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C^*-algebras over spheres with fibres noncommutative tori

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Abstract.

All C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $M_c(A_\omega)$ are constructed under the assumption that each completely irrational noncommutative torus is realized as an inductive limit of circle algebras. It is shown that each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $M_c(A_\omega)$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes M_c(A_\omega)$.

Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1} \times \mathbb{T}^{r+2}$ of which no non-trivial matrix algebra can be factored out. The spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is defined by twisting $C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}$. We prove that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of those of p.

§0. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *noncommutative torus of rank* m and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier ω is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [1, 12]). It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra. The noncommutative torus A_ω of rank m is the universal object

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for unitary ω-representations of $\mathbb{Z}^m$, so A_ω is realized as $C^*(u_1, \cdots, u_m \mid u_i u_j = e^{2\pi i \theta_{ji}} u_j u_i)$, where u_i are unitaries and θ_{ji} are real numbers for $1 \le i, j \le m$.

Boca [4] showed that almost all completely irrational noncommutative tori are isomorphic to inductive limits of circle algebras, where the term "*circle algebra*" denotes a C^*-algebra which is a finite direct sum of C^*-algebras of the form $C(\mathbb{T}^1) \otimes M_q(\mathbb{C})$. We will assume that each completely irrational noncommutative torus appearing in this paper is an inductive limit of circle algebras.

Each cd-homogeneous C^*-algebra A over M is isomorphic to the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η with base space M, fibres $M_{cd}(\mathbb{C})$, and structure group $\mathrm{Aut}(M_{cd}(\mathbb{C})) \cong PU(cd)$ (see [15, 18]). So each cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^{r+2}$ is realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^{r+2}$ with fibres $M_{cd}(\mathbb{C})$. Thus the spherical noncommutative torus $\mathbb{S}_\rho^{cd}$, defined in Section 2, is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$, where P_ρ^d is defined in Section 2.

We are going to show that the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of all spherical noncommutative tori with primitive ideal space $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ and fibres $P_\rho^d \otimes M_c(\mathbb{C})$, that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p, and that $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C})$.

§1. Homogeneous C^*-algebras over a product space of spheres

An important problem, in the bundle theory of geometry, is to compute the set $[M, BPU(cd)]$ of homotopy classes of continuous maps of a compact CW-complex M into the classifying space $BPU(cd)$ of the Lie group $PU(cd)$. The set $[M, BPU(cd)]$ is in bijective correspondence with the set of equivalence classes of principal $PU(cd)$-bundles over M, which is in bijective correspondence with the set of cd-homogeneous C^*-algebras over M (see [15, 18]). $[S^{2n}, BPU(cd)] = [S^{2n-1}, PU(cd)] \cong \mathbb{Z}$ if $n > 1$, $\cong \mathbb{Z}_{cd}$ if $n = 1$, which are the cyclic groups. So each group has a generator, and there is a unitary $U(z) \in PU(cd)$ such that the generating cd-homogeneous C^*-algebra over S^{2n} can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n} with fibres $M_{cd}(\mathbb{C})$ characterized by the unitary $U(z) \in PU(cd)$ over S^{2n-1}. If $(cd, k) = p$ $(p > 1)$, then consider the cd-homogeneous C^*-algebra over S^{2n} corresponding to each $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$ as the tensor product of $M_p(\mathbb{C})$ with a $\frac{cd}{p}$-homogeneous C^*-algebra over S^{2n}, which is given by $U(z)^{\frac{k}{p}} \in PU(\frac{cd}{p})$. Consider $U(z)^k$ as $U(z)^{\frac{k}{p}} \otimes I_p \in PU(cd)$, where I_p denotes the $p \times p$ identity matrix. Then each cd-homogeneous C^*-algebra $B_{cd,k}$ over S^{2n} can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n} with fibres $M_{cd}(\mathbb{C})$ characterized by the unitary $U(z)^k \in PU(cd)$ over S^{2n-1} for some $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$ (see [15]).

Lemma 1.1. *Every cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$, whose cd-homogeneous C^*-subalgebra restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure, is isomorphic to one of the C^*-subalgebras $A_{cd,k}$, $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$, of $C(S^{2n-1} \times [0,1], M_{cd}(\mathbb{C}))$ given as follows: if $f \in A_{cd,k}$, then the following condition is satisfied*

$$f(z,1) = U(z)^k f(z,0) U(z)^{-k}$$

for all $z \in S^{2n-1}$, where $U(z) \in PU(cd)$ is the unitary given above.

Proof. Let A be a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure. Since there is a map of degree 1 from $S^{2n-1} \times S^1$ to S^{2n}, the composite of the map of degree 1 and the map representing each element of $[S^{2n}, BPU(cd)]$ gives an element of $[S^{2n-1} \times S^1, BPU(cd)]$. Hence each element of $[S^{2n}, BPU(cd)] \cong [S^{2n-1}, PU(cd)]$ representing a cd-homogeneous C^*-algebra over S^{2n} induces an element of $[S^{2n-1}, PU(cd)] \subset [S^{2n-1} \times S^1, BPU(cd)]$, and the cd-homogeneous C^*-algebras $A_{cd,k}$ over $S^{2n-1} \times S^1$ corresponding to the cd-homogeneous C^*-algebras $B_{cd,k}$ over S^{2n} are constructed in the statement. By the assumption, the cd-homogeneous C^*-subalgebra of A restricted to the subspace $S^{2n-1} \times (0,1)$ of $S^{2n-1} \times S^1$ has the trivial bundle structure. Hence A corresponds to an element of $[S^{2n-1}, PU(cd)]$, and A is characterized by the unitary $U(z)^k \in PU(cd)$ over S^{2n-1} for some $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$. Q.E.D.

Lemma 1.2. *Let n and k be integers greater than 1. Each cd-homogeneous C^*-algebra over $S^n \times S^k$ is isomorphic to a cd-homogeneous C^*-algebra characterized by the unitary $U(z)^a$ over S^{n-1} in a cd-homogeneous C^*-algebra P_c over $e^n_+ \times S^k$ and $e^n_- \times S^k$, where*

$U(z) \in PU(cd)$ or $PU(c)$ if $M_c(\mathbb{C})$ is factored out of P_c, and e_+^n (resp. e_-^n) is the n-dimensional northern (resp. southern) hemisphere.

Proof. Since e_+^n, e_-^n are contractible, each *cd*-homogeneous C^*-algebra over $e_+^n \times S^k$ and $e_-^n \times S^k$ is essentially induced by a *cd*-homogeneous C^*-algebra over S^k. Each *cd*-homogeneous C^*-algebra over $S^n \times S^k$ is characterized by a projective unitary over the boundaries $S^{n-1} \times S^k$ of $e_+^n \times S^k$ and $e_-^n \times S^k$. But $\pi_1(S^n) = \{0\}$ and so the identification of the boundaries $S^k \hookrightarrow e_+^n \times S^k$ and $S^k \hookrightarrow e_-^n \times S^k$ does give the trivial bundle structure. Hence the *cd*-homogeneous C^*-algebra over $S^n \times S^k$ is characterized by the unitary $U(z)^a$, $a \in \mathbb{Z}$ or $a \in \mathbb{Z}_{cd}$, over S^{n-1} in the *cd*-homogeneous C^*-algebra over $e_+^n \times S^k$ and $e_-^n \times S^k$, where $U(z) \in PU(cd)$ or $PU(c)$. Q.E.D.

For a *cd*-homogeneous C^*-algebra A over S^{2n-1} there is a matrix algebra $M_q(\mathbb{C})$ such that $A \otimes M_q(\mathbb{C})$ is isomorphic to $C(S^{2n-1}) \otimes M_{cdq}(\mathbb{C})$. Since there is a map of degree 1 from S^{2n+1} to $S^{2n} \times S^1$, there are *cd*-homogeneous C^*-algebras over $S^{2n} \times S^1$ induced from *cd*-homogeneous C^*-algebras over S^{2n+1}. Also there are *cd*-homogeneous C^*-algebras over $S^{2n} \times S^1$ induced from *cd*-homogeneous C^*-algebras over S^{2n}. But the tensor product of each *cd*-homogeneous C^*-algebra over $S^{2n} \times S^1$ induced from a *cd*-homogeneous C^*-algebra over S^{2n+1} with $M_q(\mathbb{C})$ has the trivial bundle structure for some integer q big enough since $[S^{2n+1}, BPU(cdq)] \cong \{0\}$. And there is a map of degree 1 from S^{2n} to $S^{2n-1} \times S^1$, and so there are *cd*-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ induced from *cd*-homogeneous C^*-algebras over S^{2n}. Also there are *cd*-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ induced from *cd*-homogeneous C^*-algebras over S^{2n-1}. But $[S^{2n-1} \times S^1, BPU(cdq)]$ and $[S^{2n}, BPU(dq)]$ are the same for some integer q since $[S^{2n-1}, BPU(cdq)] \cong \{0\}$. So the *cd*-homogeneous C^*-subalgebra of the tensor product of a *cd*-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ with $M_q(\mathbb{C})$ restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure (see [17, 18]). From now on, we assume that each *cd*-homogeneous C^*-algebra over $S^{2n} \times S^1$ is isomorphic to the tensor product of a *cd*-homogeneous C^*-algebra over S^{2n} with $C(S^1)$, and that the *cd*-homogeneous C^*-subalgebra of a *cd*-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure.

Thomsen [19, Theorem 1.15] computed $\pi_{2n-1}(\mathrm{Aut}(M_{cdp}(\mathbb{C}) \otimes M_{q^\infty})) \cong \mathbb{Z}/cdp\mathbb{Z}$ for M_{q^∞} a *UHF*-algebra of type q^∞, and *cdp* and q relatively prime integers. Let $A_{cd,k}$ be a *cd*-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ of which no non-trivial matrix algebra can be factored out. This result implies that for any positive integer p no matrix algebra bigger than $M_p(\mathbb{C})$ can be factored out of $A_{cd,k} \otimes M_p(\mathbb{C})$. So the natural inclusion $C(S^1) \hookrightarrow A_{cd,k}$ induces the canonical homomorphism $K_0(C(S^1)) \to K_0(A_{cd,k})$ such that $[1_{C(S^1)}]$ maps to $[1_{A_{cd,k}}]$.

Lemma 1.3. *Let $A_{cd,k}$ be a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ of which no non-trivial matrix algebra can be factored out. Then $K_0(A_{cd,k}) \cong K_1(A_{cd,k}) \cong \mathbb{Z}^2$, and $[1_{A_{cd,k}}] \in K_0(A_{cd,k})$ is primitive.*

Proof. We will show later that $A_{cd,k}$ is stably isomorphic to $C(S^{2n-1} \times S^1)$. Since $K_0(C(S^{2n-1} \times S^1)) \cong K_1(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$, $K_0(A_{cd,k}) \cong K_1(A_{cd,k}) \cong \mathbb{Z}^2$. Hence it is enough to show that $[1_{A_{cd,k}}] \in K_0(A_{cd,k})$ is primitive.

No matrix algebra bigger than $M_q(\mathbb{C})$ can be factored out of $A_{cd,k} \otimes M_q(\mathbb{C})$, and so $C(S^{2n-1})$ cannot be factored out of $A_{cd,k} \otimes M_q(\mathbb{C})$. Hence the canonical embedding ϕ of $C(S^{2n-1})$ into $A_{cd,k}$ induces an isomorphism μ of $K_0(C(S^{2n-1} \times S^1))$ into $K_0(A_{cd,k})$. But the unit $1_{C(S^{2n-1})}$ maps to the unit $1_{C(S^{2n-1} \times S^1)}$ under the canonical embedding ψ of $C(S^{2n-1})$ into $C(S^{2n-1} \times S^1)$. Thus $[1_{C(S^{2n-1})}] \in K_0(C(S^{2n-1})) \cong \mathbb{Z}$ maps to $[1_{C(S^{2n-1} \times S^1)}] \in K_0(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$, primitive in $K_0(C(S^{2n-1} \times S^1))$ (see [20, 13.3.1]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^{2n-1})) & \xrightarrow{\psi_*} & K_0(C(S^{2n-1} \times S^1)) \\ {\scriptstyle (\text{identity})_*} \downarrow & & \downarrow {\scriptstyle \mu(\cong)} \\ K_0(C(S^{2n-1})) & \xrightarrow{\phi_*} & K_0(A_{cd,k}), \end{array}$$

$\mu([1_{C(S^{2n-1} \times S^1)}]) = \phi_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(S^{2n-1} \times S^1)}]) = [1_{A_{cd,k}}]$. Consequently $[1_{A_{cd,k}}]$ is the image of the primitive element $[1_{C(S^{2n-1} \times S^1)}] \in K_0(C(S^{2n-1} \times S^1))$ under the isomorphism μ. Therefore, $[1_{A_{cd,k}}] \in K_0(A_{cd,k}) \cong \mathbb{Z}^2$ is primitive.

Thus, $K_0(A_{cd,k}) \cong \mathbb{Z}^2$, $K_1(A_{cd,k}) \cong \mathbb{Z}^2$, and $[1_{A_{cd,k}}] \in K_0(A_{cd,k})$ is primitive. Q.E.D.

Lemma 1.4. *Let $B_{cd,k}$ be a cd-homogeneous C^*-algebra over S^{2n} of which no non-trivial matrix algebra can be factored out. Then $[1_{B_{cd,k}}] \in K_0(B_{cd,k}) \cong \mathbb{Z}^2$ is primitive.*

Proof. We will show later that $B_{cd,k}$ is stably isomorphic to $C(S^{2n}) \otimes M_{cd}(\mathbb{C})$. So $K_0(B_{cd,k}) \cong K_0(C(S^{2n})) \cong \mathbb{Z} \oplus \mathbb{Z}$. But $B_{cd,k}$ corresponds to $A_{cd,k}$ with respect to the conditions on sections over the boundaries S^{2n-1} of $e_+^{2n} \amalg e_-^{2n}$ and $S^{2n-1} \times [0,1]$, and the canonical embedding of $C(S^{2n-1})$ into $A_{cd,k}$ which induces the isomorphism of $K_0(C(S^{2n-1} \times S^1))$ into $K_0(A_{cd,k})$ corresponds to the imbedding ϕ of $C(S^{2n-1})$ into

$B_{cd,k}$. The canonical imbedding ϕ of $C(S^{2n-1})$ into $B_{cd,k}$ induces an isomorphism μ of $K_0(C(S^{2n}))$ into $K_0(B_{cd,k})$, where $S^{2n-1} = \partial e^{2n}_{\pm}$. The unit $1_{C(S^{2n-1})}$ maps to the unit $1_{C(S^{2n})}$ under the canonical embedding ψ of $C(S^{2n-1})$ into $C(S^{2n})$. $[1_{C(S^{2n-1})}] \in K_0(C(S^{2n-1})) \cong \mathbb{Z}$ maps to $[1_{C(S^{2n})}] \in K_0(C(S^{2n})) \cong \mathbb{Z}^2$, primitive in $K_0(C(S^{2n}))$ (see [20, 13.3.1]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^{2n-1})) & \xrightarrow{\psi_*} & K_0(C(S^{2n})) \\ {\scriptstyle (\text{identity})_*}\big\downarrow & & \big\downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S^{2n-1})) & \xrightarrow{\phi_*} & K_0(B_{cd,k}), \end{array}$$

$\mu([1_{C(S^{2n})}]) = \phi_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(S^{2n})}]) = [1_{B_{cd,k}}]$. So $[1_{B_{cd,k}}]$ is the image of the primitive element $[1_{C(S^{2n})}] \in K_0(C(S^{2n}))$ under the isomorphism μ. Hence $[1_{B_{cd,k}}] \in K_0(B_{cd,k})$ is primitive.

Therefore, $[1_{B_{cd,k}}] \in K_0(B_{cd,k}) \cong \mathbb{Z}^2$ is primitive. Q.E.D.

For each 4-dimensional factor S of $\prod^e S^2 \times \prod^{s+r+2} S^1$ every d-homogeneous C^*-algebra over S can be constructed by combining Lemma 1.1 and Lemma 1.2. If $s + r$ is odd, one can make the integer even by tensoring with $C(S^1)$. So one can assume that $s + r$ is even, and that s is greater than or equals to r and big enough. And one can rearrange $\prod_{j=1}^s S^{2k_j-1}$ and $\mathbb{T}^r$ if needed.

Theorem 1.5. *Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^r \times \mathbb{T}^2 \hookrightarrow \prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as $C(\mathbb{T}^r) \otimes A_{\frac{l}{d}} \otimes M_c(\mathbb{C})$ for $A_{\frac{l}{d}}$ a rational rotation algebra. Then A_{cd} is isomorphic to one of the C^*-subalgebras $A^{a_1,a_2,\cdots,a_e}_{b_1,b_2,\cdots,b_{\frac{s+r}{2}}}$, $a_1,\cdots,a_e,b_1,\cdots,b_{\frac{s+r}{2}} \in \mathbb{Z}$, of*

$$C(\prod_{i=1}^{e}(e_+^{2n_i} \amalg e_-^{2n_i}) \times \prod_{j=1}^{\frac{s+r}{2}}(S^{2k_j-1} \times [0,1]) \times \mathbb{T}^1 \times [0,1], M_{cd}(\mathbb{C}))$$

consisting of those functions f that satisfy

$$(f|_{e_+^{2n_i} \amalg e_-^{2n_i}})_+(z_i) = U(z_i)^{a_i}(f|_{e_+^{2n_i} \amalg e_-^{2n_i}})_-(z_i)U(z_i)^{-a_i}$$
$$(f|_{S^{2k_j-1} \times [0,1]})(w_j,1) = U(w_j)^{b_j}(f|_{S^{2k_j-1} \times [0,1]})(w_j,0)U(w_j)^{-b_j}$$
$$(f|_{\mathbb{T}^1 \times [0,1]})(x,1) = U(x)^{cl}(f|_{\mathbb{T}^1 \times [0,1]})(x,0)U(x)^{-cl}$$

for all $(z_1,\cdots,z_e,w_1,\cdots,w_{\frac{s+r}{2}},x)\in\prod_{i=1}^e S^{2n_i-1}\times\prod_{j=1}^{\frac{s+r}{2}} S^{2k_j-1}\times\mathbb{T}^1$, *one of the tensor products of homogeneous* C^**-algebras of the type above, or one of the* C^**-algebras given by replacing* $(\prod_{i=1}^e S^{2n_i}\times\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2)$ *in* $A^{a_1,\cdots,a_e}_{b_1,\cdots,b_{\frac{s+r}{2}}}$ *or the tensor products with suitable* $c'd'$*-homogeneous* C^**-algebras in the same sense as above, when* $M_{c'd'}(\mathbb{C})$ *are factored out of* $A^{a_1,\cdots,a_e}_{b_1,\cdots,b_{\frac{s+r}{2}}}$ *or the tensor products, where* $U(z_i), U(w_j)$*, and* $U(x)\in PU(cd)$ *are defined in the statement of Lemma 1.1.*

Proof. By Lemma 1.1, each cd-homogeneous C^*-algebra over $S^{2k_j-1}\times S^1$ corresponds to a cd-homogeneous C^*-algebra over S^{2k_j}. By Lemma 1.2, each cd-homogeneous C^*-algebra over the product space of two even dimensional spheres can be constructed. Combining Lemma 1.1 and Lemma 1.2 yields that replacing S^{2n_i} and S^{2k_j-1} with S^2 and S^1 does not give any change in the relation, associated with bundle structure, among the factors of $\prod_{i=1}^e S^{2n_i}\times\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2$. Hence each cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i}\times\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2$ can be given by [5, Theorem 2.5], which is exactly stated in the statement for the case $n_i=1$ and $k_j=1$. Q.E.D.

Theorem 1.6. *Let* A_{cd} *be a* C^**-algebra over* $\prod_{i=1}^e S^{2n_i}\times\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2$ *constructed in Theorem 1.5. Assume that no non-trivial matrix algebra can be factored of* A_{cd}*. Then* $K_0(A_{cd})\cong K_1(A_{cd})\cong\mathbb{Z}^{2^{e+s+r+1}}$*, and* $[1_{A_{cd}}]\in K_0(A_{cd})$ *is primitive.*

Proof. We are going to show in Lemma 3.1 that A_{cd} is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i}\times\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2)\otimes M_{cd}(\mathbb{C})$. By the Künneth theorem [2, Theorem 23.1.3]

$$\begin{aligned} K_0(A_{cd}) &\cong K_0(C(\prod_{i=1}^e S^{2n_i}\times\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2))\\ &\cong K_0(C(\prod_{i=1}^e S^{2n_i}))\otimes K_0(C(\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2))\\ &\quad\oplus K_1(C(\prod_{i=1}^e S^{2n_i}))\otimes K_1(C(\prod_{j=1}^s S^{2k_j-1}\times\mathbb{T}^r\times\mathbb{T}^2))\\ &\cong \mathbb{Z}^{2^e}\otimes\mathbb{Z}^{2^{s+r+1}}\oplus\{0\}\cong\mathbb{Z}^{2^{e+s+r+1}}. \end{aligned}$$

Similarly, one obtains that $K_1(A_{cd})\cong\mathbb{Z}^{2^{e+s+r+1}}$.

It is enough to show that $[1_{A_{cd}}]\in K_0(A_{cd})$ is primitive. But the proof is similar to the proof given in [17, Theorem 1.2]. Since the

cd-homogeneous C^*-algebra A_{cd} is just given by replacing each C^*-subalgebra $C(S^2)$ (resp. $C(S^1)$) of the cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^2 \times \prod_{i=1}^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2$ given in [17] with $C(S^{2n_i})$ (resp. $C(S^{2k_j-1})$), the proof is just given by replacing $C(S^2)$ and $C(S^1)$ given in the proof of [17, Theorem 1.2] with $C(S^{2n_i})$ and $C(S^{2k_j-1})$.

Therefore, $K_0(A_{cd}) \cong K_1(A_{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$, and $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive. Q.E.D.

§2. Spherical noncommutative tori

The noncommutative torus A_ω of rank m is obtained by an iteration of $m-1$ crossed products by actions of $\mathbb{Z}$, the first action on $C(\mathbb{T}^1)$. When A_ω is not simple, by a change of basis, A_ω is obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{l}{d}}$. Since the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ is factored out of the fibre of A_ω, A_ω can be obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on $A_{\frac{l}{d}}$, where the actions of $\mathbb{Z}$ on the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ are trivial. This assures us of the existence of such actions α_i in the definition of P_ρ^d below. So one can assume that A_ω is given by twisting $C^*(d\mathbb{Z} \times d\mathbb{Z} \times \mathbb{Z}^{m-2})$ in $A_{\frac{l}{d}} \otimes C^*(\mathbb{Z}^{m-2})$ by the restriction of the multiplier ω to $d\mathbb{Z} \times d\mathbb{Z} \times \mathbb{Z}^{m-2}$, where $\widehat{d\mathbb{Z}} \times \widehat{d\mathbb{Z}}$ is the primitive ideal space of $A_{\frac{l}{d}}$ and $C^*(d\mathbb{Z} \times d\mathbb{Z}, \text{res of } \omega) = C^*(d\mathbb{Z} \times d\mathbb{Z})$ (see [5] for details).

Definition 2.1. *Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^r \times \mathbb{T}^2 \hookrightarrow \prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as $C(\mathbb{T}^r) \otimes A_{\frac{l}{d}} \otimes M_c(\mathbb{C})$ for $A_{\frac{l}{d}}$ a rational rotation algebra. The C^*-algebra which is given by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$ is said to be a spherical noncommutative torus of rank $(e, s+r, m)$ and denoted by $\mathbb{S}_\rho^{cd}$, where $C^*(\widehat{\mathbb{T}^2}, \text{res of } \rho) = C^*(\widehat{\mathbb{T}^2})$, $\mathbb{T}^2$ is the primitive ideal space of $A_{\frac{l}{d}}$, and $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}, \rho)$ is a completely irrational noncommutative torus A_ρ.*

Then the fibre of $\mathbb{S}_\rho^d$, which is called a *generalized noncommutative torus of rank* $r+m$ and denoted by P_ρ^d, can be obtained by an iteration of $r+m-2$ crossed products by actions α_i of $\mathbb{Z}$, the first action on the rational rotation algebra $A_{\frac{l}{d}}$, where the actions α_i on the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ are trivial. Thus the spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$.

We are going to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Theorem 2.2. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus of rank $(e, s+r, m)$. Assume no non-trivial matrix algebra can be factored out of A_{cd}. Then $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.*

Proof. The proof is by induction on m. Assume that $m = 2$. We will show later that $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$, where A_ρ is a noncommutative torus of rank $r+2$. By the Künneth theorem

$$
\begin{aligned}
&K_0(C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho) \\
&\cong K_0(C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1})) \otimes K_0(A_\rho) \\
&\qquad \oplus K_1(C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1})) \otimes K_1(A_\rho) \\
&\cong \mathbb{Z}^{2^{e+s}} \otimes \mathbb{Z}^{2^{r+1}} \cong \mathbb{Z}^{2^{e+s+r+1}}.
\end{aligned}
$$

Similarly, one obtains that $K_1(C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+1}}$. So $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong K_0(C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+1}}$. It is enough to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. Combining the tricks given in Theorem 1.6 and [17, Theorem 2.2] yields that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. So $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Next, assume that the result is true for all spherical noncommutative tori with $m = i-1$. Write $\mathbb{S}_i = C^*(\mathbb{S}_{i-1}, u_i)$, where $\mathbb{S}_i = C^*(\mathbb{S}_\rho^{cd}, u_3, \ldots, u_i)$, where $\mathbb{S}_\rho^{cd}$ is the case above, $m = 2$. Then the inductive hypothesis applies to $\mathbb{S}_{i-1}$. Also, we can think of $\mathbb{S}_i$ as the crossed product by an action α of $\mathbb{Z}$ on $\mathbb{S}_{i-1}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(v_1, \cdots, v_r, u_1^d, u_2^d, u_3, \cdots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = e^{2\pi i \theta_{ji}} u_j, j \neq 1, 2$, sending u_j^d to $u_i u_j^d u_i^{-1} = e^{2\pi i d \theta_{ji}} u_j^d, j = 1, 2$, and sending v_j to $u_i v_j u_i^{-1} = e^{2\pi i \beta_{ji}} v_j$), and which acts trivially on $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes M_{cd}(\mathbb{C})$. Here $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2}, \text{res of } \rho) \cong C^*(v_1, v_2, \cdots, v_r, u_1^d, u_2^d)$. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} and β_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{S}_{i-1}$. The Pimsner-Voiculescu exact sequence for a crossed product gives an exact sequence

$$K_0(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \longrightarrow K_1(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_1(\mathbb{S}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $\alpha_* = 1$ and since the K-groups of $\mathbb{S}_{i-1}$ are free abelian, this reduces a split short exact sequence

$$\{0\} \longrightarrow K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \longrightarrow K_1(\mathbb{S}_{i-1}) \longrightarrow \{0\}$$

and similarly for K_1. So $K_0(\mathbb{S}_i)$ and $K_1(\mathbb{S}_i)$ are free abelian of rank $2 \cdot 2^{e+s+r+i-2} = 2^{e+s+r+i-1}$. Furthermore, since the inclusion $\mathbb{S}_{i-1} \to \mathbb{S}_i$ sends $1_{\mathbb{S}_{i-1}}$ to $1_{\mathbb{S}_i}$, $[1_{\mathbb{S}_i}]$ is the image of $[1_{\mathbb{S}_{i-1}}]$, which is primitive in $K_0(\mathbb{S}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. Q.E.D.

Corollary 2.3. *Let q be a positive integer. Assume that no non-trivial matrix algebra can be factored out of A_{cd}. Then $\mathbb{S}_\rho^{cd} \otimes M_q(\mathbb{C})$ is not isomorphic to $A \otimes M_{pq}(\mathbb{C})$ for any C^*-algebra A and any integer p greater than 1. In particular, no non-trivial matrix algebra can be factored out of $\mathbb{S}_\rho^{cd}$, P_ρ^{cd} and A_ρ.*

Proof. Assume $\mathbb{S}_\rho^{cd} \otimes M_q(\mathbb{C})$ is isomorphic to $A \otimes M_{pq}(\mathbb{C})$. Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes I_q$ maps to the unit $1_A \otimes I_{pq}$. So $[1_{\mathbb{S}_\rho^{cd}} \otimes I_q] = [1_A \otimes I_{pq}]$. Thus there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $q[1_{\mathbb{S}_\rho^{cd}}] = (pq)[e]$. But $K_0(\mathbb{S}_\rho^{cd})$ is torsion-free, so $[1_{\mathbb{S}_\rho^{cd}}] = p[e]$. This contradicts Theorem 2.2 if $p > 1$.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_q(\mathbb{C})$ is not isomorphic to $A \otimes M_{pq}(\mathbb{C})$. Q.E.D.

§3. The bundle structure of spherical noncommutative tori

For M a compact CW-complex the Čech cohomology group $H^3(M, \mathbb{Z})$ classifies the tensor products of *cd*-homogeneous C^*-algebras over M with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ (see [9]). The Čech cohomology group $H^3(M, \mathbb{Z})$ is isomorphic to the singular cohomology group $H^3(M, \mathbb{Z})$ when M is *triangularizable* (see [7, Theorem15.8]).

Lemma 3.1. *Each cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}) \otimes M_{cd}(\mathbb{C})$.*

Proof. Each non-trivial element in the Čech cohomology group $H^3(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}, \mathbb{Z})$ can be given by a non-trivial element in $H^3((S^1)^3, \mathbb{Z})$, $H^3(S^2 \times S^1, \mathbb{Z})$, or $H^3(S^3, \mathbb{Z})$ if there exist such factors.

First, $H^3(S^2 \times S^1, \mathbb{Z}) = \mathbb{Z}$. By the Woodward theorem [21], $[S^2 \times S^1, BPU(cd)]$ is embedded into $H^2(S^2 \times S^1, \mathbb{Z}_{cd}) \oplus H^4(S^2 \times S^1, \mathbb{Z}) \cong H^2(S^2, \mathbb{Z}_{cd}) \cong \mathbb{Z}_{cd}$. So each *cd*-homogeneous C^*-algebra over $S^2 \times S^1$ is isomorphic to the tensor product of a *cd*-homogeneous C^*-algebra over S^2 with $C(S^1)$, which is stably isomorphic to $C(S^2) \otimes C(S^1) \otimes M_{cd}(\mathbb{C})$, since $H^3(S^2, \mathbb{Z}) = \{0\}$. Thus each *cd*-homogeneous C^*-algebra over $S^2 \times S^1$ is stably isomorphic to $C(S^2 \times S^1) \otimes M_{cd}(\mathbb{C})$.

Similarly, one obtains the same result for the other cases.

Therefore, each *cd*-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}) \otimes M_{cd}(\mathbb{C})$.

Q.E.D.

We are going to show that $\mathbb{S}_\rho^{cd} \otimes \mathcal{K}(\mathcal{H})$ has the trivial bundle structure.

Theorem 3.2. *The spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. In particular, P_ρ^d is stably isomorphic to $A_\rho \otimes M_d(\mathbb{C})$.*

Proof. Let $\mathbb{S}_\rho^{cd}$ be defined by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$, where $C^*(\widehat{\mathbb{T}^2}, \text{res of } \rho) = C^*(\widehat{\mathbb{T}^2})$. By Lemma 3.1, the *cd*-homogeneous C^*-algebra A_{cd} is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2) \otimes M_{cd}(\mathbb{C})$. In particular, $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1})$ is factored out of $A_{cd} \otimes \mathcal{K}(\mathcal{H})$. By the definition of $\mathbb{S}_\rho^{cd}$, $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1})$ is factored out of $\mathbb{S}_\rho^{cd} \otimes \mathcal{K}(\mathcal{H})$. So $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes P_\rho^d \otimes M_c(\mathbb{C})$. But it was shown in [5, Theorem 3.4] that P_ρ^d is stably isomorphic to $A_\rho \otimes M_d(\mathbb{C})$.

Therefore, $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Q.E.D.

Using the fact that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive, we are going to investigate the bundle structure of the tensor products of spherical noncommutative tori $\mathbb{S}_\rho^{cd}$ with *UHF*-algebras M_{p^∞} of type p^∞.

Theorem 3.3. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus. Assume that no non-trivial matrix algebra can be factored out of A_{cd}. Then $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.*

Proof. Assume that the set of prime factors of cd is a subset of the set of prime factors of p. To show that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^\infty}$. However, there exist the C^*-algebra homomorphisms which are the canonical inclusions

$$\mathbb{S}_\rho^{cd} \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^g}(\mathbb{C})$$

and the $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho$-module maps $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow \mathbb{S}_\rho^{cd} \otimes M_{(cd)^g}(\mathbb{C})$:

$$\begin{aligned}\mathbb{S}_\rho^{cd} &\hookrightarrow C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \hookrightarrow \mathbb{S}_\rho^{cd} \otimes M_{cd}(\mathbb{C}) \\ &\hookrightarrow C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^2}(\mathbb{C}) \hookrightarrow \cdots .\end{aligned}$$

The inductive limit of the odd terms

$$\cdots \to \mathbb{S}_\rho^{cd} \otimes M_{(cd)^g}(\mathbb{C}) \to \mathbb{S}_\rho^{cd} \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots$$

is $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$, and the inductive limit of the even terms

$$\begin{aligned}\cdots &\to C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \\ &\to C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots\end{aligned}$$

is $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^\infty}$. Thus by the Elliott theorem [11, Theorem 2.1], $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^\infty}$.

Conversely, assume that

$$\mathbb{S}_\rho^{cd} \otimes M_{p^\infty} \cong C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}.$$

Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho}$ $\otimes 1_{M_{p^\infty}} \otimes I_{cd}$. So

$$\begin{aligned}
&[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}] \\
&[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{\mathbb{S}_\rho^{cd}}] \otimes [1_{M_{p^\infty}}] \\
&[1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}] \\
&\qquad = cd([1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho}] \otimes [1_{M_{p^\infty}}]).
\end{aligned}$$

Under the assumption that $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ maps to

$$1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd},$$

if there is a prime factor q of cd such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞}. So there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $[1_{\mathbb{S}_\rho^{cd}}] = q[e]$. This contradicts Theorem 2.2. Thus the set of prime factors of cd is a subset of the set of prime factors of p.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p. Q.E.D.

§4. Completely irrational noncommutative tori

It was proved in [3, Theorem 1.5] that every completely irrational noncommutative torus has real rank 0, where the "*real rank* 0" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. Combining Theorem 3.2 and [8, Corollary 3.3] yields that the generalized noncommutative torus P_ρ^d has real rank 0 since the noncommutative torus A_ρ has real rank 0. The Lin and Rørdam theorem [16, Proposition 3] says that the generalized noncommutative torus P_ρ^d is an inductive limit of circle algebras, since $P_\rho^d \otimes \mathcal{K}(\mathcal{H}) \cong A_\rho \otimes \mathcal{K}(\mathcal{H})$ is an inductive limit of circle algebras [16, Proposition]. Combining [11, Theorem 7.1] and [13, Theorem 1.3] yields that the completely irrational noncommutative tori A_ω of rank $r+m$ and the generalized noncommutative tori P_ρ^d of rank $r+m$ are isomorphic if the ranges of the traces equal.

Lemma 4.1. ([6, Lemma 4.1]) $\operatorname{tr}(K_0(P_\rho^d)) = \frac{1}{d} \cdot \operatorname{tr}(K_0(A_\rho))$.

Theorem 4.2. ([6, Theorem 4.2]) *Let A_ω be a completely irrational noncommutative torus of rank $r+m$ with $\operatorname{tr}(K_0(A_\omega)) = \frac{1}{d} \cdot \operatorname{tr}(K_0(A_\rho))$ for A_ρ a completely irrational noncommutative torus of rank $r+m$. Then A_ω is isomorphic to P_ρ^d.*

§5. C^*-algebras over spheres with fibres noncommutative tori

We are going to show that the set of all spherical noncommutative tori with primitive ideal space $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ and fibres $A_\omega \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $A_\omega \otimes M_c(\mathbb{C})$ for A_ω a completely irrational noncommutative torus.

Let A_ω be a noncommutative torus of rank m with $\widehat{S_\omega} \cong \mathbb{T}^1$. Then A_ω is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{d\mathbb{Z}}$ and fibres $C^*(\mathbb{Z}^m/S_\omega, \omega_1)$ for some totally skew multiplier ω_1, where $C^*(\mathbb{Z}^m/S_\omega, \omega_1) \cong A_\rho \otimes M_d(\mathbb{C})$ for A_ρ a completely irrational noncommutative torus of rank $m-1$ (see [1, 12]). By the definition of A_ω, $C(\mathbb{T}^1)$ and A_ρ split. Since $[\mathbb{T}^1, BPU(d)] \cong \{0\}$, $C(\mathbb{T}^1)$ and $M_d(\mathbb{C})$ split. And $M_d(\mathbb{C})$ and A_ρ also split. But by Corollary 2.3, A_ω has a non-trivial bundle structure if $d > 1$. This implies that a C^*-subalgebra of A_ρ plays a role as a base space in the bundle structure. In fact, A_ω can be obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{l}{d}}$, and the non-triviality of the bundle structure is given by a non-trivial element of $[\mathbb{T}^2, BPU(d)] \cong [\mathbb{T}^1, PU(d)] \cong \mathbb{Z}_d$, which represents $A_{\frac{l}{d}}$ canonically embedded into A_ω.

Let d be the biggest integer among the possible integers satisfying the condition $\mathrm{tr}(K_0(A_\omega)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho))$, i.e., $A_\omega \cong P_\rho^d$. For a d-homogeneous C^*-algebra A over S^{2n+1}, there is a matrix algebra $M_q(\mathbb{C})$ such that $A \otimes M_q(\mathbb{C})$ is isomorphic to $C(S^{2n+1}) \otimes M_{dq}(\mathbb{C})$. But there is a matrix subalgebra $M_q(\mathbb{C})$ big enough satisfying the above condition such that $M_q(\mathbb{C})$ is embedded into P_ρ^d, since P_ρ^d is an inductive limit of circle algebras, which is simple.

Lemma 5.1. *Each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^{2n+1} with fibres $P_\rho^1 = A_\rho$ has the trivial bundle structure.*

Proof. Let $P_\rho^1 = \varinjlim(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}))$. The C^*-algebra $\Gamma(\eta)$ is isomorphic to an inductive limit of direct sums of $p_{i(j)}$-homogeneous C^*-algebras over $S^{2n+1} \times \mathbb{T}^1$, and each $C(S^{2n+1} \times \mathbb{T}^1)$ is canonically embedded into $\Gamma(\eta)$. So there could be a canonical homomorphism of $C(S^{2n+1}) \otimes M_d(\mathbb{C})$ into the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^{2n+1} with fibres P_ρ^1 such that the non-triviality can be given by a d-homogeneous C^*-algebra over $S^{2n+1} \times \mathbb{T}^1$. Then $M_d(\mathbb{C})$ must be factored out of the circle algebra in each inductive step, and so the range of the trace of P_ρ^1 would be the form $\frac{1}{d} \cdot \mathrm{tr}(A)$ for

A a simple unital C^*-algebra, which is impossible by the assumption. We have two cases; one of them is the case that a C^*-subalgebra of P_ρ^1 plays a role as a base space in the bundle structure, and the other is not.

For the first case, when a C^*-subalgebra of P_ρ^1 plays a role as a base space in the bundle structure and P_ρ^1 is realized as a tensor product of non-trivial completely irrational noncommutative tori, the torsion-free groups in $P_\rho^1 = A_\rho$ giving simple noncommutative tori which are given by twisting the torsion-free groups by totally skew multipliers must split, so all factors of P_ρ^1 must split. The relation among factors of P_ρ^1 is different from the relation between fibres $M_d(\mathbb{C})$ and base A_ρ in the fibres of the non-simple noncommutative torus A_ω given above, and so one can assume that all factors of P_ρ^1 play roles as a base space in the bundle structure. Hence P_ρ^1 plays a role as a base space in the bundle structure, and so $\Gamma(\eta)$ is isomorphic to $C(S^{2n+1}) \otimes P_\rho^1$.

For the other case, since $P_\rho^1 = \varinjlim(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}))$, there is a matrix algebra $M_p(\mathbb{C})$ big enough which is embedded into P_ρ^1. Since $[S^{2n+1}, BPU(p)] \cong \{0\}$, $C(S^{2n+1})$ and $M_p(\mathbb{C})$ split, i.e., any p-homogeneous C^*-algebra over S^{2n+1} has the trivial bundle structure. By the same reasoning as above, $M_p(\mathbb{C})$ cannot be factored out of the circle algebras in all inductive steps. But $\Gamma(\eta)$ has a locally trivial bundle structure. Hence $C(S^{2n+1})$ and $(M_p(\mathbb{C}) \hookrightarrow)$ P_ρ^1 must split, and so $\Gamma(\eta)$ has the trivial bundle structure.

Therefore, each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^{2n+1} with fibres P_ρ^1 has the trivial bundle structure. Q.E.D.

Now we want to show that each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^1 = A_\rho$ has the trivial bundle structure.

Proposition 5.2. *Each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^1 = A_\rho$ has the trivial bundle structure.*

Proof. Let P_ρ^1 be an inductive limit of $\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C})$. For some pair $(2k_j - 1, 2k_{j'} - 1) = (2k_j - 1, 1)$, if the C^*-subalgebra of sections of a locally trivial C^*-algebra bundle over $S^{2k_j-1} \times S^1$ with fibres P_ρ^1, which is canonically embedded into $\Gamma(\eta)$, has a non-trivial bundle structure, then the factor $S^{2k_j-1} \times S^1$ can be replaced by S^{2k_j}, since there is a map of degree 1 from $S^{2k_j-1} \times S^1$ to S^{2k_j}. For each j, there is a canonical homomorphism of the C^*-subalgebra $\Gamma(\eta_j)$ of sections of a locally trivial C^*-algebra bundle η_j over S^{2k_j-1} with fibres P_ρ^1 into $\Gamma(\eta)$.

By Lemma 5.1, the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2k_j-1} with fibres P^1_ρ has the trivial bundle structure. Thus $C(S^{2k_j-1})$ are factored out of $\Gamma(\eta)$, and so $C(\prod_{j=1}^s S^{2k_j-1})$ is factored out of $\Gamma(\eta)$.

Next, $[S^{2n_i}, B(\mathrm{Aut}(P^1_\rho))] = [S^{2n_i-1}, \mathrm{Aut}(P^1_\rho)]$. But there is a map of degree 1 from S^{2n_i} to $S^{2n_i-1} \times S^1$. So for each i each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n_i} with fibres P^1_ρ is induced from the C^*-algebra $\Gamma(\zeta_i)$ of sections of a locally trivial C^*-algebra bundle ζ_i over $S^{2n_i-1} \times \mathbb{T}^1$ with fibres P^1_ρ. Consider the crossed product by the action α_θ of $\mathbb{Z}$ on $\Gamma(\zeta_i)$ for a suitable irrational number θ such that the range of the trace of $P^1_\rho \otimes A_\theta$ is not $\frac{1}{w}\times$ the range of the trace of any simple irrational noncommutative torus of rank $m+1$ for any positive integer w greater than 1, where the action α_θ on $C(S^{2n_i-1}) \otimes P^1_\rho$ is trivial and $C(\mathbb{T}^1) \times_{\alpha_\theta} \mathbb{Z}$ is the irrational rotation algebra A_θ. Then $\Gamma(\zeta_i) \times_{\alpha_\theta} \mathbb{Z}$ is obviously realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n_i-1} with fibres $P^1_\rho \otimes A_\theta$. But $\Gamma(\zeta_i) \times_{\alpha_\theta} \mathbb{Z}$ has the trivial bundle structure. So each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n_i} with fibres P^1_ρ has the trivial bundle structure. Thus $C(S^{2n_i})$ are factored out of $\Gamma(\eta)$. Hence $C(\prod_{i=1}^e S^{2n_i})$ is factored out of $\Gamma(\eta)$, and so $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1})$ is factored out of $\Gamma(\eta)$, as desired. Q.E.D.

Each cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ with fibres $M_{cd}(\mathbb{C})$, and hence $\mathbb{S}^{cd}_\rho$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P^d_\rho \otimes M_c(\mathbb{C})$.

Theorem 5.3. *The set of spherical noncommutative tori with primitive ideal space $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ and fibres $P^d_\rho \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P^d_\rho \otimes M_c(\mathbb{C})$.*

Proof. If $cd = 1$, we have obtained the result in Proposition 5.2. So assume that $cd > 1$. Then one can assume that there is a matrix subalgebra $M_{cd}(\mathbb{C})$ which is factored out of each inductive step, even though $M_d(\mathbb{C})$ is not factored out of P^d_ρ. And P^d_ρ is isomorphic to $A_{\frac{1}{d}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_{r+m}} \mathbb{Z}$. By Proposition 5.2, each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $C^*(d\mathbb{Z} \times d\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_{r+m}} \mathbb{Z}$ has the trivial bundle structure.

Hence each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is given by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by the totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$, which is a spherical noncommutative torus.

Therefore, the set of spherical noncommutative tori with primitive ideal space $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ and fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$. Q.E.D.

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