Operator means and their norms

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§1. Introduction

In his very interesting (unpublished) 1979 notes [17] A. McIntosh obtained the following arithmetic-geometric mean inequality for Hilbert space operators H, K, X:

$$(1) \qquad\qquad ||HXK|| \leq \frac{1}{2}||H^*HX + XKK^*||.$$

Among other things he also pointed out that simple alternative proofs for so-called Heinz-type inequalities ([9], and see also the discussions in §2) are possible based on this inequality. Then, about 15 years later Bhatia and Davis ([4]) noticed that the inequality remains valid for all unitarily invariant norms (including the Schatten norms $||\cdot||_p$ and so on). Recall that a norm $|||\cdot|||$ for Hilbert space operators is called unitarily invariant when $|||UXV||| = |||X|||$ for unitary operators U, V, and basic facts on these norms can be found for example in [8, 10, 19]. In recent years the arithmetic-geometric mean and related inequalities have been under active investigation by several authors, and very readable accounts on this subject can be found in [1, 3].

Motivated by all of the above, the authors have investigated simple unified proofs for known (as well as some new) norm inequalities, some refinement of the norm inequality (1) (such as the arithmetic-logarithmic-geometric mean inequality), and a general theory on operator (and/or matrix) means in a series of recent articles [15, 11, 12]. The purpose of the present notes is to give a brief survey on the topics dealt in these articles.

We will derive a variety of integral expressions for relevant operators to establish desired norm inequalities. This means that our arguments are not just for proving norm inequalities, but we are actually solving

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certain operator equations in a very explicit form. We will briefly touch this viewpoint at the end of the article, and more details were worked out in [12, §4,(A)]. Some related analysis can be found in [18], where the notion of a differential is investigated in detail. We also point out that the recent article [6] is technically closely related to our works although the main emphasis there may be different from ours.

§2.   Arithmetic-geometric mean and related inequalities

As was observed in [15] one can obtain simple and unified proofs for the norm inequalities mentioned in §1 based on the Poisson integral formula for the strip

$$S = \{z \in \mathbf{C};\ 0 \leq \mathbf{Im}\ z \leq 1\}.$$

Namely, for $0 < \theta < 1$ we set $d\mu_\theta(t) = a_\theta(t)dt$ and $d\nu_\theta(t) = b_\theta(t)dt$ with

$$a_\theta(t) = \frac{\sin(\pi\theta)}{2(\cosh(\pi t) - \cos(\pi\theta))} \quad \text{and} \quad b_\theta(t) = \frac{\sin(\pi\theta)}{2(\cosh(\pi t) + \cos(\pi\theta))}.$$

Then, for a bounded continuous function $f(z)$ on the strip S which is analytic in the interior, the well-known Poisson integral formula

$$f(i\theta) = \int_{-\infty}^{\infty} f(t)d\mu_\theta(t) + \int_{-\infty}^{\infty} f(i+t)d\nu_\theta(t)$$

is valid (see [20] for example). We point out that the total masses of the measures $d\mu_\theta(t), d\nu_\theta(t)$ are $1 - \theta, \theta$ respectively.

We begin with a simple proof for the arithmetic-geometric mean inequality (1). To this end, we may and do assume the positivity of H, K (by the standard argument on the polar decomposition). The function $f(t) = H^{1+it}XK^{-it}$ ($t \in \mathbf{R}$) extends to a bounded continuous (in the strong operator topology) function on the strip S which is analytic in the interior. Here, H^{it}, K^{-it} are understood as unitaries on the support spaces of H, K respectively. Since $d\mu_{\frac{1}{2}}(t) = d\nu_{\frac{1}{2}}(t) = \dfrac{dt}{2\cosh(\pi t)}$ (with total mass $\frac{1}{2}$), we have

$$
\begin{aligned}
H^{\frac{1}{2}}XK^{\frac{1}{2}} &= f(\tfrac{i}{2}) = \int_{-\infty}^{\infty} f(t)\, d\mu_{\frac{1}{2}}(t) + \int_{-\infty}^{\infty} f(i+t)\, d\nu_{\frac{1}{2}}(t) \\
&= \int_{-\infty}^{\infty} K^{it}(KX + XK)K^{-it} \frac{dt}{2\cosh(\pi t)}.
\end{aligned}
$$

The unitary invariance of $||| \cdot |||$ thus implies

$$|||K^{\frac{1}{2}}XK^{\frac{1}{2}}||| \leq |||HX + XK||| \times \int_{-\infty}^{\infty} \frac{dt}{2\cosh(\pi t)} = \frac{1}{2}|||HX + XK|||.$$

Heinz-type inequalities ([9]) deal with operators of forms $H^{\frac{1}{p}}XK^{\frac{1}{q}} \pm H^{\frac{1}{q}}XK^{\frac{1}{p}}$, where $p, q \geq 1$ with $1/p + 1/q = 1$. Note that the preceding argument also shows

$$(2) \quad H^{\frac{1}{p}}XK^{\frac{1}{q}} = \int_{-\infty}^{\infty} H^{it}HXK^{-it}d\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{\infty} H^{it}XKK^{-it}d\nu_{\frac{1}{q}}(t),$$

$$(3) \quad H^{\frac{1}{q}}XK^{\frac{1}{p}} = \int_{-\infty}^{\infty} H^{it}HXK^{-it}d\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{\infty} H^{it}XKK^{-it}d\nu_{\frac{1}{p}}(t).$$

We note $d\mu_{\frac{1}{q}} = d\nu_{\frac{1}{p}}$ and $d\mu_{\frac{1}{p}} = d\nu_{\frac{1}{q}}$. Hence, by summing up (2) and (3), we get

$$
\begin{aligned}
H^{\frac{1}{p}}XK^{\frac{1}{q}} + H^{\frac{1}{q}}XK^{\frac{1}{p}} &= \int_{-\infty}^{\infty} H^{it}(HX + XK)K^{-it}d\mu_{\frac{1}{q}}(t) \\
&\quad + \int_{-\infty}^{\infty} H^{it}(HX + XK)K^{-it}d\nu_{\frac{1}{q}}(t) \\
&= \int_{-\infty}^{\infty} K^{it}(HX + XK)K^{-it}d(\mu_{\frac{1}{q}} + \nu_{\frac{1}{q}})(t).
\end{aligned}
$$

This expression obviously shows

$$|||H^{\frac{1}{p}}XK^{\frac{1}{q}} + H^{\frac{1}{q}}XK^{\frac{1}{p}}||| \leq |||HX + XK|||$$

since the total mass of the measure $d(\mu_{\frac{1}{q}} + \nu_{\frac{1}{q}})(t)$ is $\frac{1}{p} + \frac{1}{q} = 1$. The "difference version"

$$(4) \qquad |||H^{\frac{1}{p}}XK^{\frac{1}{q}} - H^{\frac{1}{q}}XK^{\frac{1}{p}}||| \leq |\frac{2}{p} - 1| \times |||HX - XK|||.$$

is also valid. Indeed, by subtracting (3) from (2), we have

$$H^{\frac{1}{p}}XK^{\frac{1}{q}} - H^{\frac{1}{q}}XK^{\frac{1}{p}} = \int_{-\infty}^{\infty} H^{it}(HX - XK)K^{-it}d(\mu_{\frac{1}{q}} - \nu_{\frac{1}{q}})(t).$$

It is plain to see

$$a_{\frac{1}{q}}(t) - b_{\frac{1}{q}}(t) = \frac{\sin(\frac{\pi}{q}) \times 2\cos(\frac{\pi}{q})}{2\left(\cosh^2(\pi t) - \cos^2(\frac{\pi}{q})\right)} = \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})},$$

and consequently we have

$$H^{\frac{1}{p}}XK^{\frac{1}{q}} - H^{\frac{1}{q}}XK^{\frac{1}{p}}$$

$$= \int_{-\infty}^{\infty} H^{it}(HX - XK)K^{-it} \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})}\, dt$$

$$= \int_{-\infty}^{\infty} H^{\frac{is}{2}}(HX - XK)K^{-\frac{is}{2}} \frac{\sin(\frac{2\pi}{q})}{\cosh(\pi s) - \cos(\frac{2\pi}{q})} \times \frac{ds}{2}.$$

Let us assume $1 < p < 2$. Then, the above measure is exactly $d\mu_{\frac{q}{2}}(s)$ with the total mass $1 - \frac{2}{q} = 1 - 2(1 - \frac{1}{p}) = \frac{2}{p} - 1$ (> 0), showing (4) in this case. The opposite case $2 < p < \infty$ can be handled simply by switching H and K.

It follows from (4) that

$$|||H^{\frac{1}{2}+\varepsilon}XK^{\frac{1}{2}-\varepsilon} - H^{\frac{1}{2}-\varepsilon}XK^{\frac{1}{2}+\varepsilon}||| \leq 2\varepsilon \times |||HX - XK|||$$

is valid for $0 < \varepsilon < \frac{1}{2}$. Let us assume the invertibility of $H, K \geq 0$ here. By dividing the above by ε and then by letting $\varepsilon \searrow 0$, we easily see

$$|||(\log H)(H^{\frac{1}{2}}XK^{\frac{1}{2}}) - (H^{\frac{1}{2}}XK^{\frac{1}{2}})(\log K)||| \leq |||HX - XK|||.$$

From this we obtain the following commutator estimate:

Theorem 1 (Theorem 4, [15]).   *For operators A, B, X with A, B self-adjoint, we have*

$$|||AX - XB||| \leq |||\exp(\tfrac{A}{2})X\exp(-\tfrac{B}{2}) - \exp(-\tfrac{A}{2})X\exp(\tfrac{B}{2})|||$$

for each unitarily invariant norm $||| \cdot |||$.

A somewhat related topic is the "matrix Young inequality" due to T. Ando. In [2] he showed that for each positive matrices H, K and $1 < p < \infty$ (with the conjugate exponent q) one can find a unitary matrix U satisfying

$$|HK| \leq U(\tfrac{1}{p}H^p + \tfrac{1}{q}K^q)U^*$$

(the special case $p = q = 2$ was dealt in [5]). In particular,

$$|||HK||| \leq |||\tfrac{1}{p}H^p + \tfrac{1}{q}K^q|||$$

is valid, however in [2, §7] he pointed out that

(5) $|||HXK||| \leq |||\tfrac{1}{p}H^pX + \tfrac{1}{q}XK^q|||$ is **false**

(unless $p = 2$) for example for the operator norm $||| \cdot ||| = || \cdot ||$.

Let us try to understand this phenomenon. Assume that H ($=\exp A$), X are matrices, and let A be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbf{R}$. We set

$$Y = \int_{-\infty}^{\infty} \exp(itA)(\tfrac{1}{p}\exp(A)X + \tfrac{1}{q}X\exp(A))\exp(-itA)f(t)dt$$

with $f(t)$ to be determined. The (j,k)-component of Y is

$$Y_{j,k} = (\tfrac{1}{p}\exp(\lambda_j) + \tfrac{1}{q}\exp(\lambda_k))(\mathcal{F}f)(\lambda_j - \lambda_k)X_{j,k}.$$

Therefore, if one wants $Y = \exp(\tfrac{A}{p})X\exp(\tfrac{A}{q})$, then one must have

$$\left(\frac{\exp(\lambda_j)}{p} + \frac{\exp(\lambda_k)}{q}\right)(\mathcal{F}f)(\lambda_j - \lambda_k) = \exp(\tfrac{\lambda_j}{p})\exp(\tfrac{\lambda_k}{q}).$$

This requirement is the same as

$$(6) \qquad (\mathcal{F}f)(\lambda_j - \lambda_k) = \frac{\exp(\tfrac{\lambda_j}{p})\exp(\tfrac{\lambda_k}{q})}{\frac{\exp(\lambda_j)}{p} + \frac{\exp(\lambda_k)}{q}}$$

$$= \frac{1}{\frac{1}{p}\exp(\tfrac{\lambda_j - \lambda_k}{q}) + \frac{1}{q}\exp(-\tfrac{\lambda_j - \lambda_k}{p})},$$

that is,

$$(\mathcal{F}f)(s) = \left(\tfrac{1}{p}\exp(\tfrac{s}{q}) + \tfrac{1}{q}\exp(-\tfrac{s}{p})\right)^{-1}.$$

It is possible to compute explicitly the inverse Fourier transform of this function, and indeed we can prove

$$(7) \qquad f(t) = \frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(\frac{p}{q}\right)^{-it}}{2\cosh\left(\pi t + \frac{\pi i}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right)}.$$

By the standard approximation argument we get the next result in the special case $H = K$. Then, the general case can be handled by the well-known 2×2-matrix trick: by applying the special case to

$$\tilde{H} = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} \quad \text{and} \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix},$$

one can look at the $(1,2)$-component to get the desired conclusion.

Theorem 2 (Theorem 6, [15]). *For operators H, K, X with H, K positive and $p \in (1, \infty)$ with the conjugate exponent q we have*

$$H^{\frac{1}{p}} X K^{\frac{1}{q}} = \int_{-\infty}^{\infty} H^{it}(\tfrac{1}{p} HX + \tfrac{1}{q} XK) K^{-it} f(t) dt$$

with the function $f(t)$ defined by (7). In particular, for each unitarily invariant norm $||| \cdot |||$ we have

$$|||H^{\frac{1}{p}} X K^{\frac{1}{q}}||| \leq k_p |||\tfrac{1}{p} HX + \tfrac{1}{q} XK|||$$

with $k_p = \int_{-\infty}^{\infty} |f(t)| dt \ (< \infty)$.

Notice $\int_{-\infty}^{\infty} f(t) dt = (\mathcal{F}f)(0) = 1$. However, $f(t)$ is complex-valued and consequently $k_p = \int_{-\infty}^{\infty} |f(t)| dt > 1$ (unless $p = 2$). This fact corresponds to the failure of (5). The constant k_p can be rewritten as

$$k_p = \frac{p^{\frac{1}{p}} q^{\frac{1}{q}}}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}} \quad \text{with } \kappa = \sin\left(\frac{\pi}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right).$$

Note that k_p depends only on $p \in (1, \infty)$ (independent of the choice of $||| \cdot |||$), but unfortunately k_p blows up when either $p \searrow 1$ or $p \nearrow \infty$. On the other hand, unitarily invariant norms under which the map $A \to |A|$ is Lipschitz continuous were thoroughly analyzed in [7, 14]. For such a unitarily invariant norm $||| \cdot |||$ a constant $k = k_{||| \cdot |||}$ can be chosen in such a way that

$$|||H^{\frac{1}{p}} X K^{\frac{1}{q}}||| \leq k|||\tfrac{1}{p} HX + \tfrac{1}{q} XK|||$$

is valid for all $p \in (1, \infty)$ (see [12, Proposition 3.1]).

§3. Refinement of the arithmetic-geometric mean inequality

The logarithmic mean of positive scalars λ, μ is

$$\frac{\lambda - \mu}{\log \lambda - \log \mu} = \int_0^1 \lambda^t \mu^{1-t} dt.$$

The second integral form indicates that for operators H, K, X with $H, K \geq 0$ one can introduce their logarithmic mean by

$$L = \int_0^1 H^t X K^{1-t} dt.$$

The above right side should be understood in the weak sense, i.e.,

$$(L\xi, \eta) = \int_0^1 (H^t X K^{1-t}\xi, \eta)dt \quad \text{(for each vectors } \xi, \eta).$$

For simplicity, we set

$$G = H^{\frac{1}{2}} X K^{\frac{1}{2}} \quad \text{(geometric mean)},$$

$$A = \frac{1}{2}(HX + XK) \quad \text{(arithmetic mean)},$$

and we would like to compare the three means.

The ratios (between the relevant scalar means) are

$$\frac{\log \lambda - \log \mu}{\lambda - \mu} \times \sqrt{\lambda\mu} = g_1(\log \lambda - \log \mu),$$

$$\frac{\lambda - \mu}{\log \lambda - \log \mu} \times \frac{2}{\lambda + \mu} = g_2(\log \lambda - \log \mu)$$

with

$$g_1(s) = \frac{s/2}{\sinh(s/2)} \quad \text{and} \quad g_2(s) = \frac{\tanh(s/2)}{s/2}.$$

By repeating the argument (recall (6)) before Theorem 2 with H^{it} instead of e^{itA}, we arrive at the integral expressions

$$G = \int_{-\infty}^{\infty} H^{it} L K^{-it} \frac{\pi}{2\cosh^2(\pi t)} \, dt,$$

$$L = \int_{-\infty}^{\infty} H^{it} A K^{-it} \log\left|\coth\left(\frac{\pi t}{2}\right)\right| \frac{2dt}{\pi}.$$

The densities $\dfrac{\pi}{2\cosh^2(\pi t)}$ and $\dfrac{2}{\pi}\log\left|\coth\left(\dfrac{\pi t}{2}\right)\right|$ here arise as the inverse Fourier transforms of $g_1(s)$ and $g_2(s)$. They are positive functions (i.e., $g_i(s)$'s are positive definite thanks to the Bochner theorem) with total mass $g_i(0) = 1$. Consequently, we get the following strengthening of (1) (i.e., arithmetic-logarithmic-geometric mean inequality):

Proposition 3 (Proposition 1, [11]). *Let H, K, X be Hilbert space operators with $H, K \geq 0$. For any unitarily invariant norm $|||\cdot|||$ we have*

$$|||H^{\frac{1}{2}} X K^{\frac{1}{2}}||| \leq \left|\left|\left|\int_0^1 H^s X K^{1-s} ds\right|\right|\right| \leq \frac{1}{2}|||HX + XK|||.$$

Actually, further refinement is possible by introducing the two series
of operator means corresponding to the following natural scalar means:

$$\frac{1}{m}\sum_{k=0}^{m-1}\lambda^{\frac{k}{m-1}}\mu^{\frac{m-1-k}{m-1}}, \quad \frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{k}{n+1}}\mu^{\frac{n+1-k}{n+1}}.$$

The cases $m = 2$ and $n = 1$ correspond to the arithmetic and geo-
metric means respectively. Note that what was important in the proof
of Proposition 3 is the positive definiteness of ratios between relevant
scalar means, and this reasoning (together with some others) enables us
to prove

Theorem 4 (Theorem 5, [11]).   *Let H, K, X be Hilbert space op-
erators with H, K positive, and $|||\cdot|||$ be a unitarily invariant norm.*
(i)   For each m (≥ 1) and n (≥ 2), the following inequalities are valid:

$$|||H^{\frac{1}{2}}XK^{\frac{1}{2}}||| \leq \frac{1}{m}|||\sum_{k=1}^{m}H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}||| \leq |||\int_{0}^{1}H^{t}XK^{1-t}dt|||$$

$$\leq \frac{1}{n}|||\sum_{k=0}^{n-1}H^{\frac{k}{n-1}}XK^{\frac{n-1-k}{n-1}}||| \leq \frac{1}{2}|||HX + XK|||.$$

(ii)   The quantity $\dfrac{1}{m}|||\sum_{k=1}^{m}H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}|||$ is monotone increasing in
m, and furthermore we have the monotone convergence

$$\lim_{m\to\infty}\frac{1}{m}|||\sum_{k=1}^{m}H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}||| = |||\int_{0}^{1}H^{t}XK^{1-t}dt|||.$$

(iii)   The quantity $\dfrac{1}{n}|||\sum_{k=0}^{n-1}H^{\frac{k}{n-1}}XK^{\frac{n-1-k}{n-1}}|||$ is monotone decreasing in
n.

Notice that the assertion (ii) in the theorem is a certain monotone
convergence theorem for a norm, and more precise convergence results
(for operators) for various means are investigated in our recent article
[13].

§4.   General means for matrices

It is clear from the discussions so far that the positive definiteness of
ratios between involved scalar means is a key to establish norm inequal-
ities. This viewpoint in fact enables us to investigate norm comparison

of means in a more axiomatic fashion, which makes it possible to han-
dle various other means. In this section we explain this approach, but
for simplicity we will mainly restrict ourselves to finite-dimensional op-
erators (see Remark 6 for the infinite-dimensional case). Namely, we
introduce a certain class of binary means (for positive scalars), to each
of which one can associate a matrix mean in a natural way.

By a symmetric homogeneous mean we shall mean a continuous
positive function on $[0, \infty) \times [0, \infty)$ satisfying

   (a) $M(\lambda, \mu) = M(\mu, \lambda)$,
   (b) $M(\alpha\lambda, \alpha\mu) = \alpha M(\lambda, \mu)$ for any $\alpha > 0$,
   (c) $M(\lambda, \mu)$ is non-decreasing in λ and μ,
   (d) $\min\{\lambda, \mu\} \leq M(\lambda, \mu) \leq \max\{\lambda, \mu\}$.

We denote by $\mathfrak{M}$ the set of all such means.

For $H \in M_n(\mathbf{C})$, the $n \times n$ matrices, we write $H \geq 0$ if H is positive
semi-definite, and $H > 0$ if $H \geq 0$ is invertible. We regard $M_n(\mathbf{C})$
as a (finite-dimensional) Hilbert space equipped with the inner product
$\langle X, Y \rangle = \mathrm{Tr}(XY^*)$ $(X, Y \in M_n(\mathbf{C}))$. For $H, K \geq 0$ let $\mathbf{L}_H, \mathbf{R}_K$ be the
left multiplication by H and the right multiplication by K respectively,
i.e., $\mathbf{L}_H X = HX$ and $\mathbf{R}_K X = XK$ for $X \in M_n(\mathbf{C})$. Note that they are
commuting positive operators acting on $M_n(\mathbf{C})$, and for each $M \in \mathfrak{M}$
one can perform the functional calculus $M(\mathbf{L}_H, \mathbf{R}_K)$ (which is a positive
operator acting on $M_n(\mathbf{C})$). Thus, for each $X \in M_n(\mathbf{C})$ we can consider
$M(\mathbf{L}_H, \mathbf{R}_K)X$ $(\in M_n(\mathbf{C}))$, which will be simply denoted by $M(H, K)X$.

Assume that the spectral decompositions of $H, K \in M_n(\mathbf{C})$ are

$$H = \sum_{i=1}^{n} \lambda_i P_i, \quad K = \sum_{j=1}^{n} \mu_j Q_j$$

with eigenvalue lists $\{\lambda_i\}, \{\mu_j\}$ and rank-one projections $\{P_i\}, \{Q_j\}$ re-
spectively. Then, $M(H, K)$ is obviously given by

$$(8) \qquad M(H, K)X = \sum_{i,j=1}^{n} M(\lambda_i, \mu_j) P_i X Q_j.$$

This means that with the diagonalization

$$H = U\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U^*, \quad K = V\mathrm{diag}(\mu_1, \mu_2 \ldots, \mu_n)V^*$$

via unitary matrices U, V we have

$$(9) \qquad M(H, K)X = U\Big([M(\lambda_i, \mu_j)] \circ (U^* X V)\Big)V^*,$$

where $\circ$ means the Hadamard product (i.e., the entry-wise product).

With the interpretation of $M(H, K)X$ explained so far, we can prove

Theorem 5 (Theorem 1.1, [12]).   *For means $M, N \in \mathfrak{M}$ the following conditions are equivalent:*

(i) *one can find a symmetric probability measure ν on $\mathbf{R}$ satisfying*

$$M(H, K)X = \int_{-\infty}^{\infty} H^{is}(N(H, K)X)K^{-is} \, d\nu(s)$$

   for all matrices H, K, X (of any size) with $H, K > 0$;
(ii) *one has $|||M(H, K)X||| \leq |||N(H, K)X|||$ for each matrices H, K, X (of any size) with $H, K \geq 0$ and for each unitarily invariant norm $||| \cdot |||$;*
(iii) *one has $\|M(H, H)X\| \leq \|N(H, H)X\|$ for each matrices H, X (of any size) with $H \geq 0$;*
(iv) *for each $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ (with any n), $\left[\dfrac{M(\lambda_i, \lambda_j)}{N(\lambda_i, \lambda_j)} \right]_{1 \leq i, j \leq n}$ is a positive semi-definite matrix;*
(v) *the function $M(e^t, 1)/N(e^t, 1)$ is positive definite on $\mathbf{R}$.*

In the above, the measure ν in (i) is the representing one for the ratio $M(e^t, 1)/N(e^t, 1)$ in the Bochner theorem, i.e.,

$$M(e^t, 1)/N(e^t, 1) = \int_{-\infty}^{\infty} e^{its} \, d\nu(s).$$

We consider the following typical one-parameter families of means:

$$M_\alpha(\lambda, \mu) = \begin{cases} \frac{\alpha-1}{\alpha} \times \frac{\lambda^\alpha - \mu^\alpha}{\lambda^{\alpha-1} - \mu^{\alpha-1}} & (\lambda \neq \mu) \\ \lambda & (\lambda = \mu), \end{cases}$$

$$B_\alpha(\lambda, \mu) = \left(\frac{\lambda^\alpha + \mu^\alpha}{2} \right)^{1/\alpha}$$

with $-\infty \leq \alpha \leq \infty$. The arithmetic, logarithmic and geometric means appear as

$$M_2(\lambda, \mu) = \frac{\lambda + \mu}{2},$$

$$M_1(\lambda, \mu) = \frac{\lambda - \mu}{\log \lambda - \log \mu} \left(= \lim_{\alpha \to 1} M_\alpha(\lambda, \mu) \right),$$

$$M_{1/2}(\lambda, \mu) = \sqrt{\lambda\mu},$$

while it is easy to see

$$M_{\frac{n}{n-1}}(\lambda,\mu) \;=\; \frac{1}{n}\sum_{k=0}^{n-1}\lambda^{\frac{k}{n-1}}\mu^{\frac{k-1}{n-1}} \quad (n=2,3,\cdots),$$

$$M_{\frac{m}{m+1}}(\lambda,\mu) \;=\; \frac{1}{m}\sum_{k=1}^{m}\lambda^{\frac{k}{m+1}}\mu^{\frac{m+1-k}{m+1}} \quad (m=1,2,\cdots)$$

(which correspond to the operator means appeared in Theorem 4). On the other hand, with the special choice $\alpha = 1/n$

$$B_{1/n}(\lambda,\mu) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\lambda^{\frac{k}{n}}\mu^{\frac{n-k}{n}}$$

is the usual binomial mean.

For $-\infty \le \alpha \le \beta \le \infty$ one can prove the positive definiteness of the ratio

$$\begin{aligned}
\frac{M_\alpha(e^t,1)}{M_\beta(e^t,1)} &= \frac{(\alpha-1)\beta}{\alpha(\beta-1)} \times \frac{(e^{\alpha t}-1)(e^{(\beta-1)t}-1)}{(e^{(\alpha-1)t}-1)(e^{\beta t}-1)} \\
&= \frac{(\alpha-1)\beta}{\alpha(\beta-1)} \times \frac{\sinh(\alpha t/2)\sinh((\beta-1)t/2)}{\sinh((\alpha-1)t/2)\sinh(\beta t/2)}
\end{aligned}$$

(see [12, Theorem 2.1]). Therefore, thanks to Theorem 5 we can obtain further generalization of Theorem 4 in §3. One can also prove the positive definiteness of ratios such as

$$\begin{aligned}
\frac{M_{1/2}(e^t,1)}{B_\alpha(e^t,1)} &= \left(\frac{1}{\cosh(\alpha t/2)}\right)^{1/\alpha} \quad (\alpha>0), \\
\frac{B_{1/n}(e^t,1)}{M_2(e^t,1)} &= \frac{\cosh^n(t/2n)}{\cosh(t/2)}.
\end{aligned}$$

Note

$$\frac{1}{\cosh t} = \int_{-\infty}^{\infty} e^{its}\,\frac{ds}{2\cosh(\pi s/2)},$$

and the positive definiteness of the former is indeed a consequence of the infinite divisibility of the probability measure $(2\cosh(\pi s/2))^{-1}ds$. From these we conclude

$$|||H^{1/2}XK^{1/2}||| \le \frac{1}{2^n}|||\sum_{k=0}^{n}\binom{n}{k}H^{\frac{k}{n}}K^{\frac{n-k}{n}}||| \le \frac{1}{2}|||HX+XK|||$$

for instance (see [12, Proposition 3.3]). Some other means as well as a variety of comparison results for their norms (based on Theorem 5) are obtained in [12].

The idea behind Theorem 5 (especially the integral representation for matrix means) can be also adopted to obtain solutions to certain matrix equations in a very explicit way. To see a flavor of this application, as a typical example we consider the matrix equation

$$\int_0^1 H^t Y K^{1-t} dt = X$$

for a unknown matrix Y with positive invertible matrices H, K. The equation means $M_1(H, K)Y = X$ with the logarithmic mean $M_1(\lambda, \mu) = \dfrac{\lambda - \mu}{\log \lambda - \log \mu}$. It is plain to see that the reciprocal is $M_0(\lambda^{-1}, \mu^{-1})$, and it follows from the expression (9) that the unique solution Y is given by

$$Y = M_0(H^{-1}, K^{-1})X.$$

The comparison of M_0 with $M_{1/2}$, for instance, supplies the integral expression

$$Y = \int_{-\infty}^{\infty} H^{-\frac{1}{2}+is} X K^{-\frac{1}{2}-is} \frac{\pi ds}{2 \cosh^2(\pi s)}$$

for this solution. Furthermore, the different integral expressions

$$Y = \int_0^{\infty} (H + tI)^{-1} X (K + tI)^{-1} dt$$

and

$$Y = \int_0^{\infty} \int_0^{\infty} e^{-sH} X e^{-tK} \frac{ds dt}{s+t}$$

for the same Y are also possible based on some other tools (see [12]).

Remark 6. *It is possible to generalize Theorem 5 to infinite-dimensional operators. Namely, we simply replace $M_n(\mathbf{C})$ by the Hilbert space $\mathcal{C}_2(\mathcal{H})$ of Hilbert-Schmidt class operators. In this setting the multiplication operators $\mathbf{L}_H, \mathbf{R}_K$ (positive operators in $B(\mathcal{C}_2(\mathcal{H}))$) can be also considered for arbitrary positive operators $H, K \geq 0$. Consequently, as long as X is taken from $\mathcal{C}_2(\mathcal{H})$, the mean $M(H, K)X = M(\mathbf{L}_H, \mathbf{R}_K)X$ ($\in \mathcal{C}_2(\mathcal{H})$) makes a perfect sense. With this interpretation the theorem remains valid for Hilbert space operators. In [12, §4,(C)] the requirement $X \in \mathcal{C}_2(\mathcal{H})$ was not explicitly mentioned, and we apologize for this inaccuracy.*

A theory of means $M(H, K)X$ with $X \in B(\mathcal{H})$ is more preferable. Such a theory is developed in our recent article [13] based on the theory of double integral transformations. Roughly speaking it is a continuous version of (8), and for a very wide class of scalar means $M(\lambda, \mu)$ (including all the examples in [12]) corresponding operator means $M(H, K)X$ ($\in B(\mathcal{H})$) are completely justified for each $X \in B(\mathcal{H})$. Moreover, in the forthcoming article [16] we will obtain a variety of new norm inequalities not covered here (nor in [11, 12, 13, 15]).

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