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Extensions of quasidiagonal C^*-algebras and K-theory

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Abstract.

Let $0 \to I \to E \to B \to 0$ be a short exact sequence of C^*-algebras where E is separable, I is quasidiagonal (QD) and B is nuclear, QD and satisfies the UCT. It is shown that if the boundary map $\partial : K_1(B) \to K_0(I)$ vanishes then E must be QD also.

A Hahn-Banach type property for K_0 of QD C^*-algebras is also formulated. It is shown that every nuclear QD C^*-algebra has this K_0-Hahn-Banach property if and only if the boundary map $\partial : K_1(B) \to K_0(I)$ (from above) always completely determines when E is QD in the nuclear case.

§1. Introduction

Quasidiagonal (QD) C^*-algebras are those which enjoy a certain finite dimensional approximation property. (See [Vo2], [Br3] for surveys of the theory of QD C^*-algebras.) While these finite dimensional approximations have certainly lead to a better understanding of the structure of QD C^*-algebras, there are a number of very basic open questions. For example, assume that $0 \to I \to E \xrightarrow{\pi} B \to 0$ is a split exact sequence (i.e. there exists a *-homomorphism $\varphi : B \to E$ such that $\pi \circ \varphi = id_B$) where both I and B are QD. It is not known whether E must be QD (and, in fact, it is not even clear what to expect).

In this paper we study the extension problem for QD C^*-algebras and it's relation to some natural questions concerning K-theory of QD C^*-algebras. Our techniques rely heavily on Kasparov's theory of extensions and thus we will always need some nuclearity assumptions.

For example, adapting techniques found in [Sp] we will show (Theorem 3.4) that if $0 \to I \to E \to B \to 0$ is short exact where E is separable,

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I is QD, B is nuclear, QD and satisfies the Universal Coefficient Theorem (UCT) and the boundary map $\partial : K_1(B) \to K_0(I)$ vanishes then E must be QD also. It follows that if $K_1(B) = 0$ then E is always QD, which generalizes work of Eilers, Loring and Pedersen ([ELP]). As another application we observe that in the case that I is the compact operators our result implies that E is QD if and only if the (class of the) extension is in the kernel of the natural map $Ext(B) \to Hom(K_1(B), \mathbb{Z})$, where $Ext(B)$ denotes the classical BDF group (recall that we are assuming B is nuclear and hence $Ext(B)$ is a group). Also, we verify a conjecture of [BK], stating that an asymptotically split extension of NF algebras is NF, under the additional assumption that the quotient algebra satisfies the UCT of [RS].

We then study the general extension problem. Now let $0 \to I \to E \to B \to 0$ be exact where E is separable and nuclear, I is QD and B is QD and satisfies the UCT. Based on previous work of Spielberg ([Sp]) it is reasonable to expect that in this case E will be QD if and only if $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$, where $K_0^+(I) = \{0\}$ denotes the positive cone of $K_0(I)$. Though we are unable to resolve this question we do show that it is equivalent to some other natural questions concerning the K-theory of QD C^*-algebras and that in order to solve the general extension problem it suffices to prove the special case that $B = C(\mathbb{T})$ (see Theorem 4.11).

The first equivalent K-theory question is: If A is nuclear, separable and QD and $G \subset K_0(A)$ is a subgroup such that $G \cap K_0^+(A) = 0$ then can one always find an embedding $\rho : A \hookrightarrow C$ where C is QD and $\rho_*(G) = 0$? The condition $G \cap K_0^+(A) = 0$ is easily seen to be necessary and hence the question is whether or not it is sufficient. The second K-theory question asks whether every nuclear QD C^*-algebra satisfies what we call the *K_0-Hahn-Banach property* (see Definition 4.7). Roughly speaking this K_0-Hahn-Banach property states that if $x \in K_0(A)$ and $\pm x \notin K_0^+(A)$ then one can always find finite dimensional approximate morphisms (i.e. "functionals") which separate x from $K_0^+(A)$. (Due to possible perforation in $K_0(A)$ this statement is not quite correct, but it conveys the main idea.) Determining whether every nuclear QD algebra satisfies the K_0-Hahn-Banach property is of independent interest as our inability to understand how well finite dimensional approximate morphisms read K-theory has been a major obstacle in the classification program.

In section 2 we review the necessary theory of extensions and prove a few simple results needed later. In section 3 we handle the case when $\partial : K_1(B) \to K_0(I)$ vanishes. In section 4 we turn to the general extension problem and show equivalence with the K-theory questions described above.

The present work is related to work of Salinas [Sa1], [Sa2] and Schochet [Sch]. Those authors study the quasidiagonality of extensions $0 \to I \to E \to B \to 0$ (i.e. the question of whether or not I contains an approximate unit of projections which is quasicentral in E) whereas we study the QD of the C^*-algebra E. The two questions are different even if I is the compact operators. Indeed, while the quasidiagonality of $0 \to \mathcal{K} \to E \to B \to 0$ does imply the QD of E, the converse implication is false (see Section 3).

§2. Preliminaries and Trivial Extensions.

Most of this section is devoted to reviewing definitions, introducing notation and recalling some standard facts about extensions of C^*-algebras. However, at the end we prove a few simple facts which will be needed later. The main result states that quasidiagonality is preserved in split extensions provided that either the ideal or the quotient is a nuclear C^*-algebra (see Proposition 2.5).

For a comprehensive introduction to the aspects of extension theory which we will need we recommend looking at [Bl, Section 15]. For any C^*-algebra I we will let $M(I)$ be it's multiplier algebra and $Q(I) = M(I)/I$ be it's corona algebra. Let $\pi : M(I) \to Q(I)$ be the quotient map.

If E is any C^*-algebra containing I as an ideal and $B = E/I$ then there exists a unique *-homomorphism $\rho : E \to M(I)$ such that $\rho(I) = I$ and hence an induced *-homomorphism $\gamma : B \to Q(I)$. The map γ is injective if and only if ρ is in injective if and only if I sits as an essential ideal in E. Conversely, given a C^*-algebra B and a *-homomorphism $\gamma : B \to Q(I)$ we can construct the pullback which, by definition, is the C^*-algebra

$$E(\gamma) = \{x \oplus b \in M(I) \oplus B : \pi(x) = \gamma(b)\}.$$

This gives a short exact sequence $0 \to I \to E(\gamma) \to B \to 0$. Moreover, if $B = E/I$ with induced map $\gamma : B \to Q(I)$ then there is an induced *-isomorphism $\Phi : E \to E(\gamma)$ with commutativity in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\
 & & \Vert & & {\scriptstyle\Phi}\downarrow & & \Vert & & \\
0 & \longrightarrow & I & \longrightarrow & E(\gamma) & \longrightarrow & B & \longrightarrow & 0.
\end{array}$$

Hence there is a one to one correspondence between extensions of I by B and *-homomorphisms $\gamma : B \to Q(I)$. As is standard, we will refer to a *-homomorphism $\gamma : B \to Q(I)$ as a *Busby invariant* and freely use the above correspondence between Busby invariants and extensions.

When I is stable (i.e. $I \cong \mathcal{K} \otimes I$, where $\mathcal{K}$ denotes the compact operators on a separable infinite dimensional Hilbert space) there is a natural way of adding two extensions which we now describe. Any isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ induces an isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \otimes I \cong \mathcal{K} \otimes I$ which then gives isomorphisms $M_2(\mathbb{C}) \otimes M(\mathcal{K} \otimes I) \cong M(\mathcal{K} \otimes I)$ and $M_2(\mathbb{C}) \otimes Q(\mathcal{K} \otimes I) \cong Q(\mathcal{K} \otimes I)$. Thus if we are given two Busby invariants $\gamma_1,\ \gamma_2 : B \to Q(\mathcal{K} \otimes I)$ we can define a new Busby invariant $\gamma_1 \oplus \gamma_2$ by

$$(\gamma_1 \oplus \gamma_2)(b) = \begin{pmatrix} \gamma_1(b) & 0 \\ 0 & \gamma_2(b) \end{pmatrix} \in M_2(\mathbb{C}) \otimes Q(\mathcal{K} \otimes I) \cong Q(\mathcal{K} \otimes I).$$

Of course the Busby invariant $\gamma_1 \oplus \gamma_2$ constructed in this way will depend on the particular isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$. To remedy this we say that two Busby invariants $\gamma_1,\ \gamma_2$ are *strongly equivalent* if there exists a unitary $u \in M(I)$ such that $\mathrm{Ad}\pi(u)\big(\gamma_1(b)\big) = \pi(u)\gamma_1(b)\pi(u^*) = \gamma_2(b)$, for all $b \in B$, where $\pi : M(I) \to Q(I)$ is the quotient map. Note that if γ_1 and γ_2 are strongly equivalent then $E(\gamma_1)$ and $E(\gamma_2)$ are isomorphic C^*-algebras. Indeed, the map $E(\gamma_1) \to E(\gamma_2)$, $x \oplus b \mapsto uxu^* \oplus b$ is easily seen to be an isomorphism. Since any isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ is implemented by a unitary we see that $\gamma_1 \oplus \gamma_2$ is unique up to strong equivalence. In particular, the isomorphism class of the C^*-algebra $E(\gamma_1 \oplus \gamma_2)$ does not depend on the choice of isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$.

A Busby invariant τ is called *trivial* if it lifts to a *-homomorphism $\varphi : B \to M(I)$ (i.e. $\pi \circ \varphi = \gamma$). A Busby invariant $\gamma : B \to Q(\mathcal{K} \otimes I)$ is called *absorbing* if $\gamma \oplus \tau$ is strongly equivalent to γ for every trivial τ. Note that if γ is absorbing then so is $\tilde{\gamma} \oplus \gamma$ for any $\tilde{\gamma}$. In particular if γ is absorbing then γ is injective. Note also that if τ_1 and τ_2 are both trivial and absorbing then τ_1, $\tau_1 \oplus \tau_2$ and τ_2 are all strongly equivalent. Thus we get the following fact.

Lemma 2.1. *If $\tau_1,\ \tau_2 : B \to Q(\mathcal{K} \otimes I)$ are both trivial and absorbing then $E(\tau_1) \cong E(\tau_2)$.*

Another simple fact we will need is the following.

Lemma 2.2. *If $\gamma,\ \tau : B \to Q(\mathcal{K} \otimes I)$ are Busby invariants with τ trivial then there is a natural embedding $E(\gamma) \hookrightarrow E(\gamma \oplus \tau)$.*

Proof. Let $\varphi : B \to M(I)$ be a lifting of τ. Define a map $E(\gamma) \to E(\gamma \oplus \tau)$ by

$$x \oplus b \mapsto \begin{pmatrix} x & 0 \\ 0 & \varphi(b) \end{pmatrix} \oplus b.$$

Evidently this map is an injective $*$-homomorphism. □

The following generalization of Voiculescu's Theorem, which is due to Kasparov, will be crucial in what follows.

Theorem 2.3. ([Bl, Thm. 15.12.4]) *Assume that B is separable, I is σ-unital and either B or I is nuclear. Let $\rho : B \to B(H)$ be a faithful representation such that H is separable, $\rho(B) \cap \mathcal{K}(H) = \{0\}$ and the orthogonal complement of the nondegeneracy subspace of $\rho(B)$ (i.e. $H \ominus \overline{\rho(B)H}$) is infinite dimensional. Regarding $B(H) \cong B(H) \otimes 1 \subset M(\mathcal{K} \otimes I)$ as scalar operators we get a short exact sequence*

$$0 \to \mathcal{K} \otimes I \to \rho(B) \otimes 1 + \mathcal{K} \otimes I \to B \to 0.$$

If τ denotes the induced Busby invariant then τ is both trivial and absorbing.

We define an equivalence relation on the set of Bubsy invariants $B \to Q(\mathcal{K} \otimes I)$ by saying γ is related to $\tilde{\gamma}$ if there exist trivial Busby invariants $\tau, \tilde{\tau}$ such that $\gamma \oplus \tau$ is strongly equivalent to $\tilde{\gamma} \oplus \tilde{\tau}$. Taking the quotient by this relation yields the semigroup $Ext(B, \mathcal{K} \otimes I)$. The image of a map $\gamma : B \to Q(\mathcal{K} \otimes I)$ in $Ext(B, \mathcal{K} \otimes I)$ is denoted $[\gamma]$. Note that all trivial Busby invariants give rise to the same class denoted by $0 \in Ext(B, \mathcal{K} \otimes I)$ and this class is a neutral element (i.e. identity) for the semigroup. Note also that if $[\gamma] = 0 \in Ext(B, \mathcal{K} \otimes I)$ then it does not follow that γ is trivial. However, it does follow that if τ is a trivial absorbing Busby invariant then so is $\gamma \oplus \tau$.

We are almost ready to prove the main result of this section. We just need one more definition.

Definition 2.4. *If $0 \to I \to E \to B \to 0$ is an exact sequence with Busby invariant γ then we let $\gamma^s : \mathcal{K} \otimes B \to Q(\mathcal{K} \otimes I)$ denote the stabilization of γ. That is, γ^s is the Busby invariant of the exact sequence $0 \to \mathcal{K} \otimes I \to \mathcal{K} \otimes E \to \mathcal{K} \otimes B \to 0$.*

Note that there is always an embedding $E \cong E(\gamma) \hookrightarrow E(\gamma^s)$.

Proposition 2.5. *Let $0 \to I \to E \to B \to 0$ be exact with Busby invariant γ. If both I and B are QD, B is separable, I is σ-unital, either I or B is nuclear and $[\gamma^s] = 0 \in Ext(\mathcal{K} \otimes B, \mathcal{K} \otimes I)$ then E is also QD.*

Proof. Since quasidiagonality passes to subalgebras, it suffices to show that if $\tau : \mathcal{K}\otimes B \to Q(\mathcal{K}\otimes I)$ is a trivial absorbing Busby invariant (which exists by Theorem 2.3) then $E(\tau)$ is QD. Indeed, by Lemmas 2.1, 2.2 and the definition of $Ext(\mathcal{K}\otimes B, \mathcal{K}\otimes I)$ we have the inclusions

$$E \hookrightarrow E(\gamma^s) \hookrightarrow E(\gamma^s \oplus \tau) \cong E(\tau).$$

To prove that $E(\tau)$ is QD we may assume (again by Lemma 2.1) that τ arises from the particular extension described in Theorem 2.3. However for that extension it is easy to see that $E(\tau) \hookrightarrow (\rho(B)+\mathcal{K})\otimes \tilde{I}$, where $\tilde{I}$ is the unitization of I. But since $\rho(B)\cap\mathcal{K} = \{0\}$ it follows that $\rho(B)+\mathcal{K}$ is QD ([Br3, Thm. 3.11]). Hence $(\rho(B)+\mathcal{K})\otimes\tilde{I}$ is also QD as a minimal tensor product QD C^*-algebras ([Br3, Prop. 7.5]). □

Note that the above proposition covers the case of split extensions (i.e. when γ is trivial).

§3. When $\partial : K_1(B) \to K_0(I)$ is zero.

The main result of this section (Theorem 3.4) states that if the boundary map $\partial : K_1(B) \to K_0(I)$ coming from an exact sequence $0 \to I \to E \to B \to 0$ is zero then E will be QD whenever I is QD and B is nuclear, QD and satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet ([RS]). The main ideas in the proof are inspired by work of Spielberg ([Sp]). We also discuss a few consequences of our result, including generalization of work of Eilers-Loring-Pedersen ([ELP]) and a partial solution to a conjecture of Blackadar and Kirchberg [BK].

Definition 3.1. An embedding $I \hookrightarrow J$ is called *approximately unital* if it takes an approximate unit of I to an approximate unit of J.

In this case there is a natural inclusion $M(I) \hookrightarrow M(J)$ which induces an inclusion $Q(I) \hookrightarrow Q(J)$ [Pe, 3.12.12]. Hence for any Busby invariant $\gamma : B \to Q(I)$ there is an induced Busby invariant $\eta : B \to Q(J)$ with commutativity in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E(\gamma) & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & J & \longrightarrow & E(\eta) & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Moreover, the two vertical maps on the left are injective.

There are two ways of producing approximately unital embeddings which we will need. The first is $I \hookrightarrow I \otimes A$, for some *unital* C^*-algebra

A. If $\{e_\lambda\}$ is an approximate unit of I then, of course, $e_\lambda \otimes 1_A$ will be an approximate unit of $I \otimes A$. The other is to start with an arbitrary embedding $I \hookrightarrow J'$ and define J to be the hereditary subalgebra in J' generated by I. That is, define J to be the closure of $\cup_\lambda e_\lambda J' e_\lambda$. One easily checks that J is then a hereditary subalgebra of J' and the embedding $I \hookrightarrow J$ is approximately unital.

In the theory of separable QD C^*-algebras there are some *nonseparable* algebras which play a key role. The first is the direct product $\Pi_i M_{n_i}(\mathbb{C})$ for some sequence of integers $\{n_i\}$. This algebra is the multiplier algebra of the direct sum $\oplus_i M_{n_i}(\mathbb{C})$. If H is any separable Hilbert space then we can always find a decomposition $H = \oplus_i \mathbb{C}^{n_i}$ and then we have natural inclusions $\oplus_i M_{n_i}(\mathbb{C}) \hookrightarrow \mathcal{K}(H)$, $\Pi_i M_{n_i}(\mathbb{C}) \hookrightarrow B(H)$ and $Q(\oplus_i M_{n_i}(\mathbb{C})) \hookrightarrow Q(\mathcal{K}(H))$. Another algebra which we will need is $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$.

Lemma 3.2. *Let $J \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be a hereditary subalgebra containing $\mathcal{K}(H)$. Then $K_1(J) = 0$.*

Proof. Letting $\pi : B(H) \to Q(H)$ be the quotient map we have that $\pi(J)$ is a hereditary subalgebra of $Q(\oplus_i M_{n_i}(\mathbb{C}))$ (use the fact that if $0 \leq a \in J, b \in Q(\oplus_i M_{n_i}(\mathbb{C}))$ and $0 \leq b \leq \pi(a)$ then there exists $0 \leq c \in \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ such that $c \leq a$ and $\pi(c) = b$; [Da, Cor. IX.4.5]. Also, the exact sequence $0 \to \mathcal{K}(H) \to J \to \pi(J) \to 0$ is a *quasidiagonal extension* (i.e. $\mathcal{K}(H)$ contains an approximate unit *of projections which is quasicentral in* J). Hence [BD, Thm. 8], states that we have a short exact sequence

$$0 \to K_1(\mathcal{K}(H)) \to K_1(J) \to K_1(\pi(J)) \to 0.$$

Thus it suffices to show that $K_1(X) = 0$ for any hereditary subalgebra X of $Q(\oplus_i M_{n_i}(\mathbb{C}))$.

But if $X \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ is a hereditary subalgebra then we can find a quasidiagonal extension

$$0 \to \oplus_i M_{n_i}(\mathbb{C}) \to Y \to X \to 0,$$

where $Y \subset \Pi_i M_{n_i}(\mathbb{C})$ is a hereditary subalgebra. Applying [BD, Thm. 8] again it suffices to show that every hereditary subalgebra of $\Pi_i M_{n_i}(\mathbb{C})$ has trivial K_1-group.

But, if $Y \subset \Pi_i M_{n_i}(\mathbb{C})$ is a hereditary σ-unital subalgebra then Y has an increasing approximate unit consisting of projections, say $\{e_n\}$ ([BP]). Hence

$$K_1(Y) = \lim K_1(e_n \Pi_i M_{n_i}(\mathbb{C}) e_n),$$

since $Y = \lim e_n \Pi_i M_{n_i}(\mathbb{C}) e_n$ (by heredity). But, for each n, it is clear that $e_n \Pi_i M_{n_i}(\mathbb{C}) e_n$ is isomorphic to $\Pi_i M_{k_i}(\mathbb{C})$ for some integers $\{k_i\}$ and consequently $K_1(e_n \Pi_i M_{n_i}(\mathbb{C}) e_n) = 0$. □

Proposition 3.3. *Let I be a separable QD C^*-algebra. Then there exists an approximately unital embedding $I \hookrightarrow J$, where J is a σ-unital QD C^*-algebra with $K_1(J) = 0$.*

Proof. Let $\rho : I \to B(H)$ be a nondegenerate faithful representation such that $\rho(I) \cap \mathcal{K}(H) = \{0\}$. By [Br3, Prop. 5.2], there exists a decomposition $H = \oplus_i \mathbb{C}^{n_i}$ such that $\rho(I) \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$. Let J be the hereditary subalgebra of $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ generated by $\rho(I)$. The conclusion follows from the previous lemma. □

For the remainder of this section we will let $\mathcal{U} = \otimes_n M_n(\mathbb{C})$ be the Universal UHF algebra (i.e. the UHF algebra with $K_0(\mathcal{U}) = \mathbb{Q}$). For any Busby invariant $\gamma : B \to Q(J)$ we let $\gamma^{\mathbb{Q}}$ denote the Busby invariant coming from the short exact sequence

$$0 \to J \otimes \mathcal{U} \to E(\gamma) \otimes \mathcal{U} \to B \otimes \mathcal{U} \to 0.$$

Theorem 3.4. *Let $0 \to I \to E \to B \to 0$ be a short exact sequence where E is separable, I is QD and B is nuclear, QD and satisfies the UCT. If the induced map $\partial : K_1(B) \to K_0(I)$ is zero then E is QD.*

Proof. Let γ be the Busby invariant of the exact sequence in question. By the previous proposition we can find an approximately unital embedding $I \hookrightarrow J$, where J is QD with $K_1(J) = 0$. By the remarks following Definition 3.1 we have an inclusion $E \hookrightarrow E(\eta)$ where $\eta : B \to Q(J)$ is the induced Busby invariant. By naturality we then have that both index maps $\partial : K_1(B) \to K_0(J)$ and $\partial : K_0(B) \to K_1(J)$ are zero. Hence the index maps arising from the stabilization $\eta^s : B \otimes \mathcal{K} \to Q(J \otimes \mathcal{K})$ are also zero.

Now, if it happens that $K_0(J)$ is a divisible group then the Universal Coefficient Theorem would imply that $[\eta^s] = 0 \in Ext(B \otimes \mathcal{K}, J \otimes \mathcal{K})$ and so by Proposition 2.5 we would be done. Of course this will not be true in general and so may have to replace η^s with $(\eta^s)^{\mathbb{Q}}$. But applying naturality one more time, both boundary maps on K-theory arising from $(\eta^s)^{\mathbb{Q}}$ will also vanish. Hence the theorem follows from the inclusions $E \hookrightarrow E(\eta) \hookrightarrow E(\eta^s) \hookrightarrow E((\eta^s)^{\mathbb{Q}})$ together with Proposition 2.5 applied to $(\eta^s)^{\mathbb{Q}}$. □

In the case that the ideal is nuclear and the quotient is an AF algebra, the next result was obtained by Eilers, Loring and Pedersen ([ELP, Cor. 4.6]).

Corollary 3.5. *Assume that B is a separable nuclear QD C^*-algebra satisfying the UCT and with $K_1(B) = 0$. For any separable QD C^*-algebra I and Busby invariant $\gamma : B \to Q(I)$ we have that $E(\gamma)$ is QD.*

This corollary actually extends to the case where $K_1(B)$ is a torsion group since we can tensor any short exact sequence with $\mathcal{U}$ and $K_1(B \otimes \mathcal{U}) = 0$ in this case. For example, this would cover the case that $B = C_0(\mathbb{R}) \otimes \mathcal{O}_n$, $(2 \leq n \leq \infty)$, where $\mathcal{O}_n$ denotes the Cuntz algebra on n generators. Similarly, it is clear that Theorem 3.4 is valid under the weaker hypothesis that $\partial(K_1(B))$ is contained in the torsion subgroup of $K_0(I)$.

Definition 3.6. For any two QD C^*-algebras I, B let $Ext_{QD}(B, \mathcal{K} \otimes I) \subset Ext(B, \mathcal{K} \otimes I)$ denote the set of classes of Busby invariants γ such that $E(\gamma)$ is QD.

It is easy to check that if $[\gamma] = [\tilde{\gamma}] \in Ext(B, \mathcal{K} \otimes I)$ then $E(\gamma)$ is QD if and only if $E(\tilde{\gamma})$ is QD and hence $Ext_{QD}(B, \mathcal{K} \otimes I)$ is well defined. It is also easy to see that $Ext_{QD}(B, \mathcal{K} \otimes I)$ is a sub-semigroup of $Ext(B, \mathcal{K} \otimes I)$. Finally, we remark that in the case $I = \mathbb{C}$ we do *not* get the semigroup $Ext_{qd}(B, \mathcal{K})$ defined by Salinas; it follows from Corollary 3.7 below, however, that we do get what he called $Ext_{bqt}(B, \mathcal{K})$ in this case (see [Sa1, Definitions 2.7, 2.12 and Thm. 2.14]). One has $Ext_{qd}(B, \mathcal{K}) \subset Ext_{QD}(B, \mathcal{K})$. The elements of $Ext_{QD}(B, \mathcal{K})$ corresponds to C^*-algebras $E(\gamma)$ that are QD whereas $[\gamma] \in Ext_{qd}(B, \mathcal{K})$ if the only if the extension $0 \to \mathcal{K} \to E(\gamma) \to B \to 0$ is QD i.e. the concrete set $E(\gamma) \subset M(\mathcal{K})$ is QD.

Recall that there is a natural group homomorphism $\Phi : Ext(B, \mathcal{K} \otimes I) \to Hom(K_1(B), K_0(I))$ taking a Busby invariant to the corresponding boundary map on K-theory. From Theorem 3.4 it follows that we always have an inclusion $Ker(\Phi) \subset Ext_{QD}(B, \mathcal{K} \otimes I)$, when B is nuclear, QD and satisfies the UCT. In general this inclusion will be proper, but we now describe a class of algebras for which we have equality.

There is a natural semigroup $K_0^+(I) \subset K_0(I)$, called the *positive cone*, given by

$$K_0^+(I) = \bigcup_{n \in \mathbb{N}} \{x \in K_0(I) : x = [p], \text{ for some projection } p \in M_n(I)\}.$$

When I is unital this semigroup generates $K_0(I)$ but can also be trivial in general (e.g. if I is stably projectionless). The natural isomorphism $K_0(I) \cong K_0(\mathcal{K} \otimes I)$ induced by an embedding $I = e_{11} \otimes I \subset \mathcal{K} \otimes I$, where e_{11} is a minimal projection in $\mathcal{K}$, preserves the positive cones. We say that $K_0(I)$ is *totally ordered* if for every $x \in K_0(I)$ either x or $-x$ is an element of $K_0^+(I)$.

Corollary 3.7. *Assume I is separable, QD and $K_0(I)$ is totally ordered. For any separable, nuclear, QD algebra B which satisfies the UCT we have that $Ext_{QD}(B, \mathcal{K} \otimes I) = Ker(\Phi)$.*

Proof. We only have to show $Ext_{QD}(B, \mathcal{K} \otimes I) \subset Ker(\Phi)$. So let $[\gamma] \in Ext(B, \mathcal{K} \otimes I)$. If $E(\gamma)$ is a stably finite C^*-algebra then a result of Spielberg (see Proposition 4.1 of the next section), together with the assumption that $K_0(I)$ is totally ordered, implies that $[\gamma] \in Ker(\Phi)$. But since QD implies stably finite ([Br3, Prop. 3.19]) we have that if $[\gamma] \in Ext_{QD}(B, \mathcal{K} \otimes I)$ then $[\gamma] \in Ker(\Phi)$. □

The classic example for which $K_0(I)$ is totally ordered is the case when $I = \mathcal{K}$. In this setting the corollary above is very similar to a result of Salinas' which describes the closure of $0 \in Ext(B, \mathcal{K})$ in terms of quasidiagonality ([Sal, Thm. 2.9]). See also [Sal, Thm. 2.14] for another characterization of $Ext_{QD}(B, \mathcal{K})$ in terms of bi-quasitriangular operators. For a K-theoretical characterization of $Ext_{qd}(B, \mathcal{K})$ see [Sch, Theorem 8.3].

The class of NF algebras introduced in [BK] coincides with the class of separable QD nuclear C^*-algebras. It was conjectured in [BK, Conj. 7.1.6] that an asymptotically split extension of NF algebras is NF. We can verify the conjecture under an additional asumption.

Corollary 3.8. *Let $0 \to I \to E \to B \to 0$ be an asymptotically split extension with I and B NF algebras. If B satisfies the UCT, then E is NF.*

Proof. Both index maps are vanishing since the extension is asymptotically split. The conclusion follows from Theorem 3.4. □

§4. Extensions and K-theory

In this section we show that the general extension problem for nuclear QD C^*-algebras is equivalent to some natural K-theoretic questions.

We begin by recalling a result of Spielberg which solves the extension problem for stably finite C^*-algebras and shows that it is completely governed by K-theory.

Proposition 4.1. [Sp, Lemma 1.5] *Let $0 \to I \to E \to B \to 0$ be short exact where both I and B are stably finite. Then E is stably finite if and only if $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$, where $\partial : K_1(B) \to K_0(I)$ is the boundary map of the sequence.*

In [BK, Question 7.3.1], it is asked whether every nuclear stably finite C^*-algebra is QD. Support for an affirmative answer to this question is provided by a number of nontrivial examples ([Pi], [Sp], [Br1], [Br2]). In fact, Corollary 3.7 above also provides examples since the proof shows the equivalence of quasidiagonality and stable finiteness (in fact we did not even assume nuclearity of E in that corollary). Hence it is natural to wonder if Spielberg's criterion completely determines quasidiagonality in extensions as well. The following result gives some more evidence for an affirmative answer. If I is a C^*-algebra, let $SI = C_0(\mathbb{R}) \otimes I$ denote the suspension of I. Note that $K_0(SI)^+ = \{0\}$ since $SI \otimes \mathcal{K}$ contains no nonzero projections.

Proposition 4.2. *Let $0 \to SI \to E \to B \to 0$ be exact, where I is σ-unital and B is separable, QD, nuclear. Then E is QD.*

Proof. The suspension SI of I is QD by [Vo1]. We may assume that I is stable. Let $\alpha : SI \hookrightarrow SI$ be a null-homotopic approximately unital embedding and let $\widehat{\alpha} : Q(SI) \hookrightarrow Q(SI)$ be the corresponding $*$-monomorphism. Then for any Busby invariant $\gamma : B \to M(SI)$, $[\widehat{\alpha} \circ \gamma] = 0 \in Ext(B, SI)$ by the homotopy invariance of $Ext(B, SI)$ in the second variable [Kas]. It follows that $E(\gamma) \hookrightarrow E(\widehat{\alpha} \circ \gamma)$ is QD by Proposition 2.5. $\square$

Definition 4.3. Say that a QD C^*-algebra A has the *QD extension property* if for every separable, nuclear, QD algebra B which satisfies the UCT and Busby invariant $\gamma : B \to Q(\mathcal{K} \otimes A)$ we have that $E(\gamma)$ is QD if and only if $E(\gamma)$ is stably finite (which is if and only if $\partial(K_1(B)) \cap K_0^+(\mathcal{K} \otimes A) = \{0\}$, by Proposition 4.1).

The QD extension property is closely related to a certain embedding property for the K-theory of A which we now describe. The interest in controlling the K-theory of embeddings of C^*-algebras goes back to the seminal work of Pimsner and Voiculescu on AF embeddings of irrational rotation algebras ([PV]). Since then other authors have studied the K-theory of (AF) embeddings ([Lo], [EL], [DL], [Br1], [Br1]).

Definition 4.4. Say that a QD C^*-algebra A has the *K_0-embedding property* if for every subgroup $G \subset K_0(A)$ such that $G \cap K_0^+(A) = \{0\}$ there exists an embedding $\rho : A \hookrightarrow C$, where C is also QD, such that $\rho_*(G) = 0$.

It is not hard to see that if C is a stably finite C^*-algebra and $p \in C$ is a nonzero projection then $[p]$ must be a nonzero element of $K_0(C)$. From this remark it follows that the condition $G \cap K_0^+(A) = \{0\}$ is necessary. Hence the K_0-embedding property states that this condition is also sufficient.

A number of QD C^*-algebras have the K_0-embedding property. For example, commutative C^*-algebras, AF algebras ([Sp, Lem. 1.14]), crossed products of AF algebras by $\mathbb{Z}$ ([Br1, Thm. 5.5]) and simple nuclear unital C^*-algebras with unique trace.

Our next goal is to connect the QD extension and K_0-embedding properties. But we first need a simple lemma.

Lemma 4.5. *Let C be a hereditary subalgebra of a unital C^*-algebra D. If C has an approximate unit consisting of projections and $K_0(D)$ has cancellation then the inclusion $C \hookrightarrow D$ induces an injective map $K_0(C) \hookrightarrow K_0(D)$.*

Proof. By *cancellation* we mean that if $p, q \in M_n(D)$ are projections with $[p] = [q]$ in $K_0(D)$ then there exists a partial isometry $v \in M_n(D)$ such that $vv^* = p$ and $v^*v = q$.

Let $x = [p] - [q] \in K_0(C)$ be an element such that $x = 0 \in K_0(D)$. Since C has an approximate unit of projections, say $\{e_\lambda\}$, we may assume that p and q are projections in $(e_\lambda \otimes 1)C \otimes M_n(\mathbb{C})(e_\lambda \otimes 1)$ for sufficiently large n and λ. Since $[p] = [q]$ in $K_0(D)$ and this group has cancellation we can find a partial isometry $v \in M_n(D)$ such that $vv^* = p$ and $v^*v = q$.

We claim that actually $v \in M_n(C)$ (which will evidently prove the lemma). To see this we first note that $v = vv^*(v)v^*v = pvq$ and hence

$$v = pvq = (e_\lambda \otimes 1)pvq(e_\lambda \otimes 1) = (e_\lambda \otimes 1)v(e_\lambda \otimes 1).$$

Hence $v \in (e_\lambda \otimes 1)D \otimes M_n(\mathbb{C})(e_\lambda \otimes 1)$. But since C is hereditary in D, $C \otimes M_n(\mathbb{C})$ is hereditary in $D \otimes M_n(\mathbb{C})$ and thus

$$v \in (e_\lambda \otimes 1)D \otimes M_n(\mathbb{C})(e_\lambda \otimes 1) \subset C \otimes M_n(\mathbb{C}).$$

$\square$

Proposition 4.6. *Let A be a separable QD C^*-algebra. Then A satisfies the QD extension property if and only if A satisfies the K_0-embedding property.*

Proof. We begin with the easy direction. Assume that A has the QD extension property and let $G \subset K_0(A)$ be a subgroup such that $G \cap K_0^+(A) = \{0\}$. Since abelian C^*-algebras satisfy the UCT we can construct an extension

$$0 \to \mathcal{K} \otimes A \to E \to \oplus_{\mathbb{N}} C(\mathbb{T}) \to 0,$$

such that $\partial(K_1(\oplus_{\mathbb{N}} C(\mathbb{T}))) = \partial(\oplus_{\mathbb{N}}\mathbb{Z}) = G$. But since A has the QD extension property E must be a QD C^*-algebra. Thus the six-term K-theory exact sequence implies that A has the K_0-embedding property (i.e. the embedding into E will work).

Conversely, assume that A has the K_0-embedding property and let

$$0 \to \mathcal{K} \otimes A \to E \to B \to 0$$

be a short exact sequence where B is separable, nuclear, QD, satisfies the UCT and E is stably finite.

Let $G = \partial(K_1(B)) \subset K(\mathcal{K} \otimes A) \cong K_0(A)$. Since E is stably finite, $G \cap K_0^+(A) = \{0\}$. By the K_0-embedding property we can find a QD C^*-algebra C and an embedding $\rho : A \hookrightarrow C$ such that $\rho_*(G) = 0$. Since A is separable we may assume that C is also separable. Indeed $K_0(A)$ (and hence G) is countable. Thus it only takes a countable number of projections and partial isometries in matrices over C to kill off $\rho_*(G)$. From this observation it is easy to see that we may assume that C is also separable.

Let $\pi : C \hookrightarrow \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be an embedding (the existence of which is ensured by the separability of C) as in the proof of Proposition 3.3. Let $J \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be the hereditary subalgebra generated by $\pi \circ \rho(A)$. Since $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ has real rank zero and stable rank one it follows from Lemma 4.5 that the inclusion $J \hookrightarrow \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ induces an injective map $K_0(J) \hookrightarrow K_0(\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H))$. Since G is in the kernel of the K-theory map induced by the embedding $\pi \circ \rho : A \to \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ it follows that G is also in the kernel of the K-theory map induced by the embedding $\pi \circ \rho : A \to J$. But the embedding into J is approximately unital by construction and so we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} \otimes J & \longrightarrow & E(\eta) & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where η is the induced Busby invariant and the two vertical maps on the left are injective.

Now we are done since naturality of the boundary map implies that the homomorphism $\partial : K_1(B) \to K_0(\mathcal{K} \otimes J)$ is zero and hence $E(\eta)$ is QD by Theorem 3.4. □

We now wish to point out a connection between extensions of QD C^*-algebras and another very natural K-theoretic question. For brevity, we say a linear map $\varphi : A \to B$ is *ccp* if it is contractive and completely positive ([Pa]). We recall a theorem of Voiculescu.

Theorem 4.7. [Vo1, Thm. 1] *Let A be a separable C^*-algebra. Then A is QD if and only if there exists an asymptotically multiplicative, asymptotically isometric sequence of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ for* *some sequence of natural numbers k_n (i.e. $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ and $\|\varphi_n(a)\| \to \|a\|$ for all $a, b \in A$).*

Given this abstract characterization of QD C^*-algebras it is natural to ask how well these approximating maps capture the relevant K-theoretic data.

Definition 4.8. Say that a QD C^*-algebra A has the *K_0-Hahn-Banach property* if for each $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, where $\mathbb{Z}x = \{kx : k \in \mathbb{Z}\}$, there exists a sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) = 0$ for all n large enough.

It is easy to see that if $y \in K_0(A)$ and there exists a nonzero integer k such that $ky \in K_0^+(A)$ then *for every* asymptotically multiplicative, asymptotically isometric sequence of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have $(\varphi_n)_*(y) > 0$ (if $k > 0$) or $(\varphi_n)_*(y) < 0$ (if $k < 0$), for all sufficiently large n. Hence this K_0-Hahn-Banach property states that one can separate elements $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ from (finite subsets of) the positive cone using finite dimensional approximate morphisms.

Another way of thinking about this property is that A has the K_0-Hahn-Banach property if and only if finite dimensional approximate morphisms determine the order on $K_0(A)$ to a large extent. A more precise formulation is contained in the next proposition (not needed for the rest of the paper).

Proposition 4.9. *The K_0-Hahn-Banach property is equivalent to the following property: If $x \in K_0(A)$ and for every sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have that $(\varphi_n)_*(x) > 0$ for all large n then there exists a positive integer k such that $kx \in K_0^+(A)$.*

Proof. We first show that the (contrapositive of the) second property above follows from the K_0-Hahn-Banach property. So assume we are given an element $x \in K_0(A)$ and assume that there is *no* positive integer k such that $kx \in K_0^+(A)$. We must exhibit a sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) \leq 0$ for all sufficiently large n. There are two cases.

If there exists a negative integer k such that $kx \in K_0^+(A)$ then for every sequence $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have $(\varphi_n)_*(x) < 0$ for all sufficiently large n (see the discussion following definition 4.7). The second case is if $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. This case is obviously handled by the K_0-Hahn-Banach property.

Now we show how the second property above implies the K_0-Hahn-Banach property. So let $x \in K_0(A)$ be such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. Since no positive multiple of x is in $K_0^+(A)$ the second property implies that we can find some sequence $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) \leq 0$ for all sufficiently large n. Similarly, since no positive multiple of $-x$ is in $K_0^+(A)$ we can find a sequence $\psi_n : A \to M_{j_n}(\mathbb{C})$ such that $(\psi_n)_*(x) \geq 0$ for all sufficiently large n. If either of $\{\varphi_n\}$ or $\{\psi_n\}$ contains a subsequence with equality at 0 then we are done so we assume that $(\varphi_n)_*(x) = -s_n < 0$ and $(\psi_n)_*(x) = t_n > 0$ for all (sufficiently large) n. It is now clear what to do: we simply add up appropriate numbers of copies of φ_n and ψ_n so that these positive and negative ranks cancel. More precisely we define maps

$$\Phi_n = (\bigoplus_1^{t_n} \varphi_n) \oplus (\bigoplus_1^{s_n} \psi_n)$$

and regard these maps as taking values in the $(t_n k_n + s_n j_n) \times (t_n k_n + s_n j_n)$ matrices. □

Proposition 4.10. *If a separable QD C^*-algebra A has the QD extension property or, equivalently, the K_0-embedding property then A also has the K_0-Hahn-Banach property.*

Proof. Assume that A has the K_0-embedding property and we are given $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, where $\mathbb{Z}x = \{kx : k \in \mathbb{Z}\}$. By the K_0-embedding property we can find an embedding $\rho : A \hookrightarrow C$, where C is QD and $\rho_*(x) = 0$. As in the proof of Proposition 4.6 we may assume that C is also separable. But then take any asymptotically multiplicative, asymptotically isometric sequence of contractive completely positive maps $\varphi_n : C \to M_{k_n}(\mathbb{C})$ and we get that $(\varphi_n \circ \rho)_*(x) = 0$ for all sufficiently large n. □

We do not know if the converse of the previous proposition holds. However our final result will complete the circle for the class of nuclear C^*-algebras. Moreover, the next theorem also states that in order to prove that every separable, nuclear, QD C^*-algebra has any of the properties we have been studying, it actually suffices to consider very special cases of either the QD extension problem or K_0-embedding problem.

Theorem 4.11. *The following statements are equivalent.*

1. *Every separable, nuclear, QD C^*-algebra has the QD extension property.*
2. *Every separable, nuclear, QD C^*-algebra has the K_0-embedding property.*

3. *Every separable, nuclear, QD C^*-algebra satisfies the K_0-Hahn-Banach property.*
4. *If A is any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is QD (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*
5. *If A is any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

Proof. The proof of Proposition 4.6 carries over verbatim to show the equivalence of 1 and 2. That proof also shows the equivalence of 4 and 5. The previous proposition shows that 2 implies 3 and hence we are left to show that 3 implies 5 and 4 implies 2.

We begin with the easier implication 4 $\Longrightarrow$ 2. So, let A be any separable, nuclear, QD C^*-algebra and $G \subset K_0(A)$ be a subgroup such that $G \cap K_0^+(A) = \{0\}$. As in the proof of Proposition 4.6 we can construct a short exact sequence

$$0 \to \mathcal{K} \otimes A \to E \to \bigoplus_1^{\infty} C(\mathbb{T}) \to 0,$$

such that $\partial(K_1(\oplus_{\mathbb{N}} C(\mathbb{T}))) = \partial(\oplus_{\mathbb{N}}\mathbb{Z}) = G$. We will prove that E is QD and, by exactness of $\oplus_{\mathbb{N}}\mathbb{Z} \xrightarrow{\partial} K_0(A) \to K_1(E)$, this will show 2.

For each n there is a short exact sequence

$$0 \to \mathcal{K} \otimes A \to E_n \to \bigoplus_1^{n} C(\mathbb{T}) \to 0,$$

where each $E_n \subset E$ is an ideal and $E = \overline{\cup_n E_n}$. Note also that each E_n is nuclear since extensions of nuclear algebras are again nuclear. Since a locally QD algebra is actually QD it suffices to show that each E_n is QD. Since E_1 is stably finite (being a subalgebra of E) we have that the boundary map $\partial : K_1(C(\mathbb{T})) \to K_0(E_1)$ takes no positive values. But then the proof of Proposition 4.6 shows that if we assume 4 then E_1 will be QD. Proceeding by induction we may assume that E_{n-1} is QD. Since E_n is also stably finite, E_{n-1} is an ideal in E_n and $E_n/E_{n-1} = C(\mathbb{T})$, applying the same argument to the exact sequence $0 \to E_{n-1} \to E_n \to C(\mathbb{T}) \to 0$ we see that E_n is also QD.

We now show that 3 $\Longrightarrow$ 5, which will complete the proof. So let A be any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ be such that

$\mathbb{Z}x \cap K_0^+(A) = \{0\}$. Construct a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ such that $\partial(1) = x$. We will show that E must be QD.

We can use the K_0-Hahn-Banach property to construct an embedding $\rho : \mathcal{K} \otimes A \to Q(\oplus_i M_{n_i}(\mathbb{C}))$ such that $\rho_*(x) = 0$. Let $D \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ be the hereditary subalgebra generated by $\rho(\mathcal{K} \otimes A)$. Let $\pi : C(\mathbb{T}) \to B(H)$ be any faithful unital representation such that $\pi(C(\mathbb{T})) \cap \mathcal{K}(H) = \{0\}$. We first claim that there is an embedding of E into $(\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$, where $\tilde{D}$ is the unitization of D. Indeed, since the embedding $\rho : \mathcal{K} \otimes A \to D$ is approximately unital we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & E & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & F & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0, \end{array}$$

for some algebra F and the map $E \to F$ is injective. Since $\rho_*(x) = 0 \in K_0(D)$ (by Lemma 4.5) and $K_1(D) = 0$ (by the proof of Lemma 3.2) it follows that both boundary maps arising from the sequence $0 \to D \to F \to C(\mathbb{T}) \to 0$ are zero. Hence we may appeal to the UCT, add on a trivial absorbing extension and eventually find an embedding of F into $\pi(C(\mathbb{T})) \otimes 1 + \mathcal{K}(H) \otimes D \subset (\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$.

Since E is nuclear it now suffices to show that every nuclear subalgebra of $(\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$ is QD. Hence, by [Br3, Prop. 8.3] and the Choi-Effros lifting theorem ([CE]) it suffices to show that there exists a short exact sequence

$$0 \to J \to C \to (\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D} \to 0,$$

where C is QD and J contains an approximate unit consisting of projections which is quasicentral in C (i.e. the extension is quasidiagonal). However, this is now trivial since $D \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ implies that there is a quasidiagonal extension

$$0 \to \oplus_i M_{n_i}(\mathbb{C}) \to R \to \tilde{D} \to 0,$$

where $R \subset \Pi_i M_{n_i}(\mathbb{C})$. But since $X = \pi(C(\mathbb{T})) + \mathcal{K}(H)$ is nuclear the sequence

$$0 \to (\oplus_i M_{n_i}(\mathbb{C})) \otimes X \to R \otimes X \to \tilde{D} \otimes X \to 0$$

is exact and since X is unital the extension is also quasidiagonal. $\square$

Though Theorem 4.11 is stated for the class of nuclear QD C^*-algebras a close inspection of the proof shows that this assumption was only used in the proof of 4 $\Longrightarrow$ 2. Hence we also have the following result which applies to individual nuclear C^*-algebras.

Theorem 4.12. *Let A be a separable nuclear QD C^*-algebra and consider the following statements.*

1. *A has the QD extension property.*
2. *A has the K_0-embedding property.*
3. *A has the K_0-Hahn-Banach property.*
4. *If $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is QD (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*
5. *If $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

Then 1 $\Longleftrightarrow$ 2 $\Longrightarrow$ 3 $\Longleftrightarrow$ 4 $\Longleftrightarrow$ 5.

Remark. There is another version of Theorem 4.11 where the class of nuclear C^*-algebras is replaced by a class $\mathcal{A}$ of separable C^*-algebras with the following closure property. If $0 \to A \otimes \mathcal{K} \to E \to B \to 0$ is exact with $A \in \mathcal{A}$ and B separable abelian, then $E \in \mathcal{A}$. For instance $\mathcal{A}$ can be the class of all separable C^*-algebras or the class of all separable exact C^*-algebras. Then the statements 1-5 of Theorem 4.11 formulated for the class $\mathcal{A}$ (rather then for the class of nuclear C^*-algebras) are related as follows: 1 $\Longleftrightarrow$ 2 $\Longleftrightarrow$ 4 $\Longleftrightarrow$ 5 $\Longrightarrow$ 3.

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