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Semiprojectivity in Simple C*-Algebras

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Abstract.

We show that certain purely infinite simple C*-algebras, including the Cuntz algebra O_∞, are semiprojective. Some related results and conjectures are discussed, and some crossed product examples constructed.

§1. Introduction

The notions of projectivity and semiprojectivity were introduced in the development of shape theory for C*-algebras ([EK86], [Bla85]) as noncommutative analogs of the topological notions of absolute retract (AR) and absolute neighborhood retract (ANR) respectively. Semiprojective C*-algebras have rigidity properties which make them conceptually and technically important in several aspects of C*-algebra theory; this is reflected especially in the work of Loring and his coauthors (see, for example, [Lor97].) It is not too easy for a C*-algebra to be semiprojective, but there does seem to be a reasonable supply of such algebras.

Most known semiprojective C*-algebras are far from simple. (Indeed, a projective C*-algebra must be contractible, so cannot be simple.) In fact, the only known simple semiprojective C*-algebras have been the finite-dimensional matrix algebras and the (simple) Cuntz-Krieger algebras [Bla85]. In this paper, we will give a few more examples of simple semiprojective C*-algebras (and more are given in [Szy]), but also obtain some structure results which show that the class of infinite-dimensional semiprojective simple C*-algebras may not be too much larger than the class of Cuntz-Krieger algebras (in fact, it might consist exactly of the separable purely infinite simple nuclear C*-algebras with finitely generated K-theory.)

The work of this paper was largely inspired by the remarkable recent classification theorem of Kirchberg, also in part proved independently

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by Phillips ([Kir00], [KP00a], [KP00b], [Phi00].) The theorem asserts that the following class of C*-algebras is classified up to isomorphism by K-theory:

Definition 1.1. A separable, nuclear, simple, unital, purely infinite C*-algebra in the bootstrap class for the Universal Coefficient Theorem ([RS87], [Bla98, §23]) is called a *Kirchberg algebra.*

It was (and is) hoped that the notion of semiprojectivity, and results such as those of this paper, will lead to a simplification and clarification of the proof of this theorem. Although this hope has yet to be fully realized, there are obvious close connections between semiprojectivity and some of the ingredients of the proof; see 2.15.

Our main results are:

(1.) The Cuntz algebra O_∞ is semiprojective (3.2).
(2.) If A is simple, semiprojective, and properly infinite, then $A \otimes \mathbb{K}$ is also semiprojective (4.1).
(3.) If A is a semiprojective Kirchberg algebra, then $K_*(A)$ is finitely generated (2.11).
(4.) The class of semiprojective (simple) C*-algebras is not closed under crossed products by finite groups, even $\mathbb{Z}_2$ (6.3).

§2. Semiprojective C*-Algebras

We recall the definition of a semiprojective C*-algebra, which first appeared in this form in [Bla85] (a somewhat different, less restrictive, definition previously appeared in [EK86].)

Definition 2.1. A separable C*-algebra A is *semiprojective* if, for any C*-algebra B, increasing sequence $\langle J_n \rangle$ of (closed two-sided) ideals of B, with $J = [\cup J_n]^-$, and *-homomorphism $\phi : A \to B/J$, there is an n and a *-homomorphism $\psi : A \to B/J_n$ such that $\phi = \pi \circ \psi$, where $\pi : B/J_n \to B/J$ is the natural quotient map.

A ϕ for which such a ψ exists is said to be *partially liftable.* If there is a $\psi : A \to B$ with $\phi = \pi \circ \psi$, then ϕ is *liftable*; if every homomorphism from A is liftable, A is said to be *projective.*

Note that for convenience, we have only defined semiprojectivity for separable C*-algebras (although the same definition makes sense also for nonseparable C*-algebras, it is probably not the appropriate one.) Thus in this paper all semiprojective C*-algebras will implicitly be separable. In the definition, B is not required to be separable; however:

Proposition 2.2. *The definition of semiprojectivity does not change if in 2.1 we make any or all of the following restrictions:*

(i) *B is separable.*
(ii) *ϕ is surjective.*
(iii) *ϕ is injective.*

Proof. B can clearly be replaced by the C*-subalgebra D generated by any preimage of a dense set in $\phi(A)$, proving (i) and (ii). (One technical point: $\cup_n(D \cap J_n)$ is dense in $D \cap J$, an easy consequence of the uniqueness of norm on a C*-algebra.) To prove (iii), replace B by $A \oplus B$, J_n by $0 \oplus J_n$, J by $0 \oplus J$, and ϕ by $id_A \oplus \phi$. Q.E.D.

For the convenience of the reader, we recall some standard facts about semiprojective C*-algebras which we will need to use.

Proposition 2.3. [Bla85, 2.18] *Let B, J_n, and J be as in 2.1, and let $q_1, \ldots, q_k$ be mutually orthogonal projections in B/J. Then for sufficiently large n, there are mutually orthogonal projections $p_1, \ldots, p_k$ in B/J_n with $\pi(p_j) = q_j$ for all j. If B (and hence B/J) is unital and $q_1 + \cdots + q_k = 1$, then we may choose the p_j so that $p_1 + \cdots + p_k = 1$.*

Corollary 2.4. [Bla85, 2.16] *If A is unital, then the definition of semiprojectivity for A does not change if in 2.1 B and ϕ are required to be unital. In particular, $\mathbb{C}$ is semiprojective.*

Note that $\mathbb{C}$ is not projective (in the category of general C*-algebras and *-homomorphisms): a *-homomorphism from $\mathbb{C}$ to B/J is effectively just a choice of projection in B/J, and projections do not lift from quotients in general.

Proposition 2.5. [Bla85, 2.23] *Let B, J_n, J be as in 2.1. Let v be a partial isometry in B/J, and set $q_1 = v^*v$, $q_2 = vv^*$. Suppose there are projections $p_1, p_2 \in B/J_n$ for some n with $\pi(p_j) = q_j$. Then, after increasing n if necessary, there is a partial isometry $u \in B/J_n$ with $\pi(u) = v$ and $p_1 = u^*u$, $p_2 = uu^*$.*

Proposition 2.6. *(*[Bla85, 2.19], [Lor97]*) A finite direct sum of semiprojective C*-algebras is semiprojective.*

Proposition 2.7. *(*[Bla85, 2.28-2.29], [Lor97]*) If A is semiprojective, then $M_n(A)$ is semiprojective for all n. If A is semiprojective, then any unital C*-algebra strongly Morita equivalent to A is also semiprojective.*

The unital cases of 2.6 and 2.7 are simple consequences of 2.3 and 2.5, but the nonunital cases are more delicate.

Examples 2.8. Simple repeated applications of 2.3-2.7 show that the following C*-algebras are semiprojective:

(i) $\mathbb{M}_n = M_n(\mathbb{C})$, and more generally any finite-dimensional C*-algebra.
(ii) $C(\mathbb{T})$, where $\mathbb{T}$ is a circle (the universal C*-algebra generated by one unitary.)
(iii) Generalizing (ii), $C^*(\mathbb{F}_n)$, the full C*-algebra of the free group on n generators for n *finite* (the universal C*-algebra generated by n unitaries.)
(iv) The Toeplitz algebra $\mathcal{T}$ (the universal C*-algebra generated by an isometry.)
(v) The Cuntz-Krieger algebras O_A for a finite square $0-1$ matrix A [CK80], and in particular the Cuntz algebras O_n $(n \neq \infty)$.
(vi) Any C*-algebra which is the universal C*-algebra generated by a finite number of partial isometries, where the only relations (finitely many) are order and orthogonality relations among the source and range projections of the partial isometries; this includes all the above examples.

Some potential or actual non-examples are:

(vii) $C^*(\mathbb{F}_\infty)$, the universal C*-algebra generated by a sequence of unitaries. The problem is that, in the setting of 2.1 with B and ϕ unital, the n might have to be increased each time an additional generator is partially lifted. In fact, $C^*(\mathbb{F}_\infty)$ violates the conclusion of 2.10 (and obviously satisfies the hypothesis), so is not semiprojective.
(viii) The Cuntz algebra O_∞, the universal C*-algebra generated by a sequence of isometries with mutually orthogonal range projections, has the same potential difficulty as $C^*(\mathbb{F}_\infty)$. However, it turns out that O_∞ *is* semiprojective (3.2). (Note that $K_*(O_\infty)$ is finitely generated.)
(ix) $C(\mathbb{T}^n)$ for $n \geq 2$ is the universal C*-algebra generated by n commuting unitaries. Commutation relations are difficult to lift in general, and it can be shown that $C(\mathbb{T}^n)$ $(n \geq 2)$ fails to satisfy the conclusion of 2.9 and is thus not semiprojective.

We recall the following important approximate factorization property for semiprojective C*-algebras:

Proposition 2.9. [Bla85, 3.1] *Let A be a semiprojective C^*-algebra, and $(B_n, \beta_{m,n})$ be an inductive system of C^*-algebras with $B = \lim_\to (B_n, \beta_{m,n})$. If $\phi : A \to B$ is a homomorphism, then for all sufficiently large n there are homomorphisms $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ is homotopic* *to ϕ and converges pointwise to ϕ as $n \to \infty$, where β_n is the standard map from B_n to B.*

2.9 almost implies that a semiprojective C*-algebra has finitely generated K-theory:

Corollary 2.10. *Let A be a semiprojective C^*-algebra. If A can be written as an inductive limit of C^*-algebras with finitely generated K-theory, then A itself has finitely generated K-theory.*

Corollary 2.11. *If A is a semiprojective Kirchberg algebra, then $K_*(A)$ is finitely generated.*

Proof. If A is a Kirchberg algebra, then by the results of [Kir00] A can be written as an inductive limit of (Kirchberg) algebras with finitely generated K-theory, since $K_*(A)$ can be written as an inductive limit of finitely generated groups and every map on K-theory can be implemented by an algebra homomorphism between the corresponding Kirchberg algebras. Q.E.D.

The pointwise approximation part of 2.9 also applies to inductive limits in the generalized sense of [BK97] (it is unclear how an analog of the homotopy result might be phrased.) This generalization follows from the next fact about continuous fields, using [BK97, 2.2.4].

Proposition 2.12. *Let A be a semiprojective C^*-algebra, $\langle B(t)\rangle$ a continuous field of C^*-algebras over a locally compact Hausdorff space X, and t_0 a point of X with a countable neighborhood base. If ϕ is a homomorphism from A to $B(t_0)$, then there is a compact neighborhood Z of t_0 in X and a homomorphism ψ from A to the continuous field C^*-algebra defined by $\{B(t) : t \in Z\}$ such that $\phi = \pi_{t_0} \circ \psi$. In particular, if $x \in A$ with $\phi(x) \neq 0$, then for each t in some neighborhood of t_0 there is a homomorphism $\phi_t : A \to B(t)$ with $\phi_t(x) \neq 0$.*

Proof. Let (U_n) be a sequence of open sets in X with $Z_n = \bar{U}_n$ compact and contained in U_{n-1} for all n, and $\cap U_n = \{t_0\}$. Let B be the continuous field algebra defined by $\{B(t) : t \in X\}$, J_n the ideal of sections vanishing on Z_n, and J the sections vanishing at t_0. Apply 2.1. Q.E.D.

Corollary 2.13. *Let A be a semiprojective C^*-algebra, and $(B_n, \beta_{m,n})$ be a generalized inductive system of C^*-algebras* [BK97] *with $B = \lim_{\to}(B_n, \beta_{m,n})$. If $\phi : A \to B$ is a homomorphism, then for all sufficiently large n there are homomorphisms $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ converges pointwise to ϕ as $n \to \infty$, where β_n is the standard map from B_n to B.*

Corollary 2.14. *Let A be a semiprojective MF algebra* [BK97]. *Then A is residually finite-dimensional (has a separating family of finite-dimensional representations). If A is simple, then A is a finite-dimensional matrix algebra.*

Proof. Apply 2.12 and [BK97, 3.2.2(v)]. Q.E.D.

Another consequence of 2.12 is that every asymptotic morphism from a semiprojective C*-algebra to any other C*-algebra (in the sense of Connes-Higson E-theory ([CH90], [Bla98, §25])) can be realized up to homotopy by an actual homomorphism. This has potentially important consequences in the classification of purely infinite simple C*-algebras.

Corollary 2.15. [Bla98, 25.1.7] *Let A and B be separable C*-algebras, with A semiprojective. Then the canonical map from the set $[A, B]$ of homotopy classes of homomorphisms into the set $[[A, B]]$ of homotopy classes of asymptotic homomorphisms is a bijection.*

§3. Examples of Semiprojective Simple C*-Algebras

In this section, we show that certain purely infinite simple nuclear C*-algebras such as the Cuntz algebra O_∞ are semiprojective.

The main technical fact used in the proofs of this section and those of section 4 is the following sharpening of a well-known lifting property for unitaries (cf. [Bla98, 3.4.5].) If A is a C*-algebra, we write $A^\dagger$ for its unitization.

Proposition 3.1. *Let B be a C*-algebra, J a (closed 2-sided) ideal of B, and $\pi : B \to B/J$ the quotient map. Let q be a projection in B/J and v a unitary in $(B/J)^\dagger$ such that*

(1) $qv = vq = q$

(2) $(1-q)v = (1-q)v(1-q)$ *is in the connected component of the identity in* $\mathcal{U}((1-q)(B/J)^\dagger(1-q)$.

If there is a projection p in B with $\pi(p) = q$, then there is a unitary u in $B^\dagger$ with $\pi(u) = v$ and $pu = up = p$.

Proof. π maps $(1-p)B(1-p)$ onto $(1-q)(B/J)(1-q)$, so by [Bla98, 3.4.5] there is a unitary w in $(1-p)B^\dagger(1-p)$ with $\pi(w) = (1-q)v$. Set $u = p + w$. Q.E.D.

Theorem 3.2. *O_∞ is semiprojective.*

Proof. Let $\{s_1, s_2, \dots\}$ be the standard generators of O_∞, i.e. the s_j are isometries with mutually orthogonal ranges. Let B, J_n, J, and ϕ be as in 2.1. By 2.2 and 2.4 we may assume B is unital and ϕ is an isomorphism, and identify O_∞ with B/J. Using 2.3 and 2.5, we may partially lift any finite number of the s_j to isometries with mutually orthogonal ranges in B/J_n, for some n; the difficulty is that *a priori* we might have to increase n each time we partially lift another generator. But by using the next lemma inductively on k (with $A = O_\infty$ and $p_0 = q_0 = 0$), once we partially lift the first two generators we can lift all the rest without further increasing the n. Note that at each step we correct the provisional lift of the last of the previous generators, but do not change the lifts of the earlier ones. Q.E.D.

Lemma 3.3. *Let A be a unital C^*-algebra, q_0 a projection in A, and $\{s_1, s_2, \dots\}$ a sequence of isometries in A whose range projections are mutually orthogonal and all orthogonal to q_0. Let D be a unital C^*-algebra, and $\pi : D \to A$ a surjective homomorphism, and let $k \geq 2$. Suppose p_0 is a projection in D and $r_1, \dots, r_{k-1}, t_k$ are isometries in D whose range projections are mutually orthogonal and all orthogonal to p_0, with $\pi(p_0) = q_0$, $\pi(r_j) = s_j$ for $1 \leq j \leq k-1$, and $\pi(t_k) = s_k$. Then there are isometries r_k and t_{k+1} in D, such that the ranges of $r_1, \dots, r_k, t_{k+1}$ are mutually orthogonal and orthogonal to p_0, and $\pi(r_k) = s_k$, $\pi(t_{k+1}) = s_{k+1}$.*

Proof. We may assume A is generated by q_0 and $\{s_n\}$. Then A is isomorphic either to O_∞ (if $q_0 = 0$) or to a split essential extension of O_∞ by $\mathbb{K}$ (if $q_0 \neq 0$.) In either case, the unitary group of A, or any corner in A, is connected: this follows from [Cun81] for O_∞, and if u is a unitary in the extension, let v be the image of $\pi(u^*) \in O_\infty$ under a cross section; then v is in the connected component of 1, and so is vu since it is a unitary in $\mathbb{K}^\dagger$.

Set $p = p_0 + \sum_{j=1}^{k-1} r_j r_j^*$ and $q = q_0 + \sum_{j=1}^{k-1} s_j s_j^*$; then p and q are projections, and $\pi(p) = q$. In the copy of O_∞ in A generated by $\{s_1, s_2, \dots\}$, the range projections of the isometries $s_k s_1$ and s_k^2 are orthogonal to each other and to q, and are equivalent to $s_k s_k^*$ and $s_{k+1} s_{k+1}^*$ via partial isometries $v_1 = s_k s_1^* s_k^*$ and $v_2 = s_{k+1} s_k^{*2}$ respectively. Also, the projections $1 - q - s_k s_1 s_1^* s_k^* - s_k^2 s_k^{*2}$ and $= 1 - q - s_k s_k^* - s_{k+1} s_{k+1}^*$ are equivalent via a partial isometry v_3, since these projections are nonzero and have the same K_0-class. Set $v = q + v_1 + v_2 + v_3$. Then v is a unitary in O_∞, $qv = vq = q$, and $v s_k s_1 = s_k$, $v s_k^2 = s_{k+1}$. Also, the unitary group of $(1-q)A(1-q)$ is connected; thus by 3.1 there is a unitary u in D with $\pi(u) = v$ and $pu = up = p$. Set $r_k = u r_1 t_k$ and $t_{k+1} = u t_k^2$. Q.E.D.

We next consider a non-simple example, which will be used to obtain a generalization of 3.2.

Proposition 3.4. *Let $\mathcal{T}$ be the Toeplitz algebra, the universal C*-algebra generated by a single isometry s. Let ω be a primitive n'th root of unity, and let α be the automorphism of $\mathcal{T}$ which sends s to ωs. Then $\mathcal{T} \times_\alpha \mathbb{Z}_n$ is semiprojective.*

Proof. $A = \mathcal{T} \times_\alpha \mathbb{Z}_n$ is the universal unital C*-algebra generated by $\{s, v\}$, with relations $\{s^*s = 1, v^n = v^*v = 1, v^*sv = \omega s\}$. Let B, J_n, J be as in 2.1; as usual, assume B is unital and ϕ is an isomorphism, and identify A with B/J. We can partially lift v to a unitary $u \in B/J_m$ for some m. If $x \in B/J_m$ is a preimage of s, then $y = n^{-1}\sum_{k=1}^n \omega^k u^{-k} s u^k$ is a preimage for s with $u^*yu = \omega y$. y^*y commutes with u, and since $\pi(y^*y) = 1$ we may assume y^*y is close to 1 and therefore invertible, by increasing m if necessary. Then $t = y(y^*y)^{-1/2}$ is an isometry, $\pi(t) = s$, and $u^*tu = \omega t$, so $\{t, u\}$ generate the partial lift of A. Q.E.D.

Theorem 3.5. *Let ω be a primitive n'th root of unity, and let α be the automorphism of O_∞ such that $\alpha(s_1) = \omega s_1$ and $\alpha(s_k) = s_k$ for all $k > 1$. Let $A = O_\infty \times_\alpha \mathbb{Z}_n$. Then*

(i) *A is the (unique) Kirchberg algebra with $K_0(A) = \mathbb{Z}^n$ (with $[1] = (1, 0, \cdots, 0)$) and $K_1(A) = 0$.*

(ii) *A is semiprojective.*

Proof. (i): This can be proved directly using arguments very similar to those in [CE81]. A more elegant approach, though, is to write $O_\infty \otimes \mathbb{K}$ as a graph C*-algebra as in [Kum98, 2.3(h)]; then A is the graph C*-algebra of the skew product graph [KP99], and hence is purely infinite (and simple) by [KPR98, 3.9] and in the UCT bootstrap class by [KP99, 2.6]. The K-theory can be calculated as in [PR96].
(ii): Let w be the unitary in A or order n implementing α. Let B, J_n, J, and ϕ be as in 2.1, with B unital and ϕ an isomorphism (2.2, 2.4.) Identify A with B/J. By 3.4 we can partially lift s_1 and w to an isometry r_1 and a unitary z of order n in B/J_m for some m, with $z^*r_1z = \omega r_1$. Let D be the commutant of z in B/J_m. Then the image of D in A contains s_k for all $k > 1$, since if $x \in B/J_m$ with $\pi(x) = s_k$, then $y = n^{-1}\sum_{j=1}^n z^{-j} s_k z^j \in D$ satisfies $\pi(y) = s_k$. Also, $r_1r_1^* \in D$. By increasing m if necessary, we can find isometries $r_2, r_3 \in D$ with range projections orthogonal to $r_1r_1^*$ and with $\pi(r_j) = s_j$ for $j = 2, 3$, by 2.3 and 2.5. Now we can, by inductively using 3.3 with $q_0 = s_1s_1^*$ and $p_0 = r_1r_1^*$, find a lift $r_k \in D$ for s_k for each k, such that the range projections are all mutually orthogonal and also orthogonal to $r_1r_1^*$. Q.E.D.

This example has been generalized in [Szy] to include all Kirchberg algebras A where $K_0(A)$ is finitely generated, $K_1(A)$ is finitely generated and torsion-free, and $rank(K_1(A)) \leq rank(K_0(A))$. [See note added in proof.]

The results of this section, [Szy], 2.8, and 2.11 suggest the following conjecture:

Conjecture 3.6. A Kirchberg algebra is semiprojective if and only if its K-theory is finitely generated.

Note that if A is a Kirchberg algebra, $K_0(A)$ is finitely generated, and $K_1(A)$ is isomorphic to the torsion-free part of $K_0(A)$, then A is stably isomorphic to a Cuntz-Krieger algebra (and conversely) [Rør95], and therefore semiprojective. Thus the most important test algebras for this conjecture, besides the examples of this section, include:

> $O_n \otimes O_n$ (the Kirchberg algebra B with $K_0(B) \cong = K_1(B) \cong \mathbb{Z}_{n-1}$)
> P_∞ (the Kirchberg algebra B with $K_0(B) = 0$ and $K_1(B) = \mathbb{Z}$.)

[See note added in proof.]

The difficulty in proving that $O_n \otimes O_n$ is semiprojective is that the two copies of O_n must be partially lifted so that the lifts exactly commute. One can come frustratingly close to proving that this can be done: for example, inside O_n is a copy of O_∞ containing the first $n-1$ generators of O_n, and the subalgebra $O_\infty \otimes O_n$ can be partially lifted since it is isomorphic to O_n and is therefore semiprojective.

It appears that the results and techniques of [DE] can be used to show that a Kirchberg algebra with finitely generated K-theory is weakly semiprojective in the sense of [Lor97] (I am indebted to M. Dadarlat for this observation.) The best approach to the conjecture might be to solve the following problem (if it has a positive solution.)

Problem 3.7. If B is a Kirchberg algebra with finitely generated K-theory, find a finite presentation for B as in [Bla85], preferably with stable (partially liftable) relations.

The only Kirchberg algebras for which such a presentation is known are the (simple) Cuntz-Krieger algebras O_A and their matrix algebras. Finite tensor products of these, such as $O_n \otimes O_n$, and certain crossed products by finite groups, also have obvious finite presentations, but the relations include ones such as commutation relations, which are not (obviously) stable. No finite presentation for O_∞ or P_∞ is known (B. Neubüser has obtained a non-finite presentation of P_∞ as a graph C*-algebra.)

§4. Stable Semiprojective C*-Algebras and Hereditary Subalgebras

In this section, we examine conditions related to when a stable C*-algebra is semiprojective. Semiprojectivity in stable C*-algebras is fairly exceptional.

Recall that a unital C*-algebra A is *properly infinite* if A contains two isometries with orthogonal range projections; A then contains a unital copy of O_∞. A simple unital C*-algebra which is infinite (contains a nonunitary isometry) is automatically properly infinite ([Cun81], [Bla98, 6.11.3]). The main result of this section is:

Theorem 4.1. *Let A be a semiprojective properly infinite unital C*-algebra. Then its stable algebra $A \otimes \mathbb{K}$ is also semiprojective.*

Proof. The proof is quite similar in spirit to the proof of 3.2, based on the fact that inside A is a nicely embedded copy of $A \otimes \mathbb{K}$. Let $\{s_j\}$ be a sequence of isometries in A with mutually orthogonal ranges, and set $f_{ij} = s_i s_j^*$. Then $\{f_{ij}\}$ is a set of matrix units in A. Let $\{e_{ij}\}$ be the standard matrix units in $\mathbb{K}$.

Let B, J_n, and J be as in 2.1, with $\phi : A \otimes \mathbb{K} \to B/J$ an isomorphism. Fix an m such that there is a projection $h_{11} \in B/J_m$ with $\pi(h_{11}) = 1 \otimes e_{11}$ and such that ϕ lifts to a unital $\psi : A \otimes e_{11} \cong A \to h_{11}(B/J_m)h_{11}$, using 2.3 and semiprojectivity of A. It suffices to show that the matrix units $\{1 \otimes e_{ij}\}$ lift to matrix units $\{h_{ij}\}$ B/J_m including the chosen h_{11}.

For each i, j, let $r_j = \psi(s_j)$, and $g_{ij} = r_i r_j^* = \psi(f_{ij})$. We now inductively choose unitaries v_k and u_k, and projections h_{kk}, as follows. Let v_1 be a unitary in the connected component of the identity of $\mathcal{U}((A \otimes \mathbb{K})^\dagger)$ with $v_1^* f_{11} v_1 = 1 \otimes e_{11}$ [Bla98, 4.3.1, 4.4.1], and let $u_1 \in \mathcal{U}((B/J_m)^\dagger)$ be a lift of v_1. Increasing m if necessary, we may choose u_1 so that $u_1^* g_{11} u_1 = h_{11}$ (2.5).

If projections $h_{11}, \ldots, h_{kk} \in B/J_m$ and unitaries $v_1, \ldots, v_k \in (A \otimes \mathbb{K})^\dagger$ have been defined, with lifts $u_1, \ldots, u_k \in (B/J_m)^\dagger$, set $q = \sum_{j=1}^k 1 \otimes e_{jj}$ and $p = \sum_{j=1}^k h_{jj}$. We have that

$$v_k^* \cdots v_2^* v_1^* f_{k+1,k+1} v_1 v_2 \cdots v_k$$

is orthogonal to q. Let v_{k+1} be a unitary in the connected component of the identity in $\mathcal{U}((A \otimes \mathbb{K})^\dagger)$ with $v_{k+1} q = q v_{k+1} = q$ and

$$v_{k+1}^* v_k^* \cdots v_1^* f_{k+1,k+1} v_1 \cdots v_k v_{k+1} = 1 \otimes e_{k+1,k+1}$$

and let $u_{k+1} \in \mathcal{U}((B/J_m)^\dagger)$ be a lift of v_{k+1} with $u_{k+1}p = pu_{k+1} = p$ (3.1). Then

$$h_{k+1,k+1} = u_{k+1}^* u_k^* \cdots u_1^* g_{k+1,k+1} u_1 \cdots = u_k u_{k+1}$$

is a lift of $1 \otimes e_{k+1,k+1}$ to a projection in B_m orthogonal to $h_{11}, \ldots, h_{kk}$.

Now for each k let $w_k = g_{kk} u_k u_{k-1} \cdots u_1$. Then w_k is a partial isometry in B/J_m with $w_k^* w_k = h_{kk}$ and $w_k w_k^* = g_{kk}$. Set $z_k = w_k^* g_{k1} w_1$; then z_k is a partial isometry with $z_k^* z_k = h_{11}$ and $z_k z_k^* = h_{kk}$. We have that $(1 \otimes e_{1k})\pi(z_k)$ is a unitary y_k in $A \otimes e_{11} \cong A$, and if $h_{k1} = z_k \psi(y_k)^*$, then h_{k1} is a partial isometry in B/J_m from h_{11} to h_{kk} which is a lift of $1 \otimes e_{k1}$. For each i, j, set $h_{ij} = h_{i1} h_{j1}^*$; then the $\{h_{ij}\}$ are the desired lifts of $\{1 \otimes e_{ij}\}$. Q.E.D.

This result is false if A is stably finite: for example, $\mathbb{K} \cong \mathbb{C} \otimes \mathbb{K}$ is not semiprojective, as is easily seen from 2.9 (or 2.14). In fact, a partial converse to 4.1 (a full converse, at least stably, in the simple unital case) is a special case of the next result.

Proposition 4.2. *Let A be a nonunital semiprojective C^*-algebra. If A has an approximate identity of projections, then A contains an infinite projection.*

Proof. This follows easily from 2.9. Let $\{p_n\}$ be a strictly increasing approximate unit of projections in A. Then $A \cong \lim_{\to} p_n A p_n$, and so the identity map on A is homotopic to a homomorphism through $p_n A p_n$ for some n. In particular, p_{n+1} is homotopic, hence equivalent, to a subprojection of p_n. Q.E.D.

Corollary 4.3. *Let A be a (separable) unital C^*-algebra. If $A \otimes \mathbb{K}$ is semiprojective, then A is not stably finite.*

This result is probably not the best possible; in fact, $\mathcal{T} \otimes \mathbb{K}$ is not semiprojective (if it were, a nontrivial homomorphism from $\mathcal{T} \otimes \mathbb{K}$ to $M_n(\mathcal{T})$, and hence to $\mathbb{M}_n$, could be constructed for some n by 2.9), and the full converse (stably) of 4.1 may well hold.

Note that if A is unital and $A \otimes \mathbb{K}$ is semiprojective, then A is semiprojective (2.7). This should be true even if A is nonunital. In fact, the following conjecture seems likely:

Conjecture 4.4. Let A be a semiprojective C*-algebra. Then any full corner in A is also semiprojective.

To prove this, it would suffice (and be essentially equivalent) to prove:

Conjecture 4.5. Let $0 \to A \to B \to \mathbb{C} \to 0$ be a split exact sequence of separable C*-algebras. If A is semiprojective, then so is B.

It is plausible, but less clear, that a full hereditary C*-subalgebra of a semiprojective C*-algebra should be semiprojective. Fullness is essential: $\mathcal{T}$ has a hereditary C*-subalgebra (closed ideal) isomorphic to $\mathbb{K}$.

If it is true that full hereditary C*-subalgebras of semiprojective C*-algebras are semiprojective, then a stably finite semiprojective simple C*-algebra must be nearly projectionless: by 4.2, such a C*-algebra could not contain a strictly increasing or decreasing sequence of projections.

It is quite conceivable that there could be semiprojective simple C*-algebras which are projectionless. However, 2.14 strongly suggests that there cannot be such examples which are nuclear. Some evidence is described in the next section to suggest that semiprojective simple C*-algebras are at least exact, if not nuclear; thus there is some modest evidence for a positive answer to the following question:

Question 4.6. Is every semiprojective simple C*-algebra either a finite-dimensional matrix algebra, or a Kirchberg algebra or stabilized Kirchberg algebra with finitely generated K-theory?

Recall that every hereditary C*-subalgebra of a purely infinite simple C*-algebra is either unital or stable [Zha90].

§5. Exactness of Semiprojective Simple C*-Algebras

It is quite possible that every simple semiprojective C*-algebra is C*-exact. Recall that a C*-algebra A is (C*-)exact if forming the minimal tensor products by A preserves exact sequences, and that a C*-subalgebra of an exact C*-algebra is exact [Kir94]. If $\mathbb{F}_2$ is the free group on two generators, then $C^*(\mathbb{F}_2)$ is not exact [Was76], hence cannot be embedded in an exact C*-algebra.

Conjecture 5.1. Every semiprojective simple C*-algebra embeds in O_2 and is therefore exact.

Note that by [KP00b] every separable exact C*-algebra embeds into O_2, so the first conclusion follows from the second. However, the likely proof of 5.1 would show exactness by directly embedding A into O_2, using the following conjecture of Kirchberg, for which there seems to be good evidence:

Conjecture 5.2. Every separable C*-algebra embeds in the corona algebra

$$\left(\prod O_2\right) / \left(\bigoplus O_2\right),$$

the quotient of the bounded sequences in O_2 by the sequences converging to 0.

This conjecture can be slightly modified by replacing the corona algebra with an ultrapower of O_2. In fact, it seems likely that $\mathcal{B}(\mathcal{H})$ for separable $\mathcal{H}$ embeds in these corona algebras or ultraproducts.

To prove 5.1 from 5.2, let $B = \prod O_2$, and J_n the sequences in B which are 0 after the n'th term. Then $[\cup J_n]^- = \oplus O_2$, and $B/J_n \cong \prod O_2$. Let ϕ be an embedding of A into the corona algebra, and partially lift ϕ to an embedding of A into $\prod O_2$. Since A is simple, composing this embedding with a suitable coordinate projection gives an embedding of A into O_2.

Note that a nonsimple semiprojective C*-algebra, e.g. $C^*(\mathbb{F}_2)$, need not be C*-exact. This proof of 5.1 would show that any semiprojective C*-algebra embeds into $\prod O_2$. A separable simple C*-algebra which is not exact cannot embed into $\prod O_2$, although it would embed in the corona algebra if 5.2 is true; such a C*-algebra can be constructed by embedding $C^*(\mathbb{F}_2)$ into the hyperfinite II_1 factor and applying [Bla78, 2.2].

§6. Finite Group Actions

One might hope (or expect) from 2.7 that a crossed product of a semiprojective C*-algebra by a finite group, or more generally a subalgebra of finite Jones index in a semiprojective C*-algebra, would be semiprojective, since these operations are the "square root" of a Morita equivalence. However, we will give examples here of $\mathbb{Z}_2$-actions on a semiprojective C*-algebra such that the crossed product is not semiprojective.

In fact, we show that there are $\mathbb{Z}_2$-actions on O_2 such that the crossed product, which is a Kirchberg algebra, does not have finitely generated K-theory. It is a bit surprising that the crossed product can even have nontrivial K-theory, since O_2 is K-contractible and thus K-theoretically trivial. The first example of a symmetry of O_2 such that the crossed product has nontrivial K-theory appeared in [CE81]. This gives yet another indication that $\mathbb{Z}_2$-actions can be badly behaved from a K-theoretic point of view; cf. [Bla98, 10.7], [Bla90], [Ell95]. It may even be true that the bootstrap class for the Universal Coefficient Theorem is not closed under crossed products by $\mathbb{Z}_2$ (in fact, this appears to be equivalent to the question of whether every separable nuclear C*-algebra is in the bootstrap class).

We will use the next example, which is a special case of [Bla90, 6.3.3].

Proposition 6.1. *Let B be the UHF algebra with $K_0(B) \cong \mathbb{Q}$. Then there is a symmetry σ of B such that, if $D = B \times_\sigma \mathbb{Z}_2$, then $K_0(D) \cong \mathbb{Q}$ and $K_1(D)$ is the dyadic rationals $\mathbb{D}$.*

Theorem 6.2. *Let G_0 and G_1 be countable abelian torsion groups in which every element has odd order. Then there is a symmetry α of O_2 such that $K_n(O_2 \times_\alpha \mathbb{Z}_2) \cong G_n$ for $n = 0, 1$.*

Proof. By [ER95] there is a Kirchberg algebra A such that $K_0(A) \cong G_1$ and $K_1(A) \cong G_0$. If B is the UHF algebra of 6.1, then $A \otimes B$ has trivial K-theory by the Künneth theorem for tensor products ([Sch82], [Bla98, 23.1.3]), and hence is isomorphic to O_2 by [ER95]. Let α be the symmetry $id \otimes \sigma$ of $O_2 \cong A \otimes B$. Then $O_2 \times_\alpha \mathbb{Z}_2$ is isomorphic to $A \otimes D$ (6.1), so $K_0(O_2 \times_\alpha \mathbb{Z}_2) \cong G_0$ and $K_1(O_2 \times_\alpha \mathbb{Z}_2) \cong G_1$ by the Künneth Theorem. Q.E.D.

Corollary 6.3. *There is a simple semiprojective C*-algebra A and a symmetry α of A such that $A \times_\alpha \mathbb{Z}_2$ is not semiprojective.*

Proof. In 6.2, choose G_0 or G_1 to be not finitely generated (i.e. not finite.) Apply 2.10. Q.E.D.

It would be interesting to study and classify symmetries of O_2 (and, more generally, finite group actions on Kirchberg algebras, or subalgebras of finite index in Kirchberg algebras). It is worth noting that 6.2 exhausts the possibilities for the K-theory of the crossed product of O_2 by a symmetry if these K-groups are torsion groups:

Proposition 6.4. *Let α be a symmetry of O_2. Then the K-groups of $O_2 \times_\alpha \mathbb{Z}_2$ are 2-divisible countable abelian groups with no 2-torsion.*

Proof. The proof is very similar to the arguments in [Bla90, 2.2.1]. The groups are countable because $O_2 \times_\alpha \mathbb{Z}_2$ is separable. For the rest, first note that $K_*(O_2 \times_\alpha \mathbb{Z})$ is trivial by the Pimsner-Voiculescu exact sequence [Bla98, 10.2.1]. It then follows from the exact sequence of [Bla98, 10.7.1] that $1 - \hat{\alpha}_* : K_*(O_2 \times_\alpha \mathbb{Z}_2) \to K_*(O_2 \times_\alpha \mathbb{Z}_2)$ is an isomorphism. Then from $0 = 1 - \hat{\alpha}_*^2 = (1 + \hat{\alpha}_*)(1 - \hat{\alpha}_*)$ it follows that $1 + \hat{\alpha}_* = 0$, $\hat{\alpha}_* = -1$, so $1 - \hat{\alpha}_*$ is multiplication by 2. Q.E.D.

Question 6.5. Is the K-theory of the crossed product by $\mathbb{Z}_2$ a complete outer conjugacy or stable conjugacy invariant for symmetries of O_2?

Example 6.6. The K-theory, or even the isomorphism class, of the crossed product is not a complete conjugacy invariant for symmetries of O_2. If α is the symmetry of O_2 which sends each generator to its negative, then the fixed-point algebra is isomorphic to O_4; but if β is the stabilized version of α, i.e. $\beta = ad\,diag(1, -1) \otimes \alpha$ on $\mathbb{M}_2 \otimes O_2 \cong O_2$, then the fixed-point algebra of β is isomorphic to $M_3(O_4)$, so α and β are not conjugate. If $\gamma = id \otimes \alpha$ on $\mathbb{M}_3 \otimes O_2 \cong O_2$, then the fixed-point algebra of γ is also $M_3(O_4)$. We do not know if β and γ are conjugate or outer conjugate. The crossed products of O_2 by each of these symmetries is isomorphic to $M_3(O_4)$.

Note Added in Proof.

J. Spielberg [Spi] has recently shown that Conjecture 3.6 is true if $K_1(A)$ is torsion-free. This includes the test case of P_∞, as well as the examples of 3.5 and [Szy]. The question remains open for such Kirchberg algebras as $O_n \otimes O_n$.

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