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Finite approximations and physics over unconventional fields

Trond Digernes

In this talk we will discuss some ideas and results from 'unconventional physics', partly from the point of view of finite approximations.

Finite approximations play an important role in many areas of mathematics. In operator algebras there are several notions of approximate finiteness, for example *hyperfinite* algebras, AF-algebras, *residually finite* algebras, to mention some.

In the context of locally compact abelian groups there is a useful notion of closeness which takes into account the Weyl structures of the groups. It was shown in [4] that – with respect to this concept of closeness – any (separable) locally compact abelian group is a limit of finite abelian groups. This notion of convergence – called convergence of Weyl systems – involves *approximation from the outside*, i.e., the approximating groups need not be subgroups of the given group.

Convergence of Weyl systems takes place at the kinematical level. The deeper problem of approximating dynamical operators requires a more detailed analysis, and was treated in [6] for the case $\mathbf{R}^n$. Here it was shown that for quantum systems with potentials of 'oscillator type' (essentially those with discrete Hamilton spectrum), the finite approximands converge to the continuous system in the strongest possible sense: eigenvalues and eigenfunctions for the finite systems converge to the corresponding objects for the continuous system. These results have later been generalized to the setting of a general locally compact group [1]. (In this general setting, though, the position and momentum operators do not have obvious interpretations.)

The above approximation results may serve as motivation for studying quantum systems over fields other than $\mathbf{R}$ and $\mathbf{C}$ – like $\mathbf{Q}_p$, for instance – since, after all, such systems, too, can be obtained as limits of finite systems (where most computations will have to take place). This is,

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however, a rather modest approach to the subject of 'physics over unconventional fields and rings' (or 'unconventional physics' for short). There are other, and more profound, reasons, of which we mention just a couple here: 1) It can be argued that the structure of space-time below the Planck scale is best described by a number field with a non-archimedean metric (like $\mathbf{Q}_p$; see [13] and further references there). 2) Regularization: the well-known divergences in quantum field theory disappear when the underlying field is non-archimedean; and unbounded operators become bounded [2]. -It should be mentioned that simple quantum mechanical models, like the harmonic oscillator, can be successfully formulated over $\mathbf{Q}_p$, although it is not obvious what the nature of 'time' should be: should it be p-adic, real, or discrete? [13, 10, 7]

Since there does not seem to be a preferred prime p in nature, one eventually has to work with all the primes at the same time, which leads to the *adelic* theories (the ring of *adeles* is the restricted product of all the $\mathbf{Q}_p$'s, including $\mathbf{Q}_\infty = \mathbf{R}$). An adelic string theory has been formulated (see the excellent review article by Brekke and Freund [3] and the references therein), and Manin gives a beautiful argument for the adelic nature of our physical world in [8].

In a forthcoming paper [5] we study some other phenomena which may occur in general models. More specifically, we study irreducible models for Heisenberg groups based on compact maximal isotropic subgroups. It is shown that if both the Heisenberg group and the subgroup are 2-regular, the "vacuum sector" of the associated representation is 1-dimensional and thus gives rise to a unique vacuum state (this generalizes a result in [13] for $\mathbf{Q}_p$ with $p \neq 2$). On the other hand, if the Heisenberg group is 2-regular, but the subgroup is not, the vacuum sector exhibits a fermionic structure. This will be the case, for instance, in a quantum mechanical model built on the 2-adic numbers, with a maximal isotropic subgroup constructed from the 2-adic integers.

Finally it should be mentioned that the idea of doing physics in a setting other than ordinary space-time appears already in the works of Weyl [14, 15] and Schwinger (see, e.g., [11] and further references there; this article served as a starting point and motivation for the paper [6]). In fact, some of these ideas go back to Riemann [9] (for an English translation, see [12, page 135]).

References

[1] S. Albeverio, E. I. Gordon, and A. Yu. Khrennikov, *Finite-dimensional approximations of operators in the Hilbert spaces of functions on locally*

compact abelian groups, Acta Appl. Math. **64** (2000), no. 1, 33–73. MR **2002f:**47030

[2] Sergio Albeverio and Andrew Khrennikov, *A regularization of quantum field Hamiltonians with the aid of p-adic numbers*, Acta Appl. Math. **50** (1998), no. 3, 225–251. MR **99f:**81117

[3] Lee Brekke and Peter G. O. Freund, *p-adic numbers in physics*, Phys. Rep. **233** (1993), no. 1, 1–66. MR **94h:**11115

[4] T. Digernes, E. Husstad, and V. S. Varadarajan, *Finite approximation of Weyl systems*, Math. Scand. **84** (1999), no. 2, 261–283. MR **2001b:**22006

[5] Trond Digernes and V. S. Varadarajan, *Models for the irreducible representation of a Heisenberg group*, in preparation.

[6] Trond Digernes, V. S. Varadarajan, and S. R. S. Varadhan, *Finite approximations to quantum systems*, Rev. Math. Phys. **6** (1994), no. 4, 621–648. MR **96e:**81028

[7] Peter G. O. Freund and Mark Olson, *p-adic dynamical systems*, Nuclear Phys. B **297** (1988), no. 1, 86–102. MR **89g:**81026

[8] Yu. I. Manin, *Reflections on arithmetical physics*, Conformal invariance and string theory (Poiana Braşov, 1987), Perspect. Phys., Academic Press, Boston, MA, 1989, pp. 293–303. MR **90m:**11195

[9] Bernhard Riemann, *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge*, Springer-Verlag, Berlin, 1990, Based on the edition by Heinrich Weber and Richard Dedekind, Edited and with a preface by Raghavan Narasimhan. MR **91j:**01070b

[10] Ph. Ruelle, E. Thiran, D. Verstegen, and J. Weyers, *Quantum mechanics on p-adic fields*, J. Math. Phys. **30** (1989), no. 12, 2854–2874. MR **90k:**81241

[11] Julian Schwinger, *Unitary operator bases*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 570–579. MR 22 #6446

[12] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I*, second ed., Publish or Perish Inc., Wilmington, Del., 1979. MR **82g:**53003a

[13] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-adic analysis and mathematical physics*, Series on Soviet and East European Mathematics, vol. 1, World Scientific Publishing Co. Inc., River Edge, NJ, 1994. MR **95k:**11155

[14] Hermann Weyl, *Theory of Groups and Quantum Mechanics*, Dover, New York, 1931.

[15] ———, *Space, Time, and Matter*, Dover, New York, 1950, p. 98.

The Norwegian University of Science and Technology
7491 Trondheim
Norway
E-mail address: digernes@math.ntnu.no