mathbf{C})$. E_Q is a complete bounded map from $\mathfrak{M}$ to $\mathfrak{M}$.

Lemma 2.4. *Let φ be a translationally invariant factor state of $\mathfrak{A}$ and $\{\mathfrak{M}, V, \psi\}$ be the associated Popescu system. If $\mathfrak{M}_1$ is a von Neumann subalgebra of $\mathfrak{M}$ containing* 1 *and invariant under the operator E_Q with any $Q \in M_d(\mathbf{C})$, then $\mathfrak{M}_1 = 1$ or $\mathfrak{M}_1 = \mathfrak{M}$.*

We call that the above condition the minimality condition. This minimality is crucial in our proof of our Theorem.

Now we proceed to study of the symmetry property of pure states with split property. Here we consider $\mathfrak{A}_{[0,\infty)}$ and the Cuntz algebra implementing the shift. The gauge action γ_U of the group $U(d)$ of d by d unitary matrices is defined via the formula

$$\gamma_U(S_k) = \sum_{l=1}^{d} U_{lk} S_l,$$

where U_{kl} is the k l matrix element for U in $U(d)$. Consider the diagonal circle group $U(1) = \{z \ \ |z| = 1 \ z \in \mathbf{C}\}$. We identify the fixed point algebra ${O_d}^{U(1)}$ with $\mathfrak{A}_{[0,\infty)}$.

Let G be a compact group and $v(g)$ be a d-dimensional unitary representation of G. By β_g we denote the product action of G on the infinite tensor product $\mathfrak{A}$ induced by $v(g)$:

$$\beta_g(Q) = (....\otimes v(g)\otimes v(g)\otimes v(g)\otimes ...)Q(....\otimes v(g)^{-1}\otimes v(g)^{-1}\otimes v(g)^{-1}\otimes ...)$$

for any Q in $\mathfrak{A}$. On $\mathfrak{A}_{[0,\infty)}$ the gauge action $\gamma_{v(g)}$ of the Cuntz algebra O_d and β_g coincide, $\gamma_{v(g)}(Q) = \beta_g(Q)$ for any Q in $\mathfrak{A}_{[0,\infty)}$.

Proposition 2.5. *Let φ be a translationally invariant factor state of $\mathfrak{A}$. Suppose that φ and G invariant:*

$$\varphi(\beta_g(Q)) = \varphi(Q) \quad \textit{for any } g \textit{ in } G \textit{ and any } Q \textit{ in } \mathfrak{A}.$$

Let $\{\mathfrak{M}, V, \psi\}$ be the canonical Popescu system for φ.
(i)There exist a projective unitary representation $u(g)$ of G on $\mathfrak{K}$ and a one-dimensional unitary representation $\xi(g)$ such that V intertwines the representation as follows:

$$(\xi(g)v(g)) \otimes u(g)V = Vu(g). \tag{2.4}$$

(ii) $Ad(u(g))$ leaves $\mathfrak{M}$ invariant.
(iii) The normal state of ψ of $\mathfrak{M}$ is invariant under $Ad(u(g))$.

Proposition 2.6. *Suppose that φ is translationally invariant, pure and split. The automorphism $Ad(u(g))$ on $\mathfrak{M}$ is the inner ($u(g) \in \mathfrak{M}$).*

Suppose, further, that G is the one-dimensional torus $U(1)$ or a connected, simply connected compact semisimple Lie group. The projective representation $u(g)$ is a unitary representation.

Corollary 2.7. *Let G be a connected, simply connected compact semisimple Lie group. Let $\hat{G}$ be the set of irreducible unitary duals of the compact group G. Let $\{F_\alpha\}$ be a partition of $\hat{G}$; $F_\alpha \subset \hat{G}$, $F_\alpha \cap F_\beta =$ $(\alpha \neq \beta)$, $\cup_\alpha F_\alpha = \hat{G}$.*

Suppose that any irreducible component in the tensor of the representation $v(g)$ and F_α is contained in another different F_β.

Then there exists no translationally invariant pure split state.

Using the above Proposition we obtain our main results Theorem 1.2 and 1.4.

§3. Examples

In this section we present a few examples of ground states, which illustrate some aspects of previous results. We consider the translationally invariant Hamiltonian. For simplicity we assume that the interaction is of nearest neighbor. So suppose a selfadjoint element $h_0 = h_0^*$ in $\mathfrak{A}_{\{0,1\}}$ is given and consider the finite volume Hamiltonian $H_{[n,m]}$ in $\mathfrak{A}_{[n,m]}$ on the interval $[n, m]$ is determined by

$$H_{[n,m]} = \sum_{j=n}^{m-1} h_j,$$

where we set $\tau_j(h_0) = h_j$.

We begin with the exactly solvable XY model as an example of Theorem 1.4. The Hamiltonian H_{XY} of the XY model is determined by the equation

$$H_{XY} = -\sum_{j\in\mathbf{Z}} \left\{\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)}\right\} - 2\lambda \sum_{j\in\mathbf{Z}} \sigma_z^{(j)}, \tag{3.1}$$

where λ is a real parameter (an external magnetic field), $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ and $\sigma_z^{(j)}$ are the Pauli spin matrices at the site j. For the finite chain the XY model is equivalent to the free Fermion via the Jordan Wigner transformation. For the infinite chain the equivalence is not literally correct due to the infinite product of $\sigma_z^{(j)}$ in the Jordan Wigner transformation. Nevertheless we have obtained the following results in [5].

Theorem 3.1. *The ground state of the XY model (3.1) is unique for any real* λ.
(i) $|\lambda| \geq 1$ *it is a product state. The spectral gap is open if* $|\lambda| > 1$.
(ii) $|\lambda| < 1$ *the ground state is not a product state. The spectrum is purely absolutely continuous without gap.*

The Hamiltonian H_{XY} of the XY model is invariant under the rotation around the z axis where the infinitesimal generator of the rotation is

$$N = \sum_{j\in\mathbf{Z}} \sigma_z^{(j)}.$$

Due to uniqueness the ground state is invariant under the rotation around the z axis. As a consequence, Theorem 1.4 implies the following.

Corollary 3.2. *For the case* $|\lambda| < 1$, *the unique ground state of the XY model (3.1) is a pure state without split property.*

Next we consider a generalization of the AKLT model of [3]. Here we only give an abstract condition for Hamiltonians.

Assumption 3.3. *(i) We assume that h_j is positive: $H_{[n,m]} \geq 0$ for any $[n,m]$.*
(ii) The dimension of the kernel of $H_{[n,m]}$ (the multiplicity of zero eigenvalue of $H_{[n,m]}$ as a finite matrix) is greater than one and uniformly bounded in n and m:

$$(3.2) \qquad 1 \leq \sup_{n,m} \, dim \, ker \, H_{[n,m]} \, < \, \infty.$$

If a suitable constant is added to the Hamiltonian, AKLT model (1.5) satisfies the Assumption 3.3.

Theorem 3.4. *Suppose that Assumption 3.3 is valid. Let φ be a translationally invariant pure ground state.*

Then the state φ is split and pure. In fact, the auxiliary Hilbert space $\mathfrak{K}$ of the canonical Popescu system is finite dimensional. The two point function decays exponentially fast:

$$\sup_{j \in \mathbf{Z}} |\varphi(Q_1 \tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \, e^{m|j|} \, < \, \infty$$

for any local Q_1 and Q_2.

This result is a converse to a result of M.Fannes, B.Nachetergaele and R.Werner in [13]. They have shown that if φ is a translationally invariant pure state with the finite dimensional auxiliary Hilbert space $\mathfrak{K}$ of the associated Popescu system, there exists a projection P in $\mathfrak{A}_{loc}$ such that

$$\varphi(P_j) = 0 \;\; \text{with} \;\; P_j = \tau_j(P).$$

Thus φ is a ground state for the Hamiltonian $H = \sum_j P_j$.

Thus our Theorem 1.2 asserts impossibility of the construction of SU(2) spin half odd integer models satisfying the Assumption 3.3.

Next we consider another variant of the Heisenberg model, the antiferromagnetic XXZ model. The Hamiltonian of the XXZ model is defined by

$$(3.3) \qquad H_{XXZ} = \sum_j \{\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)} + \Delta\sigma_z^{(j)}\sigma_z^{(j+1)}\},$$

where Δ is a real parameter. We consider the antiferromagnetic region of the model, i.e., $-1 < \Delta$. The following is the standard picture on the

ground state commonly accepted by physicists.
(a) If $-1 < \Delta \leq 1$, the ground state is unique. The correlation function decays in power. There is no spectral gap of Hamiltonian.
(b) When $1 < \Delta$, there exists precisely two pure ground states φ_{even} and φ_{odd}. They are not translationally invariant, however, periodic with period two, $\varphi_{even} = \varphi_{odd} \circ \tau_1$. The Hamiltonian has the spectral gap and their two point correlation functions decay exponentially fast.

The XXZ model is exactly solved by the Yang-Baxter machinery, though none of the above assertions has not be yet proved rigorously except the case where Δ is extremely large $1 << \Delta$. The reason why the large Δ case is well understood is that the standard cluster expansion (convergent perturbation theory) works. We can verify the above claim as well as the split property of ground states. It is easy to see that any product state is not the ground state of the XXZ model. Theorem 1.4 suggests the following implication.

Theorem 3.5. *Consider the XXZ model H_{XXZ} of (3.3) in the antiferromagnetic region. Then, one of the following is valid:*
(i) The ground state is unique and it is not split;
(ii) There exists two pure ground states which are not translationally invariant.

Theorem 3.5 is a complementary result to the one due to I.Affleck and E.Lieb ([2]) saying: there is no spectral gap if the ground state is unique. Their argument does not yield any information on correlation function while our Theorem 3.5 (i) asserts lack of certain uniform clustering.

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