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A notion of Morita equivalence between subfactors

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Abstract.

We will review a notion of Morita equivalence between subfactors, which is a variation of Morita equivalence in ring and module theory. The main result is stated as follows: for arbitrary two Morita equivalent subfactors of hyperfinite II_1 factors with finite Jones index and finite depth we can always choose a finite dimensional non-degenerate commuting square which gives rise to the subfactors isomorphic to the original ones. As an application of Morita equivalence between subfactors in connection with recent developments of theory of finite dimensional weak C^*-Hopf algebras, we will make a brief comment about the 3-dimensional topological quantum field theories obtained from subfactors with finite index and finite depth.

§1. Introduction

A basic tool to construct inclusions of hyperfinite II_1 factors with finite Jones index would be the method of the commuting squares. From a finite dimensional non-degenerate commuting square

$$\begin{array}{ccc} R_{00} & \subset & R_{01} \\ \cap & & \cap \\ R_{10} & \subset & R_{11} \end{array}$$

we can make the double sequence of finite dimensional C^*-algebras $\{R_{ij}\}$ by iterating the Jones' basic construction. Then we get horizontal and vertical inclusions of hyperfinite II_1 factors with finite Jones indices $N = \vee_{n=0}^{\infty} R_{0n} \subset \vee_{n=0}^{\infty} R_{1n} = M$ and $P = \vee_{n=0}^{\infty} R_{n0} \subset \vee_{n=0}^{\infty} R_{n1} = Q$ under the assumption that the Bratteli diagrams of the initial inclusions in the commuting square are connected.

A natural question is whether there is any relationship between these two inclusions. In particular, when one of the inclusions is of finite depth, then it is of great interest to know whether so is the other. Such questions were first raised by V. Jones in 1995. This question was answered

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affirmatively using techniques of paragroup theory and the answer says these inclusions have the same "size as C^*-tensor categories" [S1].

When we have an inclusion of hyperfinite II_1 factors with finite Jones index and finite depth, we can construct the topological quantum field theory in three dimensions (TQFT) based on the triangulation of the given 3-dimensional manifold V. This method was first done by Ocneanu. In the case of a closed manifold, we get a complex number. The simplest example will be the three dimensional sphere S^3. In this case, the value of the theory is given by 1/(the size of C^*-tensor category). With this observation and the above relationship between the vertical and horizontal inclusions, we can prove TQFT's constructed from the above vertical and horizontal inclusions are complex conjugate to each other [S2]. This gives a finer answer to Jones' question.

There should be a reasonable explanation for the above phenomenon. We will see this can be achieved by the notion of "Morita equivalence" between subfactors, which tells us that equivalent inclusions have "equivalent representation theory" to each other. Actually, the notion of Morita equivalence between subfactors is formulated in a quite similar way as the one in ring and module theory. With this equivalence, we will see there always exists some symmetry described by Morita equivalence on the commuting squares in question.

§2. Definition and examples of Morita equivalence between subfactors

The following observation is fundamental for our formalism.

Let $N \subset M$ be an inclusion of II_1 factors with finite Jones index and finite depth and let $N \subset M \subset M_1 \subset M_2 \subset \cdots \subset M_k \subset \cdots$ be the Jones' tower of II_1 factors obtained by the basic construction. By the assumption of finite depth, we have finitely many irreducible bimodules when decomposing a P-Q bimodule ${}_PL^2(M_k)_Q$, where P and Q are either of M or N. Hence, one gets a finite system of graded bimodules (called a graded fusion rule algebra) consisting of irreducible N-N, N-M, M-N and M-M bimodules in the sense that it is closed under the operations of the relative tensor product, conjugation and direct sum decomposition of bimodules. We call a pair of these finite systems of four kinds of bimodules and additional information about homomorphisms between bimodules (called quantum $6j$-symbols) a finite paragroup of (M, N)-type. In other words, for finite systems of M-M and N-N bimodules, a finite paragroup of (M, N)-type is defined when there exists an N-M bimodule which generates a finite system of four kinds of graded bimodules.

This gives rise to the following definition. We denote a finite system of P-Q bimodules with finite index by ${}_P\mathcal{M}_Q$.

Definition 2.1. *Let A and B be II_1 factors. We say that a finite system of A-A bimodules ${}_A\mathcal{M}_A$ is* Morita equivalent *to that of B-B bimodules ${}_B\mathcal{M}_B$, if there exist A-B bimodules and the four kinds of bimodules A-A, B-B, A-B, B-A make a finite system. We denote this relation by ${}_A\mathcal{M}_A \sim {}_B\mathcal{M}_B$. (This is an equivalence relation.)*

Note that finite system of four kinds of bimodules in Definition 2.1 gives a finite paragroup of $(A,\ B)$-type.

We are ready to introduce the notion of Morita equivalence between subfactors. See [K] in the case of strongly amenable paragroups.

Definition 2.2. *Let $A \subset B$ and $C \subset D$ be inclusions of II_1 factors with finite Jones index and finite depth. We say that these two subfactors are* Morita equivalent *if ${}_B\mathcal{M}_B \sim {}_C\mathcal{M}_C$.*

Since we have finite paragroups of (A, B)- and (C, D)-type from the inclusions $A \subset B$ and $C \subset D$ respectively, we get ${}_A\mathcal{M}_A \sim {}_B\mathcal{M}_B \sim {}_C\mathcal{M}_C \sim {}_D\mathcal{M}_D$. This means that one may use, for instance, ${}_A\mathcal{M}_A \sim {}_D\mathcal{M}_D$ instead in Definition 2.2. And the standard arguments in subfactor theory give the equalities $\dim_A\mathcal{M}_A = \dim_B\mathcal{M}_B = \dim_C\mathcal{M}_C = \dim_D\mathcal{M}_D$, where $\dim_A\mathcal{M}_A = \sum_{X\in {}_A\mathcal{M}_A}(\dim_A X_A)^2$ (Here, summation is taken over a representative set of irreducible A-A bimodules) is a global index (or dimension) for ${}_A\mathcal{M}_A$.

Typical examples are in order.

Example 1 Let G be a finite group and H be its subgroup which is relatively simple. Two subfactors $N \subset N \rtimes G$ and $N \rtimes H \subset N \rtimes G$ obtained by the crossed product of an outer action of G on a II_1 factor N are Morita equivalent because the systems of the $N \rtimes G$-$N \rtimes G$ bimodules arising from both subfactors are identified by the Mackey machine of Ocneanu [KY].

Example 2 More generally, let $P \subset Q \subset R$ be inclusions of II_1 factors. Assume that the inclusion $P \subset R$ has finite Jones index and finite depth. It is known that the intermediate inclusion of $P \subset R$ is also of finite depth. Then the system of the R-R bimodules arising from $Q \subset R$ is a subsystem of that arising from $P \subset R$. Moreover, assume that both of the system of the R -R bimodules have the same global indices. Then these two subfactors are Morita equivalent since both systems of the R-R bimodules are identified. Hence, $P \subset R$ and $Q \subset R$ are Morita equivalent in this case.

Example 3 Let $N \subset M$ be an inclusion of II_1 factors with finite index and finite depth. Then, inclusions $N \subset M$ and $N \subset M_k$ are clearly Morita equivalent.

When we take k to be the least integer such that $N \subset M$ becomes of depth two (this is possible since we are dealing with a finite depth subfactor), one finds that there is an action of a finite dimensional biconnected weak C^*-Hopf algebra A on N and a crossed product algebra $N \rtimes A$ is isomorphic to M_k, by a characterization of a depth two subfactor due to Nikshych and Vainerman [NV1]. Roughly speaking, a finite dimensional weak C^*-Hopf algebra is a finite dimensional C^*-algebra which satisfies axioms of a C^*-Hopf algebra with non-unital coproduct and counit. See [BNS] for details. In [NV2], they proved a finite system of N-N bimodules arising from $N \subset N \rtimes A$ is equivalent to the category of unitary representations $\mathrm{Rep}(A^*)$ as monoidal categories. Hence, a finite system of M-M bimodules arising from $N \subset M$ and the finite system of $\mathrm{Rep}(A^*)$ are Morita equivalent in a broad sense.

Morita equivalence of subfactors is naturally associated with a finite dimensional non-degenerate commuting square

$$\begin{array}{ccc} R_{00} & \subset & R_{01} \\ \cap & & \cap \\ R_{10} & \subset & R_{11}. \end{array}$$

As in Introduction, iterating the basic construction in the horizontal and vertical directions, we get two subfactors $M_0 \subset M_1$ and $Q_0 \subset Q_1$, respectively. Instead of the language of commuting squares, we use the paragroup theoretical one. Then, we have the biunitary connection on the four finite bipartite connected graphs corresponding to the inclusion matrices in the initial commuting square. And by the string algebra construction, we have the following double sequence of finite dimensional C^*-algebras.

$$\begin{array}{ccccccccc} \mathbb{C} = A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset M_0 = A_{0,\infty} \\ \cap & & \cap & & \cap & & & \cap \\ A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset M_1 = A_{1,\infty} \\ \cap & & \cap & & \cap & & & \cap \\ A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset M_2 = A_{2,\infty} \\ \cap & & \cap & & \cap & & & \cap \\ \vdots & & \vdots & & \vdots & & & \vdots \\ \cap & & \cap & & \cap & & & \\ Q_0 = A_{\infty,0} & \subset & Q_1 = A_{\infty,1} & \subset & Q_2 = A_{\infty,2} & \subset & \cdots & \end{array}$$

Assume that either $M_0 \subset M_1$ or $Q_0 \subset Q_1$ is of finite depth. (Hence, by [S1, Corollary 2.2], both of them are of finite depth.) By Ocneanu's compactness argument, the higher relative commutant $M_0' \cap M_k$ of the inclusion $M_0 \subset M_1$ is contained in $A_{k,0}$ for each k. Denote $\vee_{k=0}^{\infty} M_0' \cap M_k$ by B_k. See the following diagram.

$$\begin{array}{ccccccccccc}
B_0 & \subset & A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & M_0 \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
B_1 & \subset & A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & M_1 \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
B_2 & \subset & A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset & M_2 \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\cap & & \cap & & \cap & & \cap & & & & \\
B_\infty & \subset & Q_0 & \subset & Q_1 & \subset & Q_2 & \subset & \cdots & &
\end{array}$$

Then one can show that the system of the B_∞^{op}-B_∞^{op} bimodules arising from $B_\infty^{op} \subset Q_1^{op}$ is identified with that of the M_0-M_0 bimodules arising from $M_0 \subset M_1$. This implies that the subfactor $B_\infty^{op} \subset Q_1^{op}$ is Morita equivalent to $M_0 \subset M_1$. Since one can verify that $B_\infty^{op} \subset Q_0^{op} \subset Q_1^{op}$ satisfy the assumption of example 2, $Q_0^{op} \subset Q_1^{op}$ is Morita equivalent to $M_0 \subset M_1$.

§3. Reconstruction of a commuting square from the equivalent subfactors

In Section 2, we saw that horizontal and vertical inclusions of hyperfinite II_1 factors obtained from a finite dimensional non-degenerate commuting square are "opposite equivalent". Now, our main theorem claims that opposite equivalence of subfactors has enough information to insure that they certainly come from a finite dimensional non-degenerate commuting square.

Let ${}_D g_C$ be the D-C bimodule ${}_D D_C$ of the finite paragroup of (C, D)-type which canonically arises from $C \subset D$, ${}_A h_B$ be the A-B bimodule ${}_A B_B$ of the A-B paragroup which canonically arises from $A \subset B$. Denote the direct sum of the unitarily inequivalent irreducible C-A bimodules by ${}_C X_A$. Construct the following inclusions of finite dimensional C^*-algebras.

$$\begin{array}{ccc}
\mathrm{End}(g(\bar{g}g)^{n-1} \otimes_C X_A \otimes (h\bar{h})^{n-1}h) & \subset & \mathrm{End}(g(\bar{g}g)^{n-1} \otimes_C X_A \otimes (h\bar{h})^{n}) \\
\cap & & \cap \\
\mathrm{End}((\bar{g}g)^{n} \otimes_C X_A \otimes (h\bar{h})^{n-1}h) & \subset & \mathrm{End}((\bar{g}g)^{n} \otimes_C X_A \otimes (h\bar{h})^{n})
\end{array}$$

Then, one can show that this is a non-degenerate commuting square of period two when n is large enough. Thus we have the biunitary connection arising from the above commuting square. By the string algebra construction, we get the following double sequences of finite dimensional C^*-algebras $A_{k,l} = \mathrm{End}(\underbrace{\cdots \otimes \bar{g} \otimes g}_{k-\text{folds}} \otimes {}_C*_A \otimes \underbrace{h \otimes \bar{h} \cdots}_{l-\text{folds}})$, where ${}_C*_A$ is an irreducible C-A bimodule. See the following diagram.

$$\begin{array}{ccccccccc}
A_{0,0}(= \mathrm{End}({}_C*_A)) & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & M_0 = A_{0,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & M_1 = A_{1,\infty} \\
\cap & & \cap & & \cap & & & & \cap \\
A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset & M_2 = A_{2,\infty} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cap & & \cap & & \cap & & & & \\
Q_0 = A_{\infty,0} & \subset & Q_1 = A_{\infty,1} & \subset & Q_2 = A_{\infty,2} & \subset & \cdots & &
\end{array}$$

Then we have our main theorem.

Theorem 3.1. [S3] *In the above notations, the subfactors $M_0 \subset M_1$ and $Q_0 \subset Q_1$ are isomorphic to $C^{op} \subset D^{op}$ and $A \subset B$, respectively.*

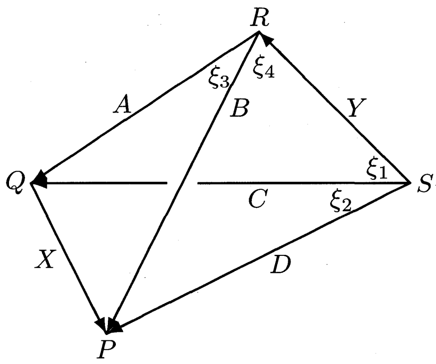
Example Assume that the inclusion $M_0 \subset M_1$ obtained in the theorem is isomorphic to the inclusion $R^G \subset R$, where R is the hyperfinite II_1 factor and G is a finite group acting freely on R by $\alpha \in \mathrm{Aut}(R)$. Then, we know that the finite system of B_∞^{op}- B_∞^{op} bimodules arising from $B_\infty^{op} \subset Q_1^{op}$ is isomorphic to that of R^G-R^G bimodules arising from $R^G \subset R$. In such a situation, Schaflitzel proved that $B_\infty^{op} \subset Q_1^{op}$ is isomorphic to $R^G \subset (R \otimes M_n(\mathbb{C}))^H$, where H ($\subset G$) is a subgroup acting on $R \otimes M_n(\mathbb{C})$ by $\alpha|_H \otimes \mathrm{Ad}\Psi$ (Ψ is a projective representation of H on $M_n(\mathbb{C})$) [Sc]. This characterization gives a restriction of the inclusion $Q_0^{op} \subset Q_1^{op}$. Namely, it is isomorphic to $R^G \subset (R \otimes M_n(\mathbb{C}))^H$ in a good situation (i.e., $*$-flat case) and in general it is an intermediate inclusion of $R^G \subset (R \otimes M_n(\mathbb{C}))^H$, i.e., $R^G \subset Q_0^{op} \subset Q_1^{op} = (R \otimes M_n(\mathbb{C}))^H$.

§4. A comment on topological quantum field theories in 3 dimensions constructed from subfactors

From an inclusion $N \subset M$ of type II_1 factors with finite index and finite depth, we have two kinds of 3-dimensional topological quantum field theories (TQFT for short). Let us recall briefly how they are obtained.

Since we have a finite paragroup of (M, N)-type, we have a graded fusion rule algebra. A space of homomorphisms $\mathrm{Hom}({}_P X_Q \otimes_Q Y_R, {}_P Z_R)$ for irreducible X, Y and Z is a finite dimensional Hilbert space and has an orthonormal basis with respect to the inner product defined by $(\xi, \eta) = \xi \cdot \eta^* \in \mathrm{End}(Z) = \mathbb{C}$. Then, for each orthonormal basis of homomorphisms $\xi_1 \in \mathrm{Hom}({}_Q A_R \otimes_R Y_S, {}_Q C_S)$, $\xi_2 \in \mathrm{Hom}({}_P X_Q \otimes_Q C_S, {}_P D_S)$,

$\xi_3 \in \mathrm{Hom}({}_P X_Q \otimes {}_Q A_R, {}_P B_R)$, $\xi_4 \in \mathrm{Hom}({}_P B_R \otimes {}_R Y_S, {}_P D_S)$, where P, Q, R, S are either of M or N and ${}_P X_Q$, ${}_R Y_S$, ${}_Q A_R$, ${}_P B_R$, ${}_Q C_S$, ${}_P D_S$ are irreducible bimodules, we have a complex number $W(A, B, C, D, X, Y | \xi_1, \xi_2, \xi_3, \xi_4)$ defined by $\xi_4 \cdot (\xi_3 \otimes \mathrm{id}) \cdot (\mathrm{id} \otimes \xi_1)^* \cdot \xi_2^* \in \mathrm{End}({}_P D_S) = \mathbb{C}$. Then, the quantum $6j$-symbol Z is defined by $Z(A, B, C, D, X, Y | \xi_1, \xi_2, \xi_3, \xi_4) = [B]^{-1/4}[C]^{-1/4} W(A, B, C, D, X, Y | \xi_1, \xi_2, \xi_3, \xi_4)$. Now, we associate each tetrahedron with a quantum $6j$-symbol. See the following figure.



Let V be a compact 3-dimensional manifold without boundaries and $\mathcal{T}$ be a triangulation of V. For each vertex of $(V, \mathcal{T})$, we assign the label either of M or N and fix it. Let C_e be the possible assignment of irreducible bimodules in a finite paragroup to each edge of triangulation. Namely, we consider a face of a tetrahedron as a space of homomorphisms. For each assignment of C_e, let C_f be the assignment of an orthonormal basis of homomorphisms to each face of a tetrahedron. Now, $\zeta(V, \mathcal{T})$ is defined to be a complex number $(\dim\mathcal{P})^{-a} \sum_{C_e} \sum_{C_f} \Pi_{X:\mathrm{edges}} [X]^{\frac{1}{2}} \Pi Z$ (tetrahedron), where $\dim\mathcal{P}$ is the global index of the finite paragroup $\mathcal{P}$, a is the number of vertices and $[X] = (\dim_N X)(\dim X_N)$. The important point here is that $\zeta(V, \mathcal{T})$ does not depend on neither the triangulation or labelings of N and M on each vertex. Hence, we get a topological invariant of V and we may write $\zeta(V)$ instead of $\zeta(V, \mathcal{T})$. (When V has a boundary, some modification is needed.) Moreover, $\zeta(V)$ can be extended to satisfy the axioms of a (unitary) TQFT in the sense of M. Atiyah [A]. Namely, ζ is a functor from the category of cobordisms of surfaces to the category of finite dimensional Hilbert spaces with some properties for cut and gluing. This type of construction was first achieved by Turaev and Viro in the case of data from a quantum group of $U_q(sl_2)$ [TV] and the present formalism using a paragroup is due to A. Ocneanu [O1]. We call our ζ the Turaev-Viro type TQFT obtained from a finite paragroup.

The notion of Morita equivalence between subfactors is remarkably efficient when one constructs topological quantum field theory of Turaev-Viro type. Denote by ζ_{M-M} (resp. ζ_{N-N}) Turaev-Viro type TQFT constructed from the data of M-M(resp. N-N) fusion rules and associated quantum $6j$-symbols obtained from $N \subset M$. Then, ζ_{M-M} and ζ_{N-N} give rise to the same TQFT's. More generally, we have the same TQFT's for finite paragroups arising from Morita equivalent subfactors.

Ocneanu introduced a new construction of an inclusion of II_1 factors with finite depth and finite index, the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$, from an inclusion of II_1 factors $N \subset M$ with finite index and finite depth, where $M_\infty = \vee_{n=1}^{\infty} M_n$. He had noticed that the finite system $\mathcal{M}_\infty$ of M_∞-M_∞ bimodules arising from the asymptotic inclusion was an analogue of the quantum double in quantum group theory. Actually, he claimed that the finite system $\mathcal{M}_\infty$ has a non-degenerate braiding through TQFT of Turaev-Viro type [O1], [EK]. In fact, one can prove that $\mathcal{M}_\infty$ satisfies the axioms of modular tensor category. Hence, with a general machinery of Turaev, one can construct a Reshetikhin-Turaev type TQFT based on Dehn surgery of 3-dimensional manifolds [O2], [T].

By the definition of the Morita equivalence of two finite depth subfactors, they give rise to the isomorphic Turaev-Viro type topological quantum field theories. Since, by Example 3 in Section 2, a finite depth subfactor $N \subset M$ is always Morita equivalent to an inclusion of the form $N \subset N \rtimes A$, where A is a finite dimensional weak C^*-Hopf algebra, one knows that TQFT's obtained from the former and the latter are isomorphic on one hand. On the other hand, a system of N-N bimodules arising from the latter inclusion is equivalent to the category of finite dimensional unitary representations $\mathrm{Rep}(A^*)$ of the dual weak C^*-Hopf algebra A^* of A. Hence, we may conclude that Turaev-Viro type TQFT obtained from a finite depth subfactor is always obtained from the data of the category of finite dimensional unitary representations of a weak C^*-Hopf algebra with quantum $6j$-symbols.

Keeping the situation in the previous paragraph, when we take the asymptotic inclusions for both $N \subset M$ and $N \subset N \rtimes A$, one can prove that the tensor category of M_∞-M_∞ bimodules arising from the former inclusion is isomorphic to that of the latter because the original subfactors are Morita equivalent and modular tensor categories obtained from the asymptotic inclusions are described by TQFT's of Turaev-Viro type [O2], [EK]. Moreover, one can prove the latter tensor category is isomorphic to the category $\mathrm{Rep}D(A^*)$ of finite dimensional unitary representations of $D(A^*)$ as unitary modular tensor categories, where $D(A^*)$ is the quantum double weak C^*-Hopf algebra of A^* [BS], [NTV]. As a consequence, the Reshetikhin-Turaev type TQFT obtained from $N \subset M$ through the asymptotic inclusion is isomorphic to the one obtained from the category of finite dimensional unitary representations of the quantum double of a weak C^*-Hopf algebra. Hence, TQFT's of both Turaev-Viro and Reshetikhin-Turaev type constructed from subfactors with finite Jones index and finite depth are obtained within the category of finite dimensional unitary representations of finite dimensional weak C^*-Hopf algebras. See [S4] for more details.

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