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Contact Weyl Manifold over a Symplectic Manifold

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Abstract.

We give a brief review on Weyl manifolds and thier Poincaré-Cartan classes. A Weyl manifold is a Weyl algebra bundle over a symplectic manifold which is a geometrization of deformation quantization and the Poincaré-Cartan class is a complete invariant of Weyl manifolds.

We introduce a concept of a contact Weyl manifold, which is a contact algebra bundle over a symplectic manifold containing a Weyl manifold as a subbundle. We show the existence of contact Weyl manifolds for a symplectic manifold.

We construct a connection on a contact Weyl manifold which gives a Fedosov connection when it is restricted to a Weyl manifold. With the help of the connection, we show that the cohomology class given by the curvature of Fedosov connection coincides with the Poincaré-Cartan class.

§1. Introduction

A contact manifold is embedded into a symplectic manifold compatible with the symplectic structure (cf. the concept of symplectification and contactification in [AG], §2). However, as to the converse direction, we have some obstruction as follows. A linear symplectic manifold $(\mathbf{R}^{2n}, \sigma_0)$ is embedded into a canonical contact manifold $(\mathbf{R}^{2n+1}, \theta_0)$ compatible with the contact structure $d\theta_0 = \sigma_0$, but in general an arbitrary symplectic manifold is not necessarily embedded into a contact manifold in a compatible way. For such an embedding, we need at least a vanishing cohomology class of the symplectic structure.

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In this paper, we show in quantum world the embedding is always possible, namely, a quantized symplectic manifold is embedded into a quantized contact manifold. Here a quantized symplectic manifold means a Weyl manifold defined by Omori-Maeda-Yoshioka ([OMY1]), which is a Weyl algebra bundle over a symplectic manifold, and a quantized contact manifold means a contact Weyl manifold, a contact algebra bundle over a symplectic manifold containing the Weyl manifold as a subbundle.

The purpose of this paper is two fold. First we give a brief review on Weyl manifold and its Poincaré-Cartan class, which are given in [OMY1] and in Omori-Maeda-Miyazaki-Yoshioka [OMMY1], respectively.

The second purpose is to give a concept of contact Weyl manifold, and to show the existence of contact Weyl manifold. We also construct a connection on a contact Weyl manifold, which is an extension of Fedosov connection. Using this connection, we show the Poincaré-Cartan class is equal to the cohomology class of the curvature of Fedosov connection.

The Weyl algebra W is the space of all formal power series of elements $\nu, Z^1, \ldots, Z^{2n}$ with coefficients in $\mathbf{R}$ having the Moyal type product $\widehat{*}$ (see equation (2.3) below). The algebra W is equipped with the formal power series topology under which $(W, \widehat{*})$ is a complete topological algebra.

In a word, a Weyl manifold W_M is a locally trivial fiber bundle over a symplectic manifold (M, σ) with fibers consisting of the Weyl algebra W and the transition functions of local trivializations are given by Weyl diffeomorphisms (see Definition 2.9). Locally trivial bundles can be regarded as quantized Darboux charts and Weyl diffeomorphisms can be regarded as quantized symplectomorhpisms. In this way, a Weyl manifold W_M is considered as a quantization of symplectic manifold.

It is shown in Theorem 6.1 in [OMY1], a star product is made from sections of a Weyl manifold and conversely a Weyl manifold is constructed by a star product. In this sense a Weyl manifold W_M is also viewed as a geometrization of deformation quantization. It is also proved that every symplectic manifold has Weyl manifolds over itself, which yields the existence of deformation quantization for a symplectic manifold (see, Theorem A and Theorem B in [OMY1]).

For constructing W_M, one needs to handle the center of W. In order to extract information of the center, the contact algebra is introduced in [OMY1]. The *contact algebra* C is a Lie algebra given as the direct sum $C = \mathbf{R}\tau \oplus W$, where τ is an element such that $[\tau, \nu] = 2\nu^2$, $[\tau, Z^i] = \nu Z^i$. In [OMMY1], by means of the contact algebra, it is also shown that the equivalence classes of the bundle W_M have a bijection to the set of all formal power series in ν^2 with coefficients in $H^2(M)$ of

the form

$$c = [\sigma] + \nu^2 c_2 + \cdots + \nu^{2k} c_{2k} + \cdots \in H^2(M)[[\nu^2]].$$

The element $c(W_M) \in H^2(M)[[\nu^2]]$ corresponding to W_M is called a *Poincaré-Cartan class* of W_M.

In this paper, we will establish the following. Using a Čech 2-cocycle giving the class $c(W_M)$ we extend the transition functions of W_M to gluing maps of locally trivial contact algebra bundles and construct a contact algebra bundle C_M over M with fiber C in §3. The bundle C_M contains W_M as a subbundle and will be called a *contact Weyl manifold*, and then we regard C_M as a contactification of a Weyl manifold W_M. Thus we have

Theorem A. *For every symplectic manifold there exists a contact Weyl manifold C_M.*

This theorem means that in quantum world contactification is always possible for a symplectic manifold.

On the other hand, we can take a closed 2-form $\Omega_M \in \Lambda^2(M)[[\nu^2]]$ such that $[\Omega_M] = c(W_M) \in H^2(M)[[\nu^2]]$ according to the deRham theorem. We will construct a connection on C_M having a curvature form Ω_M in §3.2. We show the connection gives a Fedosov connection if it is restricted to a subbundle W_M, which indicates that Ω_M is equal to the curvature of Fedosov connection.

Theorem B. *On C_M there is a connection ∂ whose curvature form is Ω_M. When restricted to a Weyl manifold W_M, the connection ∂ gives a Fedosov connection.*

Then, using this connection we prove that a Poincaré-Cartan class $c(W_M)$ coincides with the class given by the curvature form of Fedosov connection (cf. the conjecture in §Introduction of [OMMY1]).

Theorem C. *The deRham cohomomology class of the curvature Ω_M of Fedosov connection is equal to the Poincaré-Cartan class $c(W_M)$ of the Weyl manifold W_M.*

We remark that Theorems A, B and C are already given in Yoshioka [Y1]. Also in [Y2] we gave a proof of Theorem C. In this paper, we describe contructions in more detail. Especially, we improve the description of the transformation from Weyl charts to classical charts of [Y2] explicitly, which shows the restriction of the connection ∂ to W_M is equal to the Fedosov connection in §4.

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§2. Weyl manifold

A Weyl manifold emerges naturally from deformation quantization ([OMY1]) and it is considered as a quantized symplectic manifold. In this section, we give a review on Weyl manifolds.

2.1. Deformation quantization

Deformation quantization is proposed by Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer ([BFFLS]), which is an idea to quantize classical mechanical systems on a Poisson manifold without using operators in the following way.

2.1.1. Definition. Let $(M, \{\ ,\ \})$ be a Poisson manifold. Introduce a parameter ν and consider the space of formal power series $\mathfrak{a}_\nu(M) = C^\infty(M)[[\nu]]$ with coefficients in $\mathfrak{a}(M) = C^\infty(M)$, the space of all real valued smooth functions on M. Here we only consider real valued smooth functions for simplicity, although the argument is directly extened to complex valued functions. Let us consider a $\mathbf{R}[[\nu]]$-bilinear product $\mathfrak{a}_\nu(M) \times \mathfrak{a}_\nu(M) \to \mathfrak{a}_\nu(M)$.

Definition 2.1. A product $*$ is called a **star product** if

(i) For any $f, g \in \mathfrak{a}(M) = C^\infty(M)$, the product $f * g$ is expanded as

$$f * g = fg + \frac{\nu}{2}\{f, g\} + \cdots + \nu^k \pi_k(f, g) + \cdots \tag{2.1}$$

where fg is the pointwise multiplication of functions on M, $\{f, g\}$ is the Poisson bracket and $\pi_k \colon \mathfrak{a}(M) \times \mathfrak{a}(M) \to \mathfrak{a}(M)$ is a bidifferential operator $(k = 2, 3, \ldots)$,

(ii) $f * 1 = 1 * f = f$ for $\forall f \in \mathfrak{a}(M)$,

(iii) $*$ is associative.

For a star product $*$, the associative algebra $(\mathfrak{a}_\nu(M), *)$ is called a **deformation quantization** of the Poisson manifold $(M, \{\ ,\ \})$.

The existence of star products is proved by De Wilde-Lecomte for symplectic manifolds ([DL]) and for general Poisson manifolds by Kontsevich ([K]).

In this paper, we consider the case where M is a symplectic manifold with symplectic structure σ. A typical example is the Moyal product

on the canonical symplectic manifold $(\mathbf{R}^{2n}, \sigma_0)$. We write the canonical coordinates as $z = (z^1, \dots, z^{2n})$ and the canonical symplectic structure as $\sigma_0 = \frac{1}{2}\sum_{i,j} \omega_{ij} dz^i \wedge dz^j$ where ω_{ij} are the components of the constant $2n \times 2n$ matrix $\omega = (\omega_{ij}) = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$. The Poisson bracket is then written as

$$\{f, g\}_0 = \sum_{i,j} \Lambda^{ij} \partial_{z^i} f \partial_{z^j} g, \quad f,\ g \in \mathfrak{a}(\mathbf{R}^{2n}) = C^\infty(\mathbf{R}^{2n})$$

where $\Lambda = (\Lambda^{ij}) = -\omega^{-1}$. With the notation

$$\{f, g\}_0 = f\Big(\sum_{i,j} \Lambda^{ij} \overleftarrow{\partial_{z^i}}\, \overrightarrow{\partial_{z^j}}\Big) g = f \overleftarrow{\partial_z} \wedge \overrightarrow{\partial_z} g,$$

the Moyal product $*_0$ is a star product given by

$$\begin{aligned} (2.2) \qquad f *_0 g &= f\Big(\exp \frac{\nu}{2} \overleftarrow{\partial_z} \wedge \overrightarrow{\partial_z}\Big) g \\ &= fg + \frac{\nu}{2}\{f, g\}_0 + \cdots + \frac{1}{n!}\Big(\frac{\nu}{2}\Big)^n f\big(\overleftarrow{\partial_z} \wedge \overrightarrow{\partial_z}\big)^n g + \cdots . \end{aligned}$$

We sometimes refer to the deformation quantization $(\mathfrak{a}_\nu(\mathbf{R}^{2n}), *_0)$ as the *Moyal algebra.*

A star product can be restricted to an open subset $V \subset M$ and gives a star product on $\mathfrak{a}_\nu(V) = C^\infty(V)[[\nu]]$, since π_k in (2.1) are bidifferential operators. Due to the following theorem, star products on a symplectic manifold are locally isomorphic to the Moyal product (cf. Gutt [G], Lichnerowicz [L]).

Theorem 2.2. *Assume U be an open subset of $\mathbf{R}^{2n}$ with $H^2(U) = 0$. Then a star product $*$ on U is equivalent to $*_0$, that is, there exists an $\mathbf{R}[[\nu]]$-linear isomorphism $T\colon \mathfrak{a}_\nu(U) \to \mathfrak{a}_\nu(U)$ satisfying $f * g = T^{-1}(Tf *_0 Tg)$ for $\forall f,\ g \in \mathfrak{a}(U)$, where T is given in the form*

$$Tf = f + \nu T_1(f) + \cdots + \nu^k T_k(f) + \cdots, \quad \forall f \in \mathfrak{a}(U)$$

and $T_k\colon \mathfrak{a}(U) \to \mathfrak{a}(U)$, $(k = 1, 2, \dots)$ are differential operators. Thus, deformation quantizations $(\mathfrak{a}_\nu(U), *)$ and $(\mathfrak{a}_\nu(U), *_0)$ are isomorphic.

2.1.2. System of the Moyal algebras. By Theorem 2.2, a star product induces a system of local Moyal algebras and their isomorphisms as follows. Let $\{(V_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a symplectic atlas: $\bigcup_\alpha V_\alpha = M$ and each $\varphi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbf{R}^{2n}$ is a homeomorphism such that $\varphi_\alpha = (z^1_\alpha, z^2_\alpha, \dots, z^n_\alpha)$ is a canonical coordinate system of U_α and $\varphi_\alpha^* \sigma_{\alpha,0} = \sigma$ where $\sigma_{\alpha,0} = \frac{1}{2}\sum \omega_{ij} dz^i_\alpha \wedge dz^j_\alpha$ is the canonical

symplectic structure. Here one may assume $H^2(V_\alpha) = 0$ for every $\alpha \in A$. A star product $*$ on M is reduced to every V_α and produces a deformation quantization $(\mathfrak{a}_\nu(U_\alpha), *_\alpha)$ of a linear symplectic manifold $(U_\alpha, \sigma_{\alpha,0})$ by the local coordinate expression. Theorem 2.2 yields an isomorphism $T_\alpha\colon (\mathfrak{a}_\nu(U_\alpha), *_\alpha) \to (\mathfrak{a}_\nu(U_\alpha), *_0)$. Then we have a Moyal algebra isomorphism $T_{\alpha\beta} = T_\beta \circ T_\alpha^{-1}\colon (\mathfrak{a}_\nu(U_{\alpha\beta}), *_0) \to (\mathfrak{a}_\nu(U_{\beta\alpha}), *_0)$, where $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$. Thus, a star product $*$ on M gives a picture that a system of local Moyal algebras $\{(\mathfrak{a}_\nu(U_\alpha), *_0)\}_{\alpha\in A}$ is glued together by a system of algebra isomorphisms $\{T_{\alpha\beta}\colon (\mathfrak{a}_\nu(U_{\alpha\beta}), *_0) \to (\mathfrak{a}_\nu(U_{\beta\alpha}), *_0)\}$.

We can consider the local Moyal algebra $(\mathfrak{a}_\nu(U_{\alpha\beta}), *_0)$ as a quantized Darboux coordinate and the transformation $T_{\alpha\beta}\colon (\mathfrak{a}_\nu(U_{\alpha\beta}), *_0) \to (\mathfrak{a}_\nu(U_{\beta\alpha}), *_0)$ as the quantized symplectormorphism.

2.1.3. Motivation of Weyl manifold. Using the Weyl algebra W, we can attach to the system $\{((\mathfrak{a}_\nu(U_\alpha), *_0), T_{\alpha\beta})\}$ a geometric picture, that is, a bundle over M and its sections. Although the details will be given in the next sections, we see here an idea of Weyl manifold. We consider a locally trivial bundle $W_{U_\alpha} = U_\alpha \times W$ and consider the space of all smooth sections of W_{U_α}, which is denoted by $\Gamma(W_{U_\alpha})$. By the pointwise multiplication, $\Gamma(W_{U_\alpha})$ is an associative algebra. We will see that there exists a subalgebra $\mathcal{F}(W_{U_\alpha}) \subset \Gamma(W_{U_\alpha})$, whose elements are called local *Weyl functions*, isomorphic to the local Moyal algebra $(\mathfrak{a}_\nu(U_\alpha), *_0)$. With the identification of $(\mathfrak{a}_\nu(U_\alpha), *_0)$ and $\mathcal{F}(W_{U_\alpha})$, the isomorphism $T_{\alpha\beta}$ of local Moyal algebras naturally induces an algebra isomorphism $\widehat{T}_{\alpha\beta}\colon \mathcal{F}(W_{U_{\alpha\beta}}) \to \mathcal{F}(W_{U_{\beta\alpha}})$. This algebra isomorphism is given as the pullback map of certain algebra bundle isomorphism $\Phi_{\beta\alpha}\colon W_{U_{\beta\alpha}} \to W_{U_{\alpha\beta}}$, $\Phi_{\beta\alpha}^* = \widehat{T}_{\alpha\beta}$ (see for a proof, Lemma 3.2 in [OMY1]). Such a bundle isomorphism will be called a *Weyl diffeomorphism* (see §2.2.3). Then, a Weyl manifold W_M will be given as a bundle over M by gluing trivial bundles $\{W_{U_\alpha}\}$ with Weyl diffeomorphisms $\{\Phi_{\alpha\beta}\}$. Since the local Weyl function algebra $\mathcal{F}(W_{U_\alpha}) \cong (\mathfrak{a}_\nu(U_\alpha), *_0)$ can be regarded as a quantized Darboux chart and also the Weyl diffeomorphism $\Phi_{\alpha\beta}$ can regarded as a quantized symplectomorphism, the Weyl manifold W_M is considered as a quantized symplectic manifold, and also considered as a geometric picture of a star product on M.

2.2. Review of Weyl manifold

In this section, we give a brief review of Weyl manifold defined in [OMY1]. Let us consider a $2n$-dimensional symplectic manifold M with symplectic structure σ.

2.2.1. Weyl algebra. Introducing $2n+1$ elements ν, Z^1, $Z^2, \ldots, Z^{2n}$, we consider a formal power series with coefficients in $\mathbf{R}$, $a = \sum a_{l\alpha} \nu^l Z^\alpha$, where $a_{l\alpha} \in \mathbf{R}$, $l = 0, 1, 2, \ldots$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ is a multi-index.

We put the set of all formal power series

$$W = \Big\{ a = \sum a_{l\alpha} \nu^l Z^\alpha \;\Big|\; a_{l\alpha} \in \mathbf{R} \Big\}.$$

We introduce the formal power series topology in W and W is complete under this topology. Using a $2n \times 2n$ constant matrix $\Lambda = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$ where 1_n is the $n \times n$ identity matrix, we put a $\mathbf{R}[[\nu]]$-bilinear Poisson bracket in W

$$\{a, b\} = \sum_{ij} \Lambda^{ij} \partial_{Z^i} a \partial_{Z^j} b = a \overleftarrow{\partial}_Z \wedge \overrightarrow{\partial}_Z b.$$

Similarly as in (2.2), we consider a $\mathbf{R}[[\nu]]$-bilinear product $\hat{*}$ in W of Moyal type

$$a \hat{*} b = a\Big(\exp \frac{\nu}{2} \overleftarrow{\partial}_Z \wedge \overrightarrow{\partial}_Z\Big) b = ab + \frac{\nu}{2}\{a, b\} + \cdots . \tag{2.3}$$

The product $\hat{*}$ is continuous. We call the space W with $\hat{*}$ the *Weyl algebra*.

We can introduce an anti-involution $a \mapsto \overline{a}$ in W by putting

$$\overline{a \hat{*} b} = \overline{b} \hat{*} \overline{a}, \quad \overline{\nu} = -\nu, \quad \overline{Z^i} = Z^i, \quad \overline{a}_{l\alpha} = a_{l\alpha}. \tag{2.4}$$

It is easy to see the generators satisfy the canonical commutation relations (CCR for short)

$$\begin{cases} [\nu, Z^i] = \nu \hat{*} Z^i - Z^i \hat{*} \nu = 0 \\ [Z^i, Z^j] = Z^i \hat{*} Z^j - Z^j \hat{*} Z^i = \nu \Lambda^{ij}. \end{cases} \tag{2.5}$$

Remark 2.3. By (2.3), we have $Z^i Z^j = Z^i \hat{*} Z^j - \frac{\nu}{2}\Lambda^{ij}$, and inductively we see that every monomial $\nu^l Z^\alpha$ is written as a linear combination of the $\hat{*}$ products of the genrators ν, $Z^1, \ldots, Z^{2n}$. Thus, the Weyl algebra W is also described as an algebra over $\mathbf{R}$ formally generated by the elements ν, Z^1, $Z^2, \ldots, Z^{2n}$ satisfying the CCR relations (2.5) (see [OMY1], §1.1–1.2).

We introduce the degrees of monomials of W. We put the degrees for generators and monomials as

$$d(\nu) = 2, \quad d(Z^i) = 1, \ (i = 1, \ldots, 2n), \quad d(\nu^l Z^\alpha) = 2l + |\alpha|, \tag{2.6}$$

respectively. We set $W_0 = W$ and

$$W_k = \left\{ a \in W \;\middle|\; a = \sum_{2l+|\alpha| \geq k} a_{l\alpha} \nu^l Z^\alpha \right\}, \quad k = 1, \ 2, \ldots . \tag{2.7}$$

Using the conjugation, we can decompose W as a dirct sum of the set of hermitian, skewhermitian elements, respectively a

$$W = W^+ \oplus W^-,\ W^+ = \{a \in W \mid \overline{a} = a\},\ W^- = \{a \in W \mid \overline{a} = -a\}.$$

It is obvious the center is $\mathbf{R}[[\nu]]$, the set of all formal power seires in ν. We also put the set of noncentral elements

$$W^\circ = \left\{ a \in W \;\middle|\; a = \sum_{|\alpha|>0} a_{l\alpha} \nu^l Z^\alpha \right\}. \tag{2.8}$$

We set the intersections as

$$W_k^{\circ +} = W_k \cap W^\circ \cap W^+, \quad W_k^{\circ -} = W_k \cap W^\circ \cap W^-. \tag{2.9}$$

Definition 2.4. An $\mathbf{R}$-linear isomorphism $\Phi\colon W \to W$ satisfying

$$\text{(i) } \Phi(a \mathbin{\widehat{*}} b) = \Phi(a) \mathbin{\widehat{*}} \Phi(b), \quad \forall a,\ b \in W, \qquad \text{(ii) } \Phi(\nu) = \nu$$

is called a **ν-automorphism** of the Weyl algebra W.

A ν-automorphism Φ is **hermitian** if and only if

$$\overline{\Phi(a)} = \Phi(\overline{a}), \quad \forall a \in W. \tag{2.10}$$

We give two basic examples. Let $A = (a_{ij})$ be a $2n \times 2n$ symplectic matrix. We set $AZ^i = \sum_j a_{ij} Z^i$ and $A\nu = \nu$. Then it holds $[AZ^i, AZ^j] = A[Z^i, Z^j]$ and $[A\nu, AZ^i] = A[\nu, Z^i]$ and hence the matrix A naturally acts on W as a ν-automorphism. Also let us consider $F \in W_3$. Notice $[F, W] \subset \nu W_2$, and $[F, G]$ can be divided by ν for every $G \in W$. Then $\frac{1}{\nu} \operatorname{ad} F = \frac{1}{\nu}[F, \ ]$ gives a derivation of W. By exponentiating this derivation we have a ν-automrophism $\exp \frac{1}{\nu} \operatorname{ad} F$, which satisfies $\exp \frac{1}{\nu} \operatorname{ad} F(Z^i) = Z^i + O(2)$ where $O(2)$ means the collection of the terms belonging to W_2.

Now as to the structure of ν-automorphisms, we have

Proposition 2.5. *For a ν-automorphism $\Phi\colon W \to W$, there exist uniquely a $2n \times 2n$ symplectic matrix $A = (a_{ij})$ and $F \in W_3^\circ$ such that*

$$\Phi = A \circ \exp \frac{1}{\nu} \operatorname{ad} F.$$

If Φ has the hermitian property (2.10), *then $F \in W_3^{\circ +}$.*

Proof. Notice W_1 is the maximal ideal of W and then $\Phi(W_1) \subset W_1$. We put

$$\Phi(Z^i) = \sum_{j=1}^{2n} a_{ij} Z^j + O(2), \quad a_{ij} \in \mathbf{R}, \quad i = 1,\ 2, \ldots, 2n.$$

Form the identity $[\Phi(Z^i), \Phi(Z^j)] = \Phi([Z^i, Z^j])$, one sees $A = (a_{ij})$ is a $2n \times 2n$ symplectic matrix. Consider the ν-automorphism $A^{-1} \circ \Phi$ and apply to each Z^i. We have

$$A^{-1} \circ \Phi(Z^i) = Z^i + g^i_{(2)} + O(3), \quad i = 1, \ldots, 2n$$

where $g^i_{(2)}$ is the term of homogeneous degree 2. Then the identity $[A^{-1} \circ \Phi(Z^i), A^{-1} \circ \Phi(Z^j)] = A^{-1} \circ \Phi([Z^i, Z^j])$ gives the equation $[Z^i, g^j_{(2)}] = [Z^j, g^i_{(2)}]$, $i, j = 1, 2, \ldots, 2n$. The Poincaré lemma yields an element $F_{(3)}$ of homogenous degree 3 such that $\frac{1}{\nu}[Z^i, F_{(3)}] = g^i_{(2)}$, $i = 1, \ldots, 2n$. Thus we have

$$\Phi(Z^i) = A \circ \exp \frac{1}{\nu} \operatorname{ad} F_{(3)}(Z^i) + O(3), \quad i = 1, \ldots, 2n.$$

Repeating this process, we have a sequence $\{F_{(k)}\}$ of elements of homogenous degree k such that

$$\Phi(Z^i) = A \circ \exp \frac{1}{\nu} \operatorname{ad} F_{(3)} \circ \cdots \circ \exp \frac{1}{\nu} \operatorname{ad} F_{(k)}(Z^i) + O(k),$$

for $i = 1, \ldots, 2n$. By Campbell-Hausdorff formula there is an element $F \in W_3^\circ$ such that $\lim_{k \to \infty} \Pi_{l=3}^k \exp \frac{1}{\nu} \operatorname{ad} F_{(l)} = \exp \frac{1}{\nu} \operatorname{ad} F_3$ which completes the existence proof of A and F. The uniqueness is inductively checked by looking at the lawest degree term of $\exp \frac{1}{\nu} \operatorname{ad} F_{(k)}(Z^i) - Z^i$, $i = 1, 2, \ldots, 2n$. For the case Φ is hermitian, we obtain a proof by the similar manner. Q.E.D.

2.2.2. Weyl functions. Suppose $U \subset \mathbf{R}^{2n}$ be an open subset and let $W_U = U \times W$ be a trivial bundle over U. We set $\Gamma(W_U)$ the space of all smooth sections of W_U. By the pointwise multiplication, $\Gamma(W_U)$ is equipped with the associative product $\widehat{*}$. Under the smooth topology, $\Gamma(W_U)$ becomes a complete topological algebra. Consider a formal power series in ν with coefficients in $C^\infty(U)$

$$\widetilde{f} = f_0 + \nu f_1 + \cdots \in \mathfrak{a}_\nu(U) = C^\infty(U)[[\nu]].$$

Definition 2.6. We define a section $\widetilde{f}^\# \in \Gamma(W_U)$ in a Taylor expansion fashion

$$\widetilde{f}^\#(z) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_z^\alpha f(z) Z^\alpha, \quad (z \in U).$$

We call $\widetilde{f}^\#$ a **Weyl continuation** of $\widetilde{f}$.

We put the space $\mathcal{F}(W_U) = \{\widetilde{f}^{\#} \mid \widetilde{f} \in \mathfrak{a}_\nu(U)\}$ and we call an element of $\mathcal{F}(W_U)$ a **Weyl function**.

We have the following (see Theorem 2.6, [OMY1]).

Theorem 2.7.

(i) *The Weyl continuation gives an* $\mathbf{R}[[\nu]]$*-linear isomorphism* $\#\colon \mathfrak{a}_\nu(U) \to \mathcal{F}(W_U)$.

(ii) $(\widetilde{f} *_0 \widetilde{g})^{\#} = \widetilde{f}^{\#} \mathbin{\widehat{*}} \widetilde{g}^{\#}$ *for* $\forall \widetilde{f},\ \widetilde{g} \in \mathfrak{a}_\nu(U)$.

The above theorem indicates $\mathcal{F}(W_U)$ is a subalgebra of $\Gamma(W_U)$ and $\#$ is an algebra isomorphism between the local Moyal algebra $(\mathfrak{a}_\nu(U), *_0)$ and the algebra of local Weyl functions $(\mathcal{F}(W_U), \widehat{*})$. It is easy to see

Proposition 2.8. *Let* F *be a section of* $\Gamma(W_U)$ *satisfying*

$$[F, g^{\#}] \in \mathcal{F}(W_U) \quad \text{for} \quad \forall g^{\#}.$$

Then there exist a local Weyl function $f^{\#} \in \mathcal{F}(W_U)$ *and a formal power series* $a \in \mathfrak{a}_\nu(U)$ *such that*

$$F = f^{\#} + a.$$

If F *is a hermitian element,* $\overline{F} = F$*, then we can take as hermitian, i.e.,* $f = f(\nu^2)$, $a = a(\nu^2) \in \mathfrak{a}_{\nu^2}(U) = C^\infty(U)[[\nu^2]]$.

Proof. Set $g^{i\#} = \frac{1}{\nu}[F, z^{i\#}]$, $i = 1, \dots, 2n$. Then the Jacobi identity of the commutator togher with the relation $[z^{i\#}, z^{j\#}] = \nu\Lambda^{ij}$ yields $[z^{i\#}, g^{j\#}] = [z^{j\#}, g^{i\#}]$, which is equivalent to $\sum_l \Lambda^{il}\partial g^i/\partial z^l = \sum_l \Lambda^{jl}\partial g^j/\partial z^l$ for $i,\ j = 1, \dots, 2n$. Then, the Poincaré lemma shows there exists $f \in \mathfrak{a}_\nu(U)$ such that $g^i = \{f, z^i\}$. Hence $F - f^{\#} \in \Gamma(W_U)$ belongs to the center and we have $F = f^{\#} + a$ for certain $a \in \mathfrak{a}_\nu(U)$. If F is hermitian, it is obvious that we can take f and a as elements of $\mathfrak{a}_{\nu^2}(U)$. Q.E.D.

2.2.3. Weyl diffeomorphism. Consider a bundle isomorphism $\Phi\colon W_U \to W_{U'}$.

Definition 2.9. Φ is called a **Weyl diffeomorphism** if and only if it satisfies the following three conditions.

(i) $\Phi_z\colon W_z \to W_{\varphi(z)}$ is a ν-automorphism for every $z \in U$ where W_z is a fiber of W_U at z and $\varphi\colon U \to U'$ is the induced diffeomorphism.

(ii) The pullback map Φ^* satisfies $\Phi^*\mathcal{F}(W_{U'}) = \mathcal{F}(W_U)$.

(iii) Φ has the hermitian property $\overline{\Phi(a)} = \Phi(\overline{a})$, $\forall a \in W_U$.

As to the induced map, we have (cf. Lemma 3.3, [OMY1])

Lemma 2.10. *The induced map $\varphi\colon U \to U'$ of a Weyl diffeomorphism is a symplectic diffeomorphism.*

On the other hand, the converse direction also holds (see Theorem 3.7, [OMY1]).

Theorem 2.11. *For a symplectic diffeomorphism $\varphi\colon U \to U'$, there exists a Weyl diffeomorphism $\Phi\colon W_U \to W_{U'}$ whose induced diffeomorphism is φ.*

2.2.4. Definition of Weyl manifold. Now, gluing $\{W_U = U \times W\}$ with Weyl diffeomorphisms we can define a Weyl algebra bundle over M called a *Weyl manifold* in the following way.

Suppose we have a locally trivial bundle $W_M \to M$ with fibers isomorphic to the Weyl algebra. Let $\{(V_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be an atlas of M such that $\varphi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbf{R}^{2n}$ is a local canonical coordinate for every $\alpha \in A$. We denote by $\Phi_\alpha\colon W_{V_\alpha} \to W_{U_\alpha} = U_\alpha \times W$ a local bundle chart and by $\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ the overlap map, where $W_{V_\alpha = \pi^{-1}(V_\alpha)}$ and $W_{U_{\alpha\beta}} = \Phi_\alpha(V_\alpha \cap V_\beta) = U_{\alpha\beta} \times W$, $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$.

Definition 2.12. A locally trivial Weyl algebra bundle $W_M \xrightarrow{\pi} M$ is called a **Weyl manifold** if the overlap maps $\Phi_{\alpha\beta}$ are Weyl diffeomorphisms.

The sets $\{(W_{V_\alpha}, W_{U_\alpha}, \Phi_\alpha\colon W_{V_\alpha} \to W_{U_\alpha})\}_\alpha$ is called local Weyl charts.

We showed the existence of Weyl manifolds (Theorem A, [OMY1]).

Theorem 2.13. *For a symplectic manifold M, there exists a Weyl manifold W_M.*

The set of all smooth sections $\Gamma(W_M)$ becomes an algebra by the pointwise multiplication at each fiber. The overlap map $\Phi_{\alpha\beta}$ of a Weyl manifold preserves the class of local Weyl functions and then we can introduce a concept of global Weyl functions of W_M as follows.

Definition 2.14. A smooth section $F \in \Gamma(W_M)$ is called a **Weyl function** of W_M if $\Phi_\alpha^{*-1}F \in \mathcal{F}(W_{U_\alpha})$ for every $\alpha \in A$.

We denote by $\mathcal{F}(W_M)$ the set of all Weyl functions of W_M. It is easy to see $\mathcal{F}(W_M)$ forms a subalgebra of $\Gamma(W_M)$.

2.2.5. Weyl manifold and deformation quantization. As is seen in Theorem 2.7, the algebra of local Weyl functions $\mathcal{F}(W_{U_\alpha})$ is isomorphic to the local Moyal algebra $(\mathfrak{a}_\nu(U_\alpha), *_0)$. Also for globally defined

Weyl functions, we have an $\mathbf{R}[[\nu]]$-linear isomorphism with the following property (see, Theorems 3.10 and 6.1, [OMY1]).

Theorem 2.15. *For a Weyl manifold W_M, there exists an $\mathbf{R}[[\nu]]$-linear isomorphism $\rho\colon \mathfrak{a}_\nu(M) \to \mathcal{F}(W_M)$ such that*

$$\rho^{-1}(\rho(f)\,\hat{*}\,\rho(g)) = fg + \frac{\nu}{2}\{f,g\} + \cdots + \nu^k \pi_k(f,g) + \cdots,$$

for $f,\ g \in \mathfrak{a}_\nu(M)$ where $\{f,g\}$ is the Poisson bracket of the symplectic manifold M and π_k is a bidifferential operator on M.

If we define a product on $\mathfrak{a}_\nu(M)$ by

$$f * g = \rho^{-1}\big(\rho(f)\,\hat{*}\,\rho(g)\big), \quad f,\ g \in \mathfrak{a}_\nu(M), \tag{2.11}$$

then $*$ is obviously associative and becomes a star product, which induces an existence of star products on M (Theorem B, [OMY1]). Thus, by virtue of Theorem 2.13 and Theorem 2.15, we have obtained another proof of the existence of deformation quantization for symplectic manifold by De Wilde-Lecomte [DL].

2.3. Poincaré-Cartan class

In [OMMY1], we obtained a complete invariant $c(W_M)$ of Weyl manifolds over a symplectic manifold M, called the *Poincaré-Cartan class*. The invariant $c(W_M)$ of a Weyl manifold W_M is an element of $H^2(M)[[\nu^2]]$, the set of all formal power series in ν^2 with coefficients in $H^2(M)$. We derived a Čech 2-cocycle with values in the hermitian center $\mathbf{R}[[\nu^2]]$ through patching $\{W_{U_\alpha}\}$ and this 2-cocycle gave the Poincaré-Cartan class. Thus, we need to extract certain information from the center $\mathbf{R}[[\nu]]$ of W in order to define $c(W_M)$, and the contact algebra C was indeed introduced as a tool for this purpose. In this section, we recall the definition of the contact algebra C and give a review on the Poincaré-Cartan class.

2.3.1. Contact algebra. Let us introduce an element τ and set relations

$$[\tau,\nu] = 2\nu^2, \quad [\tau, Z^i] = \nu Z^i, \quad (i = 1,\ 2, \ldots, 2n). \tag{2.12}$$

It is easy to see

$$[\tau,[\nu,Z^i]] = [[\tau,\nu],Z^i] + [\nu,[\tau,Z^i]]$$

and

$$[\tau,[Z^i,Z^j]] = [[\tau,Z^i],Z^j] + [Z^i,[\tau,Z^j]].$$

Then the bracket $[\tau, \ ]$ is extended on the Weyl algebra as a derivation $a \mapsto [\tau, a]$, $a \in W$. We then consider a direct sum

$$C = \mathbf{R}\tau \oplus W \tag{2.13}$$

and define a bracket by

$$[\lambda_1\tau + a_1, \lambda_2\tau + a_2] = \lambda_1[\tau, a_2] - \lambda_2[\tau, a_1] + [a_1, a_2] \tag{2.14}$$

where $\lambda_i \in \mathbf{R}$, $a_i \in W$ and $[a_1, a_2] = a_1 \hat{*} a_2 - a_2 \hat{*} a_1$ is the commutator of the Weyl algebra W. The derivation property yields the Jacobi identity of $[\ ,\ ]$ and $(C, [\ ,\ ])$ becomes a Lie algebra.

Definition 2.16. The Lie algebra $(C, [\ ,\ ])$ is called a **contact algebra**.

We put the product topology of $\mathbf{R}$ and W into $C = \mathbf{R}\tau \oplus W$ and $(C, [\ ,\ ])$ is a complete topological algebra. We consider an anti-involution of C by setting

$$\overline{\tau} = \tau, \quad \overline{\lambda\tau + F} = \lambda\tau + \overline{F}, \quad \lambda \in \mathbf{R}, \quad F \in W. \tag{2.15}$$

We remark here the derivation $a \mapsto [\tau, a]$ of W counts the degree of monomials, i.e., it holds

$$[\tau, \nu^l Z^\alpha] = \nu(2l + |\alpha|)\nu^l Z^\alpha$$

and hence we have

$$[\tau, F] = 2\nu^2 \partial_\nu F + \nu \sum_{k=1}^{2n} Z^k \partial_{Z^k} F, \quad F \in W. \tag{2.16}$$

Now, we consider an automorphism group of the contact algebra. First we consider a derivation; for $F \in W$, we set

$$\operatorname{ad} \frac{1}{\nu} F(a) = \frac{1}{\nu} \operatorname{ad} F(a), \quad \forall a \in W, \tag{2.17}$$

$$\operatorname{ad} \frac{1}{\nu} F(\tau) = 2F + \frac{1}{\nu}[F, \tau]. \tag{2.18}$$

Definition 2.17. An algebra isomorphsim $\Psi\colon C \to C$ is called a **ν-autormorphism** if it gives a ν-automorphism of Weyl algebra when restricted to W. A ν-automorphism Ψ is **hermitian** if it satisfies $\overline{\Psi(P)} = \Psi(\overline{P})$, $P \in C$.

We have

Proposition 2.18. *For a hermitian ν-automorphism $\Psi: C \to C$, there exist uniquely a $2n \times 2n$ symplectic matrix A, a hermitian central element $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ and $F \in W_3^{\circ +}$ such that*

$$\Psi = A \circ \exp \operatorname{ad} \frac{1}{\nu}(F + c(\nu^2)).$$

Proof. For the restriction $\Psi|_W$, Proposition 2.5 gives a symplectic matrix A and $F \in W_3^{\circ +}$ such that $\Psi|_W = A \circ \exp \frac{1}{\nu} \operatorname{ad} F$. Then, the ν-automorphisms Ψ and $\psi = A \circ \exp \operatorname{ad}\left(\frac{1}{\nu}F\right)$ coincide when restricted to W, and it holds $\psi^{-1} \circ \Psi(Z^i) = Z^i$, $i = 1, \dots, 2n$. As to $\psi^{-1} \circ \Psi(\tau) \in C = \mathbf{R}\tau \oplus W$, we apply $\psi^{-1} \circ \Psi$ to the identities $[\tau, Z^i] = \nu Z^i$ and we have $\psi^{-1} \circ \Psi(\tau) = \tau + b$ for certain central element b. The the hermitian property induces

$$b = b(\nu^2) = b_0 + \nu^2 b_2 + \cdots + \nu^{2k} b_{2k} + \cdots \in \mathbf{R}[[\nu^2]].$$

A central element $c(\nu^2) = \sum_{k=0}^{\infty} \nu^{2k} c_{2k}$ with $c_{2k} = 1/(2(1-2k))b_{2k}$ satisfies $\exp \operatorname{ad} \frac{1}{\nu} c(\nu^2)(\tau) = \tau + b(\nu^2)$, which shows the existence. The uniqueness is a direct consequence of Proposition 2.5 and the uniqueness of $c(\nu^2)$. Q.E.D.

2.3.2. Contact Weyl diffeomorphism. Let U be an open subset of $\mathbf{R}^{2n}$ and consider a trivial bundle $C_U = U \times C$. We denote by $\Gamma(C_U)$ the set of all smooth sections of C_U. Then $\Gamma(C_U)$ forms a Lie algebra by the pointwise multiplication and becomes a complete topological Lie algebra under smooth topology. We consider a section $\tau_U \in \Gamma(C_U)$ such that

$$\tau_U(z) = \tau + \sum_{i,j=1}^{2n} \omega_{ij} z^i Z^j, \quad (z \in U). \tag{2.19}$$

Recall the derivation $[\tau, \ ]$ satisfies $[\tau, F] = 2\nu^2 \partial_\nu F + \nu \sum_{k=1}^{2n} Z^k \partial_{Z^k} F$ in (2.16) and notice $[\sum_{ij} \omega_{ij} z^i Z^j, F] = \nu \sum_k z^k \partial_{Z^k} F$, $F \in W$ for each $z \in U$. Then we see easily the fiberwise derivation $[\tau_U, \ ]$ acts on $\Gamma(W_U)$ in the form

$$[\tau_U(z), F(z)] = 2\nu^2 \partial_\nu F(z) + \nu \sum_{k=1}^{2n} z^{k\#} \partial_{Z^k} F(z), \quad F \in \Gamma(W_U). \tag{2.20}$$

The identity $\partial_{Z^k} f^{\#} = (\partial_{z^k} f)^{\#}$ yields

$$[\tau_U(z), f^{\#}(z)] = 2\nu^2 \partial_\nu f^{\#}(z) + \nu \sum_{k=1}^{2n} (z^k \partial_{z^k} f)^{\#}(z), \quad f^{\#} \in \mathcal{F}(W_U).$$

Here we use the identity $z^{k\#}(\partial_{z^k} f)^\# = (z^k \partial_{z^k} f)^\#$. In fact, using the definition of $\widehat{*}$ we calculate $z^{k\#} \widehat{*} g^\# = z^{k\#} g^\# + \frac{\nu}{2} \sum_m \Lambda^{km} (\partial_{z^m} g)^\#$, and using the formula given in Proposition 2.7 we see

$$z^{k\#} \widehat{*}\, g^\# = (z^k *_0 g)^\# = (z^k g)^\# + \frac{\nu}{2} \sum_m \Lambda^{km} (\partial_{z^m} g)^\#$$

which shows $z^{k\#} g^\# = (z^k g)^\#$ for $\forall g^\# \in \mathcal{F}(W_U)$. Thus, we have

Lemma 2.19.

$$[\tau_U(z), f^\#(z)] = 2\nu^2 \partial_\nu f^\#(z) + \nu (Ef)^\#(z), \quad f^\# \in \mathcal{F}(W_U),$$

where $E = \sum_{k=1}^{2n} z^k \partial_{z^k}$ *is the Euler vector field.*

Now remark W_U is a subbundle of C_U.

Definition 2.20. We call a bundle isomorphism $\widetilde{\Phi}\colon C_U \to C_{U'}$ a **contact extension of Weyl diffeomorphism**, or **CEWD** for short, if the restriction $\widetilde{\Phi}|_{W_U}$ is a Weyl diffeomorphism.

For a contact extension of a Weyl diffeomorphism $\widetilde{\Phi}$, the pullback $\widetilde{\Phi}^* \tau_{U'}$ is obviously a section of C_U and is written in the form

$$\widetilde{\Phi}^* \tau_{U'} = \lambda \tau_U + H, \quad \lambda \in C^\infty(U) \text{ and } H \in \Gamma(W_U).$$

Applying $\widetilde{\Phi}^*$ to the identity $[\tau_{U'}, \nu] = 2\nu^2$ induces $\lambda \equiv 1$. Applying $\widetilde{\Phi}^*$ to the identity $[\tau_{U'}, z^{i\#}] = \nu z^{i\#}$ shows $[H, \widetilde{\Phi}^* z^{i\#}] = \nu \widetilde{\Phi}^* z^{i\#}$ and hence $[\widetilde{\Phi}^{*-1} H, z^{i\#}] = \nu z^{i\#}$, $i = 1, \ldots, 2n$ which induces $[\widetilde{\Phi}^{*-1} H, \mathcal{F}(W_{U'})] \subset \nu \mathcal{F}(W_{U'})$. Proposition 2.8 then shows $\widetilde{\Phi}^{*-1} H$ is a sum of a certain Weyl function and an element of $\mathfrak{a}_{\nu^2}(U')$. Thus, for a contact extension of Weyl diffeomorphism $\widetilde{\Phi}\colon C_U \to C_{U'}$, we have

Lemma 2.21. *The pullback of* $\tau_{U'}$ *is written as* $\widetilde{\Phi}^* \tau_{U'} = \tau_U + f^\# + a(\nu^2)$, *for certain* $f^\# \in \mathcal{F}(W_U)$ *with* $\overline{f^\#} = f^\#$ *and* $a(\nu^2) \in \mathfrak{a}_{\nu^2}(U)$.

Further, if $a(\nu^2)$ is a constant, i.e., $a(\nu^2) \in \mathbf{R}[[\nu^2]]$, we can view $f^\# + a(\nu^2) = \big(f^\# + a(\nu^2)\big)^\#$ as a Weyl function. We define (cf. Definition 4.6, [OMY1])

Definition 2.22. A contact extension of Weyl diffeomorphism $\Psi\colon C_U \to C_{U'}$ is called a **contact Weyl diffeomorphism** if and only if $\Psi^* \tau_{U'} = \tau_U + f^\#$ for certain $f^\# \in \mathcal{F}(W_U)$.

Notice a contact Weyl diffeomorphism Ψ yields a Weyl diffeomorphism $\Psi|_{W_U}$, and then by Lemma 2.10 the induced map $\varphi\colon U \to U'$ is

a symplectic diffeomorphism. As to the existence we have (see Theorem 4.7, [OMY1])

Theorem 2.23. (i) *For a Weyl diffeomorphism* $\Phi\colon W_U \to W_{U'}$, *there exists a contact Weyl diffeomorphism* $\Psi\colon C_U \to C_{U'}$ *such that* $\Psi|_{W_U} = \Phi$. (ii) *For a symplectic diffeomorphism* $\varphi\colon U \to U'$, *there exists a contact Weyl diffeomorphism* $\Psi\colon C_U \to C_{U'}$ *whose induced map is* φ.

Now we proceed to consider a contact Weyl diffeomorphism with the identity base map. Consider a hermitian, local Weyl function with the constant leading term

$$g^{\#}(\nu^2) = g_0 + \nu^2 g_2^{\#} + \cdots + \nu^{2k} g_{2k}^{\#} + \cdots, \tag{2.21}$$

where $g_0 \in \mathbf{R}$, $g_{2k} \in C^\infty(U)$, $(k = 1, 2, \ldots)$. Obviously we have a decomposition $g^{\#} = a(\nu^2) + F$ with $a(\nu^2) \in \mathfrak{a}_{\nu^2}(U)$, $F \in \Gamma(U \times W_3^{\circ +})$. Then due to Proposition 2.18, $\Psi = \exp \mathrm{ad}\big(\frac{1}{\nu} g^{\#}\big)$ gives a bundle isomorphism of C_U which clearly satisfies $\Psi^* \mathcal{F}(W_U) = \mathcal{F}(W_U)$. It is easy to see

$$\Psi^* \tau_U = \tau_U + 2g_0 + \nu^2 A_{(1)}^{\#}(\nu^2), \quad \Psi^* z^{i\#} = z^{i\#} + \nu^2 A_{(2)}^{\#}(\nu^2)$$

where $A_{(i)}^{\#}(\nu^2) = A_{(i),0}^{\#} + \nu^2 A_{(i),2}^{\#} + \cdots + \nu^{2k} A_{(i),2k}^{\#} + \cdots$, $A_{(i),2k} \in C^\infty(U)$, $i = 1, 2$. Thus, we have contact Weyl diffeomorphism Ψ with the identity base map. Moreover, one sees that a contact Weyl diffeomorphism of the above form is general as follows (cf. Corollary 2.5, [OMMY1]).

Proposition 2.24. (i) *If a contact Weyl diffeomorphism* $\Psi\colon C_U \to C_U$ *indueces the identity base map, there exists uniquely a Weyl function* $g^{\#}(\nu^2)$ *of the form* (2.21) *such that* $\Psi = \exp \mathrm{ad}\big(\frac{1}{\nu} g^{\#}(\nu^2)\big)$. (ii) *If a contact Weyl diffeomorphism* Ψ *yields the identity map* $\widetilde{\Phi}|_{W_U} = 1$ *on* W_U, *then there exists uniquley a hermitian central element*

$$c(\nu^2) = c_0 + \nu^2 c_2 + \cdots + \nu^{2k} c_{2k} + \cdots, \quad c_{2k} \in \mathbf{R} \tag{2.22}$$

such that $\Psi = \exp \mathrm{ad}\big(\frac{1}{\nu} c(\nu^2)\big)$.

In what follows we consider the relation between contact extensions of Weyl diffeomorphism and contact Weyl diffeomorphisms. Let $\widetilde{\Phi}\colon C_U \to C_{U'}$ be a contact extension of Weyl diffeomorphism, CEWD. Then Lemma 2.21 gives that $\widetilde{\Phi}^* \tau_{U'} = \tau_U + f^{\#} + b(\nu^2)$ for certain $f^{\#} \in \mathcal{F}(W_U)$ and $b(\nu^2) = b_0 + \nu^2 b_2 + \cdots + \nu^{2k} b_{2k} + \cdots$, $b_{2k} \in C^\infty(U)$. It is easy to see $a(\nu^2) = \frac{1}{2} b_0 - \frac{\nu^2}{2} b_2 + \cdots + \frac{\nu^{2k}}{2(1-2k)} b_{2k} + \cdots \in \mathfrak{a}_{\nu^2}(U)$ satisfies $\exp \mathrm{ad}\big(\frac{1}{\nu} a(\nu^2)\big) \tau_U = \tau_U + b(\nu^2)$. Then the composition

$\widetilde{\Psi} = \widetilde{\Phi} \circ \exp\bigl(-\operatorname{ad} \frac{1}{\nu} a(\nu^2)\bigr) \colon C_U \to C_U$ satisfies $\widetilde{\Psi}^* \tau_{U'} = \tau_U + f^{\#}$ and $\widetilde{\Psi}$ is a contact Weyl diffeomorphism. Thus, we have

Proposition 2.25. *For a contact extension of Weyl diffeomorphism $\widetilde{\Phi} \colon C_U \to C_{U'}$, there exists a contact Weyl diffeomorphism $\widetilde{\Psi} \colon C_U \to C_{U'}$ and $a(\nu^2) \in \mathfrak{a}_{\nu^2}(U)$ such that $\widetilde{\Phi} = \widetilde{\Psi} \circ \exp \operatorname{ad}\bigl(\frac{1}{\nu} a(\nu^2)\bigr)$.*

2.3.3. Čech 2-cocycle of W_M and the Poincaré-Cartan class. Consider a Weyl manifold $W_M \xrightarrow{\pi} M$. We have a system of local Weyl charts $\Phi_\alpha \colon \pi^{-1}(V_\alpha) = W_{V_\alpha} \to W_{U_\alpha} = U_\alpha \times W$ and a system of overlap maps $\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1} \colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$, $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$.

According to Theorem 2.23, we take a contact Weyl diffeomorphism $\widetilde{\Phi}_{\alpha\beta} \colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ such that $\widetilde{\Phi}_{\alpha\beta}|_{W_{U_{\alpha\beta}}} = \Phi_{\alpha\beta}$ for each overlap map. Here we may assume $\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} = 1$ on $C_{U_{\alpha\beta}}$. In fact, $\Phi_{\beta\alpha} \circ \Phi_{\alpha\beta} = 1$ on $W_{U_{\alpha\beta}}$ and Proposition 2.24 yields $\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} = \exp \operatorname{ad}\bigl(\frac{1}{\nu} d_{\alpha\beta}(\nu^2)\bigr)$ for certain $d_{\alpha\beta}(\nu^2) \in \mathbf{R}[[\nu^2]]$. Notice

$$\begin{aligned} \widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} &= \exp \operatorname{ad}\Bigl(\frac{1}{\nu} d_{\beta\alpha}(\nu^2)\Bigr) \circ \widetilde{\Phi}_{\alpha\beta} \\ &= \widetilde{\Phi}_{\alpha\beta} \circ \exp \operatorname{ad}\Bigl(\frac{1}{\nu} d_{\alpha\beta}(\nu^2)\Bigr) \end{aligned}$$

which shows $d_{\beta\alpha}(\nu^2) = d_{\alpha\beta}(\nu^2)$. Then we can assume $\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} = 1$ by replacing $\widetilde{\Phi}_{\alpha\beta}$ by $\widetilde{\Phi}_{\alpha\beta} \circ \exp\bigl(-\operatorname{ad} \frac{1}{2\nu} d_{\alpha\beta}(\nu^2)\bigr)$ on $C_{U_{\alpha\beta}}$.

We set a bundle isomorphism $\widetilde{\Phi}_{\alpha\beta\gamma} = \widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\beta\gamma} \circ \widetilde{\Phi}_{\alpha\beta}$ of $C_{U_{\alpha\beta\gamma}}$ where $U_{\alpha\beta\gamma} = \varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$, $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$. Since the restriction $\widetilde{\Phi}_{\alpha\beta\gamma}|_{W_{U_{\alpha\beta\gamma}}} = \Phi_{\gamma\alpha} \circ \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta}$ is the identity of $W_{U_{\alpha\beta\gamma}}$, there exists $c_{\alpha\beta\gamma}(\nu^2) \in \mathbf{R}[[\nu^2]]$ such that $\widetilde{\Phi}_{\alpha\beta\gamma} = \exp \operatorname{ad}\bigl(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\bigr)$. The identities $\widetilde{\Phi}_{\alpha\beta\gamma} \circ \widetilde{\Phi}_{\alpha\gamma\beta} = 1$ and $\widetilde{\Phi}_{\alpha\beta\gamma} = \widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\gamma\alpha\beta} \circ \widetilde{\Phi}_{\alpha\gamma}$ induce

$$c_{\alpha\beta\gamma}(\nu^2) + c_{\alpha\gamma\beta}(\nu^2) = 0, \quad c_{\alpha\beta\gamma}(\nu^2) = c_{\gamma\alpha\beta}(\nu^2),$$

respectively, which mean $\{c_{\alpha\beta\gamma}(\nu^2)\}$ is a Čech 2-cochain for the covering $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$. An easy calculation yields

$$\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\beta\delta\gamma} \circ \widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\alpha\delta\beta} \circ \widetilde{\Phi}_{\alpha\gamma\delta} \circ \widetilde{\Phi}_{\alpha\beta\gamma} = 1$$

on $C_{U_{\alpha\beta\gamma\delta}}$, $U_{\alpha\beta\gamma\delta} = \varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma \cap V_\delta)$ which induces

$$c_{\beta\delta\gamma}(\nu^2) + c_{\alpha\delta\beta}(\nu^2) + c_{\alpha\gamma\delta}(\nu^2) + c_{\alpha\beta\gamma}(\nu^2) = -\delta c_{\alpha\beta\gamma\delta}(\nu^2) = 0.$$

Thus $\{c_{\alpha\beta\gamma}(\nu^2)\}$ is a Čech 2-cocycle with values in $\mathbf{R}[[\nu^2]]$, or each $\{c_{\alpha\beta\gamma,2k}\}$ of the expansion $c_{\alpha\beta\gamma}(\nu^2) = \sum_{k=0}^{\infty} \nu^{2k} c_{\alpha\beta\gamma,2k}$ is a Čech

2-cocycle with values in $\mathbf{R}$ ($k = 0, 1, 2, \ldots$). Then we obtain an element of $H^2(M)[[\nu^2]]$ which is denoted by $c(W_M)$ and is called the **Poincaré-Cartan class** of the Weyl manifold W_M. We write the expansion as

$$c(W_M) = c_0(W_M) + \nu^2 c_2(W_M) + \cdots + \nu^{2k} c_{2k}(W_M) + \cdots,$$

where $c_{2k}(W_M) \in H^2(M)$. As to $c(W_M)$ we have the following (Theorem 3.5, [OMMY1]).

Proposition 2.26. (i) *The leading term $c_0(W_M)$ of $c(W_M)$ is equal to $[\sigma]$, the cohomology class of the symplectic structure σ of M.* (ii) *$c(W_M)$ depends on the equivalence class of W_M, i.e., if W'_M is equivalent to W_M as Weyl manifold, then $c(W'_M) = c(W_M)$.*

Theorem 2.27. *The Poincaré-Cartan class is a complete invariant of Weyl manifolds, i.e., the map $[W_M] \mapsto c(W_M)$ is bijective from all equivalence classes of Weyl manifolds to the set of all elements of $H^2(M)[[\nu^2]]$ with the leading term $[\sigma]$.*

§3. Contact algebra bundle and connection

In this section, using a Poincaré-Cartan class $c(W_M)$ of Weyl manifold W_M, we will construct a contact algebra bundle $C_M \to M$ and a connection ∂ on C_M.

Let $\{c_{\alpha\beta\gamma}(\nu^2)\}$ be a Čech 2-cocycle giving $c(W_M)$. Then by definition, contact Weyl diffeomorphisms $\widetilde{\Phi}_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ for overlap maps of local Weyl charts $\Phi_{\alpha\beta}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ satisfy

$$\widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\beta\gamma} \circ \widetilde{\Phi}_{\alpha\beta} = \exp \operatorname{ad}\Big(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big),$$

which is not the identity transformation in general. Hence the system $\{\widetilde{\Phi}_{\alpha\beta}\}$ is not useful for gluing $\{C_{U_{\alpha\beta}}\}$. Our idea for constructing C_M is to use contact extensions of Weyl diffeomorphisms instead of $\{\widetilde{\Phi}_{\alpha\beta}\}$. By means of $\{c_{\alpha\beta\gamma}(\nu^2)\}$, we construct an appropriate 1-cochain $\{h_{\alpha\beta}(\nu^2)\}$, $h_{\alpha\beta}(\nu^2) \in C^\infty(U_{\alpha\beta})[[\nu^2]]$ and using this cochain and the system of contact Weyl diffeomorphisms $\{\widetilde{\Phi}_{\alpha\beta}\}$, we obtain a certain system of contact extensions of Weyl diffeomorphisms $\{\Psi_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}\}$. We will then have a contact algebra bundle by gluing $\{C_{U_\alpha}\}$ by means of $\{\Psi_{\alpha\beta}\}$.

As to the connection ∂, we first consider a closed 2-form $\Omega_M(\nu^2) \in \Lambda^2_M[[\nu^2]]$ whose cohomology class is equal to the Poincaré-Cartan class $[\Omega_M(\nu^2)] = c(W_M)$. Then we take a system of local 1-forms $\{\xi_\alpha(\nu^2)\}$, $\xi_\alpha \in \Lambda^1(V_\alpha)[[\nu^2]]$ such that $d\xi_\alpha(\nu^2) = \Omega_M(\nu^2)$ on V_α. Using this system $\{\xi_\alpha(\nu^2)\}$, we construct a connection 1-form and then we will obtain a connection ∂ whose curvature form is $\Omega_M(\nu^2)$.

3.1. Construction of C_M; proof of Theorem A

Let W_M be a Weyl manifold and $c(W_M)$ be its Poincaré-Cartan class. Suppose $\{c_{\alpha\beta\gamma}(\nu^2)\}$ is a Čech 2-cocycle giving $c(W_M)$. Recall the cocycle has the form $c_{\alpha\beta\gamma}(\nu^2) = c_{\alpha\beta\gamma,0}+\nu^2 c_{\alpha\beta\gamma,2}+\cdots+\nu^{2k}c_{\alpha\beta\gamma,2k}+\cdots$, $c_{\alpha\beta\gamma,2k} \in \mathbf{R}$. We denote by $\Phi_{\alpha\beta}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ the overlap map of local Weyl charts and by $\widetilde{\Phi}_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ its lift as a contact Weyl diffeomorphism. Then by definition, we have the identity

$$\widetilde{\Phi}_{\alpha\beta\gamma} = \widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\beta\gamma} \circ \widetilde{\Phi}_{\alpha\beta} = \exp \operatorname{ad}\Big(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big)$$

$$\widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\beta\alpha} = 1_{U_{\alpha\beta}}.$$

In what follows, we construct Čech 1-cocycle $\{H_{\alpha\beta}(\nu^2)\}$ and a system of 1-forms $\{\xi_\alpha(\nu^2)\}$ on M related to $\{c_{\alpha\beta\gamma}(\nu^2)\}$ by the standard argument. The 1-cocycle $\{H_{\alpha\beta}(\nu^2)\}$ is used in this section to construct a gluing map system of contact extension of Weyl diffeomorphisms $\{\Psi_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}\}$ for the contact algebra bundle C_M. The system $\{\xi_\alpha(\nu^2)\}$ will be used for constructing a connection ∂ on C_M in the next section.

Now we define a formal power seires in ν^2 with coefficients in $\mathfrak{a}(M) = C^\infty(M)$ by

$$H_{\alpha\beta}(\nu^2) = \sum_\lambda c_{\alpha\beta\lambda}(\nu^2)\chi_\lambda \in \mathfrak{a}_M[[\nu^2]], \tag{3.1}$$

where $\{\chi_\lambda\}_\lambda$ is a partition of unity subbordinate to the covering $\{V_\lambda\}_\lambda$ of M. Then we have

$$H_{\alpha\beta}(\nu^2) = -H_{\beta\alpha}(\nu^2) \tag{3.2}$$

$$\delta H_{\alpha\beta\gamma}(\nu^2) = H_{\alpha\beta}(\nu^2) + H_{\beta\gamma}(\nu^2) + H_{\gamma\alpha}(\nu^2) = c_{\alpha\beta\gamma}(\nu^2). \tag{3.3}$$

We set a formal power seiries of one forms on M

$$\Xi_\alpha(\nu^2) = \sum_\lambda dH_{\alpha\lambda}(\nu^2)\chi_\lambda \in \Lambda^1_M[[\nu^2]]. \tag{3.4}$$

Then the identity (3.3) shows

Lemma 3.1. $\Xi_\beta(\nu^2) - \Xi_\alpha(\nu^2) = dH_{\beta\alpha}(\nu^2)$.

We set the local coordinate expressions,

$$h_{\alpha\beta}(\nu^2) = \varphi_\alpha^{-1*} H_{\alpha\beta}(\nu^2) \in \mathfrak{a}_{U_\alpha}[[\nu^2]]. \tag{3.5}$$

Then the identities for $H_{\alpha\beta}$ (3.2) and (3.3) gives

Lemma 3.2.

(i) $$h_{\alpha\beta}(\nu^2) = -\varphi^*_{\alpha\beta} h_{\beta\alpha}(\nu^2)$$

(ii) $$h_{\alpha\beta}(\nu^2) + \varphi^*_{\alpha\beta} h_{\beta\gamma}(\nu^2) + \varphi^*_{\alpha\gamma} h_{\gamma\alpha}(\nu^2) = c_{\alpha\beta\gamma}(\nu^2).$$

We also set the local coordinate expression

$$\xi_\alpha(\nu^2) = \varphi_\alpha^{-1*} \Xi_\alpha(\nu^2) \in \Lambda^1_{U_\alpha}[[\nu^2]]. \tag{3.6}$$

The identity $\Xi_\beta(\nu^2) - \Xi_\alpha(\nu^2) = dH_{\beta\alpha}(\nu^2)$ induces

$$\xi_\beta(\nu^2) = \varphi^*_{\beta\alpha} \xi_\alpha(\nu^2) + dh_{\beta\alpha}(\nu^2). \tag{3.7}$$

Now using $h_{\alpha\beta}(\nu^2) = \varphi_\alpha^{-1*} H_{\alpha\beta}(\nu^2)$ we set a contact extension of Weyl diffeomorphism (CEWD) of $\Phi_{\alpha\beta}$ as

$$\Psi_{\alpha\beta} = \widetilde{\Phi}_{\alpha\beta} \circ \exp\Big(-\operatorname{ad} \frac{1}{\nu} h_{\alpha\beta}(\nu^2)\Big) : C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}} \tag{3.8}$$

where $\widetilde{\Phi}_{\alpha\beta}$ is a contact Weyl diffeomorhpism lift of $\Phi_{\alpha\beta}$. Then we have

$$\Psi_{\alpha\beta} \circ \Psi_{\beta\alpha} = 1_{U_{\alpha\beta}} : C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}. \tag{3.9}$$

In fact, the skew symmetry of $h_{\alpha\beta}(\nu^2)$ in Lemma 3.2 (i) yields

$$\begin{aligned}
\Psi_{\alpha\beta} \circ \Psi_{\beta\alpha} &= \widetilde{\Phi}_{\alpha\beta} \circ \exp\Big(-\operatorname{ad} \frac{1}{\nu} h_{\alpha\beta}(\nu^2)\Big) \circ \widetilde{\Phi}_{\beta\alpha} \circ \exp\Big(-\operatorname{ad} \frac{1}{\nu} h_{\beta\alpha}(\nu^2)\Big) \\
&= \widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\beta\alpha} \circ \exp\Big(-\operatorname{ad} \frac{1}{\nu} h_{\beta\alpha}(\nu^2) + \frac{1}{\nu} \varphi^*_{\beta\alpha} h_{\alpha\beta}(\nu^2)\Big) \\
&= 1_{U_{\alpha\beta}}.
\end{aligned}$$

The cyclic condition (ii) in the same lemma also gives

$$\begin{aligned}
\Psi_{\alpha\beta\gamma} &= \Psi_{\gamma\alpha} \circ \Psi_{\beta\gamma} \circ \Psi_{\alpha\beta} \\
&= \widetilde{\Phi}_{\alpha\beta\gamma} \circ \exp\Big(-\operatorname{ad} \frac{1}{\nu} h_{\alpha\beta}(\nu^2) + \frac{1}{\nu} \varphi^*_{\alpha\beta} h_{\beta\gamma}(\nu^2) + \frac{1}{\nu} \varphi^*_{\alpha\gamma} h_{\gamma\alpha}(\nu^2)\Big) \\
&= \exp \operatorname{ad}\Big(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big) \circ \exp\Big(-\operatorname{ad} \frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big) = 1_{U_{\alpha\beta\gamma}}.
\end{aligned}$$

Thus, by gluing local trivial contact algebra bundles $\{C_{U_\alpha}\}$ by means of the system $\{\Psi_{\alpha\beta}\}$ we have a locally tirivial contact algebra bundle $C_M \to M$. By construction it is obvious that C_M contains the Weyl manifold W_M as a subbundle and hence we have a proof of Theorem A.

Finally we prepare for some basic identities for $\{\tau_{U_\alpha}\}$. The identities will be used in the next sections for contructing a connection

on C_M, where $\tau_U = \tau + \sum_{ij} \omega_{ij} z^i Z^j$ (see (2.19)). For simplicity, we write $\tau_\alpha = \tau_{U_\alpha}$.

Proposition 3.3. *We have on* $U_{\alpha\beta}$

$$\Psi_{\alpha\beta}^* \tau_\beta = \tau_\alpha + f_{\alpha\beta}^{\#} - \widehat{h}_{\alpha\beta}(\nu^2)$$

where $f_{\alpha\beta}^{\#}$ *is a local Weyl function given by the contact Weyl diffeomorphism* $\widetilde{\Phi}_{\alpha\beta}$ *as*

$$\widetilde{\Phi}_{\alpha\beta}^* \tau_\beta = \tau_\alpha + f_{\alpha\beta}^{\#} \tag{3.10}$$

and $\widehat{h}_{\alpha\beta}(\nu^2) \in \mathfrak{a}_{U_\alpha}[[\nu^2]]$ *is given by*

$$\operatorname{ad}\Big(\frac{1}{\nu} h_{\alpha\beta}(\nu^2)\Big)\tau_\alpha = \widehat{h}_{\alpha\beta}(\nu^2). \tag{3.11}$$

Proof. It is obvious by $\Psi_{\alpha\beta} = \widetilde{\Phi}_{\alpha\beta} \circ \exp\big(-\operatorname{ad} \frac{1}{\nu} h_{\alpha\beta}(\nu^2)\big)$. Q.E.D.

3.2. Construction of connection ∂: proof of Thoerem B

The connection ∂ is defined as a twisted exterior derivation. For this, we introduce a tensor product bundle $\Lambda_M \otimes C_M$, where Λ_M is the exterior algebra bundle over M, similarly as Fedosov [F].

3.2.1. Tensor product bundles. We consider the tensor product bundles $\Lambda_M \otimes W_M$ and $\Lambda_M \otimes C_M$, where Λ_M is the exterior algebra bundle over M. Obviously $\Lambda_M \otimes W_M$ is a subbundle of $\Lambda_M \otimes C_M$. Local trivializations are given by $\Lambda_{U_\alpha} \otimes W_{U_\alpha} \subset \Lambda_{U_\alpha} \otimes C_{U_\alpha}$ for each $\{U_\alpha\}$, and gluing maps are given by

$${}^t d\varphi_{\alpha\beta} \otimes \Phi_{\alpha\beta} \colon \Lambda_{U_{\alpha\beta}} \otimes W_{U_{\alpha\beta}} \to \Lambda_{U_{\beta\alpha}} \otimes W_{U_{\beta\alpha}}$$

and

$${}^t d\varphi_{\alpha\beta} \otimes \Psi_{\alpha\beta} \colon \Lambda_{U_{\alpha\beta}} \otimes C_{U_{\alpha\beta}} \to \Lambda_{U_{\beta\alpha}} \otimes C_{U_{\beta\alpha}},$$

respectively.

The algebra structure of these bundles is given in the following way: Let $U \subset \mathbf{R}^{2n}$ be an open subset and let us consider elements $P, Q \in \Lambda_U \otimes C_U$ given by $P = \sum_I dz^I P_I$, $Q = \sum_J dz^J Q_J$, $P_I, Q_J \in C_U$, $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_k}$ where $I = \{i_1, \ldots, i_k\}$, $k = |I|$, etc. We introduce a bracket

$$[P, Q] = \sum_{I,J} dz^I \wedge dz^J [P_I, Q_J]. \tag{3.12}$$

For $F = \sum_I dz^I F_I$, $G = \sum_J dz^J G_J \in \Lambda_U \otimes W_U$ we also set a product

$$F \mathbin{\hat{*}} G = \sum_{I,J} dz^I \wedge dz^J F_I \mathbin{\hat{*}} G_J. \tag{3.13}$$

Then we have the following super algebra identities:

Lemma 3.4. *Let P, Q, R be monomials of $\Lambda_U \otimes C_U$ and $|P|$, $|Q|$, $|R|$ be their degrees as forms, respectively. Then we have the skewsymmetry*

$$[P, Q] = (-1)^{|P|\cdot|Q|+1}[Q, P] \tag{i}$$

and the super-Jacobi identity

$$\begin{aligned}[P,[Q,R]] + (-1)^{|P||R|+|Q||R|}[R,[P,Q]] \\ + (-1)^{|P||Q|+|Q||R|}[Q,[R,P]] = 0.\end{aligned} \tag{ii}$$

Proof. For (i), consider $dz^I \wedge dz^J = (-1)^{|I||J|} dz^J \wedge dz^I$ and the skewsymmetry of the bracket of C. The second identity (ii) is obtained by using

$$\begin{aligned} dz^P \wedge dz^Q \wedge dz^R &= (-1)^{pr+qr} dz^R \wedge dz^P \wedge dz^Q \\ &= (-1)^{pq+qr} dz^Q \wedge dz^R \wedge dz^P \end{aligned}$$

and the Jacobi identity. Q.E.D.

3.2.2. Derivations. Now we consider derivations on $\Lambda_{U_\alpha} \otimes C_{U_\alpha}$. Let δ_α be a fiberwise derivation defined by

$$\delta_\alpha = \operatorname{ad}\left(\frac{1}{\nu} \sum_{ij} dz_\alpha^i \omega_{ij} Z^j\right) : \Lambda^p_{U_\alpha} \otimes C_{U_\alpha} \to \Lambda^{p+1}_{U_\alpha} \otimes C_{U_\alpha} \tag{3.14}$$

for each $p = 0, 1, \ldots, 2n$. It is easy to see

Lemma 3.5. *For every $dz_\alpha^I = dz_\alpha^{i_1} \wedge \cdots \wedge dz_\alpha^{i_k}$, $I = \{i_1, \ldots, i_k\}$, it holds for any $Q \in \Lambda_{U_\alpha} \otimes C_{U_\alpha}$*

$$\delta_\alpha(dz_\alpha^I \wedge Q) = (-1)^{|I|} dz_\alpha^I \wedge \delta_\alpha Q. \tag{3.15}$$

For $P \in \Lambda^p_{U_\alpha} \otimes C_{U_\alpha}$, it holds for any $Q \in \Lambda_{U_\alpha} \otimes C_{U_\alpha}$

$$\delta_\alpha[P, Q] = [\delta_\alpha P, Q] + (-1)^p [P, \delta_\alpha Q]. \tag{3.16}$$

Proof. First equation is a direct consequence of definition of the bracket (3.12). As to the second, set $P = dz_\alpha^I P_I \in \Lambda^{|I|}_{U_\alpha} \otimes C_{U_\alpha}$,

$Q = dz_\alpha^J \wedge Q_J \in \Lambda_{U_\alpha}^{|J|} \otimes C_{U_\alpha}$ and use also the definiton (3.12) then we have $[P, Q] = dz_\alpha^I \wedge dz_\alpha^J [P_I, Q_J]$. A direct calculation gives

$$\begin{aligned}\delta_\alpha[P,Q] &= (-1)^{|I|+|J|} dz_\alpha^I \wedge dz_\alpha^J \wedge \{[\delta_\alpha P_I, Q_J] + [P_I, \delta_\alpha Q_J]\} \\ &= [(-1)^{|I|} dz_\alpha^I \wedge \delta_\alpha P_I, dz_\alpha^J Q_J] \\ &\quad + (-1)^{|I|} [dz_\alpha^I P_I, (-1)^{|J|} dz_\alpha^J \wedge \delta_\alpha Q_J].\end{aligned}$$

Then the first identity shows the desired relation. Q.E.D.

Now take an arbitrary 1-form $\kappa_\alpha(\nu^2) \in \Gamma(\Lambda_{U_\alpha}^1[[\nu^2]])$ and consider a derivation $\partial_\alpha \colon \Gamma(\Lambda_{U_\alpha}^p \otimes C_{U_\alpha}) \to \Gamma(\Lambda_{U_\alpha}^{p+1} \otimes C_{U_\alpha})$, $(p = 0, 1, \ldots, 2n)$, given by

$$\partial_\alpha = d - \delta_\alpha + \mathrm{ad}\Big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\Big). \tag{3.17}$$

Obviously ∂_α induces a derivation from $\Gamma(\Lambda_{U_\alpha}^p \otimes W_{U_\alpha})$ to $\Gamma(\Lambda_{U_\alpha}^{p+1} \otimes W_{U_\alpha})$ when restricted the subbundle $\Lambda_{U_\alpha} \otimes W_{U_\alpha}$. We see easily

Proposition 3.6.

(a) $$\partial_\alpha f = df, \quad f \in C^\infty(U_\alpha)$$

(b) $$\partial_\alpha Z^i = -dz_\alpha^i, \quad i = 1, 2, \ldots, 2n$$

(c) $$\partial_\alpha \nu = 0$$

(d) $$\partial_\alpha \tau = -\sum_{ij} dz_\alpha^i \omega_{ij} Z^j + \widehat{\kappa}_\alpha(\nu^2)$$

where $\widehat{\kappa}_\alpha(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)\tau \in \Gamma(\Lambda_{U_\alpha}^1[[\nu^2]])$.

Proof. The identities (a) and (c) are obvious. For (b) we calculate as

$$\begin{aligned}\partial_\alpha Z^i &= \Big(d - \delta_\alpha + \mathrm{ad}\Big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\Big)\Big) Z^i = -\delta_\alpha Z^i + \frac{1}{\nu}[\kappa_\alpha(\nu^2), Z^i] \\ &= -\frac{1}{\nu}\sum_{kl} dz_\alpha^k \omega_{kl} [Z^l, Z^i] = -\sum_{kl} dz_\alpha^k \omega_{kl} \Lambda^{li} = -dz_\alpha^i.\end{aligned}$$

As to (d), we remark the identity (2.18) and we see

$$\Big[\frac{1}{\nu}\sum_{kl} dz_\alpha^k \omega_{kl} Z^l, \tau\Big] = \sum_{kl} dz_\alpha^k \omega_{kl} Z^l.$$

Thus, we have

$$\partial_\alpha \tau = -\delta_\alpha \tau + \mathrm{ad}\Big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\Big)\tau = -\sum_{kl} dz_\alpha^k \omega_{kl} Z^l + \widehat{\kappa}_\alpha(\nu^2),$$

which gives (d). Q.E.D.

We have also the following formulae:

Lemma 3.7. *Let us consider a p-form* $P \in \Gamma(\Lambda^p_{U_\alpha} \otimes C_{U_\alpha})$. *Then it holds for any* $Q \in \Gamma(\Lambda_{U_\alpha} \otimes C_{U_\alpha})$

$$\partial_\alpha[P,Q] = [\partial_\alpha P, Q] + (-1)^p[P, \partial_\alpha Q]. \tag{i}$$

For $F \in \Gamma(\Lambda^f_{U_\alpha} \otimes W_{U_\alpha})$, *it holds for any* $G \in \Gamma(\Lambda_{U_\alpha} \otimes W_{U_\alpha})$

$$\partial_\alpha(F \mathbin{\widehat{*}} G) = \partial_\alpha F \mathbin{\widehat{*}} G + (-1)^f F \mathbin{\widehat{*}} \partial_\alpha G. \tag{ii}$$

Proof. We remark

$$\partial_\alpha(dz^I_\alpha P_I) = (-1)^{|I|} dz^I_\alpha \wedge \partial_\alpha P_I. \tag{3.18}$$

For (i), we show first for monomials $P = dz^I_\alpha P_I$ and $Q = dz^J_\alpha Q_J$. Using the identity above we calculate as $[P,Q] = dz^I_\alpha \wedge dz^J_\alpha [P_I, Q_J]$ and

$$\partial_\alpha[P,Q] = (-1)^{|I|+|J|} dz^I_\alpha \wedge dz^J_\alpha \wedge \partial_\alpha[P_I, Q_J].$$

For sections of $\Gamma(C_{U_\alpha})$, ∂_α acts as derivation and hence

$$\partial_\alpha[P_I, Q_J] = [\partial_\alpha P_I, Q_J] + [P_I, \partial_\alpha Q_J].$$

Thus,

$$\begin{aligned}\partial_\alpha[P,Q] = {} & [(-1)^{|I|} dz^I_\alpha \wedge \partial_\alpha P_I, dz^J Q_J] \\ & + (-1)^{|I|}[dz^I_\alpha P_I, (-1)^{|J|} dz^J_\alpha \wedge \partial_\alpha Q_J]\end{aligned}$$

which yields the desired result. For (ii), replacing $[P_I, Q_J]$ with $F_I \mathbin{\widehat{*}} G_J$ gives the equation similarly. Q.E.D.

3.2.3. Construction of connection. First we set

$$\widehat{\xi}_\alpha(\nu^2) = \mathrm{ad}\Big(\frac{1}{\nu}\xi_\alpha(\nu^2)\Big)\tau_\alpha \in \Lambda^1_{U_\alpha}[[\nu^2]] \tag{3.19}$$

where we take $\xi_\alpha(\nu^2) = \varphi_\alpha^{-1*}\Xi_\alpha(\nu^2)$ and $\Xi_\alpha(\nu^2) = \sum_\lambda dH_{\alpha\lambda}(\nu^2)\chi_\lambda$ in (3.6), (3.4) respctively. Then we have a relation

Lemma 3.8.

$$\widehat{\xi}_\beta(\nu^2) = \varphi^*_{\beta\alpha}\widehat{\xi}_\alpha(\nu^2) + d\widehat{h}_{\beta\alpha}(\nu^2)$$

where $\widehat{h}_{\beta\alpha}(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}h_{\beta\alpha}(\nu^2)\big)\tau_\beta \in \mathfrak{a}_{U_{\beta\alpha}}[[\nu^2]]$.

Proof. Recall $\xi_\beta(\nu^2) = \varphi^*_{\beta\alpha}\xi_\alpha(\nu^2) + dh_{\beta\alpha}(\nu^2)$ in (3.7). Then we calculate as

$$\widehat{\xi}_\beta(\nu^2) = \left[\frac{1}{\nu}\xi_\beta(\nu^2), \tau_\beta\right] = \left[\frac{1}{\nu}\varphi^*_{\beta\alpha}\xi_\alpha(\nu^2), \tau_\beta\right] + \left[\frac{1}{\nu}dh_{\beta\alpha}(\nu^2), \tau_\beta\right].$$

Notice that $\left[\frac{1}{\nu}\varphi^*_{\beta\alpha}\xi_\alpha(\nu^2), \tau_\beta\right] = \Psi^*_{\beta\alpha}\left[\frac{1}{\nu}\xi_\alpha(\nu^2), \Psi^*_{\alpha\beta}\tau_\beta\right]$. Since $\xi_\alpha(\nu^2)$ is a one form with values in the center of the algebra W, the identity $\Psi^*_{\alpha\beta}\tau_\beta = \tau_\alpha + f^{\#}_{\alpha\beta} - h_{\alpha\beta}(\nu^2)$ shows that $[\frac{1}{\nu}\xi_\alpha(\nu^2), \Psi^*_{\alpha\beta}\tau_\beta]$ is equal to

$$\left[\frac{1}{\nu}\xi_\alpha(\nu^2), \tau_\alpha + f^{\#}_{\alpha\beta} - h_{\alpha\beta}(\nu^2)\right] = \left[\frac{1}{\nu}\xi_\alpha(\nu^2), \tau_\alpha\right] = \widehat{\xi}_\alpha(\nu^2).$$

Using $\left[\frac{1}{\nu}dh_{\beta\alpha}(\nu^2), \tau_\beta\right] = d\widehat{h}_{\beta\alpha}(\nu^2)$, we obtain the desired result. Q.E.D.

For an arbitrary 1-form $\kappa_\alpha(\nu^2) \in \Lambda^1_{U_\alpha}[[\nu^2]]$ we have

Lemma 3.9.

$$\partial_\alpha\tau_\alpha = -2\theta_\alpha + \widehat{\kappa}_\alpha(\nu^2)$$

where $\theta_\alpha = \frac{1}{2}\sum_{ij} z^i_\alpha\omega_{ij}dz^j_\alpha$ *is a canonical* 1*-form and* $\widehat{\kappa}_\alpha(\nu^2) \in \Lambda^1_{U_\alpha}[[\nu^2]]$ *is given by* $\widehat{\kappa}_\alpha(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)\tau_\alpha$.

Proof. By Proposition 3.6 (a) and (d), we see $\partial_\alpha\tau_\alpha = \partial_\alpha\tau + \partial_\alpha\sum_{ij} z^i_\alpha\omega_{ij}Z^j$ is computed as

$$\begin{aligned}\partial_\alpha\tau_\alpha = &-\sum_{ij} dz^i_\alpha\omega_{ij}Z^j + \widehat{\kappa}_\alpha(\nu^2)\\ &+\sum_{ij}\partial_\alpha z^i_\alpha\omega_{ij}Z^j + \sum_{ij} z^i_\alpha\omega_{ij}\partial_\alpha Z^j\end{aligned}$$

which gives the desired result. Q.E.D.

Now we can set a connection. For $\widehat{\xi}_\alpha(\nu^2)$ defined in (3.19) we consider to find $\kappa_\alpha(\nu^2)$ so that the derivation $\partial_\alpha = d - \delta_\alpha + \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)$ satisfies

$$\partial_\alpha\tau_\alpha = \widehat{\xi}_\alpha(\nu^2). \tag{3.20}$$

We have

Lemma 3.10. *The equation* (3.20) *has a unique solution* $\kappa_\alpha(\nu^2)$.

Proof. By Lemma 3.9, the equation is equivalent to

$$\widehat{\kappa}_\alpha(\nu^2) = 2\theta_\alpha + \widehat{\xi}_\alpha(\nu^2), \tag{3.21}$$

where $\widehat{\kappa}_\alpha(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)\tau_\alpha$. Since

$$\mathrm{ad}\left(\frac{1}{\nu}\sum_{k=0}^{\infty}\nu^{2k}\kappa_{\alpha,2k}\right)\tau_\alpha = \sum_{k=0}^{\infty}(2-4k)\nu^{2k}\kappa_{\alpha,2k},$$

a 1-form $\kappa_\alpha(\nu^2) = \sum_{k=0}^{\infty}\nu^{2k}\kappa_{\alpha,2k}$ is uniquely determined from (3.21). Q.E.D.

We have

Lemma 3.11.

(i) *The restriction of* ∂_α *satisfies* $\partial_\alpha^2|_{W_{U_\alpha}} = 0$.

(ii) *A section* $F \in \Gamma(W_{U_\alpha})$ *satisfies* $\partial_\alpha F = 0$ *if and only if* $F \in \mathcal{F}(W_{U_\alpha})$.

(iii) *It holds* $\partial_\alpha^2\tau_\alpha = \widehat{\Omega}_M(\nu^2)$, *where* $\widehat{\Omega}_M(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\Omega_M(\nu^2)\big)\tau_\alpha$.

Proof. The first statement is a dirct consequence of Proposition 3.6. For (iii), we use the equation (3.20) and we see

$$\partial_\alpha^2\tau_\alpha = \partial_\alpha\widehat{\xi}_\alpha = d\widehat{\xi}_\alpha = \widehat{\Omega}_M(\nu^2) = \mathrm{ad}\Big(\frac{1}{\nu}\Omega_M(\nu^2)\Big)\tau_\alpha.$$

As to (ii), we calculate for $F = \sum_\mu \frac{1}{\mu!}F_\mu(\nu)Z^\mu$, $F_\mu(\nu) \in \mathfrak{a}_{U_\alpha}[[\nu]]$,

$$\partial_\alpha F = \sum_\mu \frac{1}{\mu!}\,dF_\mu(\nu)Z^\mu + \sum_\mu \frac{1}{\mu!}F_\mu(\nu)\partial_\alpha Z^\mu.$$

Notice here $\partial_\alpha Z^\mu = -\sum_i \mu_i\, dz_\alpha^i Z^{\mu-e_i}$, and hence

$$\sum_\mu \frac{1}{\mu!}F_\mu(\nu)\partial_\alpha Z^\mu = \sum_\mu\sum_{i=1}^{2n}\frac{1}{\mu!}F_{\mu+e_i}(\nu)\,dz_\alpha^i Z^\mu.$$

We then have

$$\partial_\alpha F = \sum_\mu\sum_{i=1}^{2n}\frac{1}{\mu!}\Big(\frac{\partial}{\partial z_\alpha^i}F_\mu(\nu) - F_{\mu+e_i}(\nu)\Big)dz_\alpha^i Z^\mu$$

which yields the desired result. Q.E.D.

In what follows, we fix $\kappa_\alpha(\nu^2)$ given by (3.20) for each α. Then, we have a derivation $\partial_\alpha\colon \Gamma(\Lambda^p_{U_\alpha}\otimes C_{U_\alpha}) \to \Gamma(\Lambda^{p+1}_{U_\alpha}\otimes C_{U_\alpha})$ for each local trivialization.

We set a transformation as

$$\Psi_{\alpha\beta *}\partial_\alpha = \Psi_{\alpha\beta}^{-1*}\partial_\alpha\Psi_{\alpha\beta}^*\colon \Gamma(\Lambda^p_{U_{\beta\alpha}}\otimes C_{U_{\beta\alpha}}) \to \Gamma(\Lambda^{p+1}_{U_{\beta\alpha}}\otimes C_{U_{\beta\alpha}}). \tag{3.22}$$

Then we have

Proposition 3.12. $\Psi^*_{\alpha\beta}\partial_\alpha = \partial_\beta$

A proof follows from the following identities:

Lemma 3.13.

(i) $(\Psi_{\alpha\beta *}\partial_\alpha)f = df, \quad f \in \mathfrak{a}_{U_{\beta\alpha}}[[\nu^2]],$

(ii) $(\Psi_{\alpha\beta *}\partial_\alpha)Z^j = -dz^j_\beta, \quad j = 1,\ 2, \ldots, 2n,$

(iii) $(\Psi_{\alpha\beta *}\partial_\alpha)\nu = 0,$

(iv) $(\Psi_{\alpha\beta *}\partial_\alpha)\tau_\beta = \widehat{\xi}_\beta.$

Proof. For the first identity, we apply the definiton of the transformation (3.22) and we see

$$(\Psi_{\alpha\beta *}\partial_\alpha)f = \Psi^{-1*}_{\alpha\beta}\partial_\alpha(\varphi^*_{\alpha\beta}f) = \varphi^{-1*}_{\alpha\beta}\, d\varphi^*_{\alpha\beta}f = df.$$

For (ii), we notice $Z^j = z^{j\#}_\beta - z^j_\beta$, $j = 1, \ldots, 2n$ and we calculate

$$(\Psi_{\alpha\beta *}\partial_\alpha)Z^j = \Psi^{-1*}_{\alpha\beta}\partial_\alpha\Psi^*_{\alpha\beta}(z^{j\#}_\beta - z^j_\beta) = -\Psi^{-1*}_{\alpha\beta}\partial_\alpha\Psi^*_{\alpha\beta}z^j_\beta = -dz^j_\beta$$

since $\Psi^*_{\alpha\beta}z^{j\#}_\beta$ is a Weyl function and then vanished by ∂_α. The third one is obvious. As to (iv), we notice the identity in Proposition 3.3 and we have

$$(\Psi_{\alpha\beta *}\partial_\alpha)\tau_\beta = \Psi^{-1*}_{\alpha\beta}\partial_\alpha(\tau_\alpha + f^\#_{\alpha\beta} - \widehat{h}_{\alpha\beta}) = \Psi^{-1*}_{\alpha\beta}(\widehat{\xi}_\alpha - d\widehat{h}_{\alpha\beta}).$$

Hence Lemma 3.2 (i) gives $\Psi^{-1*}_{\alpha\beta}(\widehat{\xi}_\alpha - d\widehat{h}_{\alpha\beta}) = \varphi^*_{\beta\alpha}\widehat{\xi}_\alpha + d\widehat{h}_{\beta\alpha}$. Thus, Lemma 3.8 yields the desired equation. Q.E.D.

Now, by virtue of Proposition 3.12 we define

Definition 3.14. We set a globally defined derivation

$$\partial\colon \Gamma(\Lambda^p_M \otimes C_M) \to \Gamma(\Lambda^{p+1}_M \otimes C_M)$$

by

$$\partial F = \Psi^{-1}_{\alpha *}\partial_\alpha F = \Psi^*_\alpha\partial_\alpha\Psi^{-1*}_\alpha F, \quad F \in \Gamma(\Lambda_M \otimes C_M).$$

Now, we consider the curvature of the covariant exterior derivative ∂. From Lemma 3.11, we have

Theorem 3.15.

(i) $\partial^2|_{W_{U_\alpha}} = 0.$

(ii) *A section* $F \in \Gamma(W_M)$ *satisfies* $\partial F = 0$ *if and only if* $F \in \mathcal{F}(W_M)$.
(iii) $\partial^2 = \mathrm{ad}\left(\frac{1}{\nu}\Omega_M(\nu^2)\right)$, *i.e., the curvature form of* ∂ *is equal to* $\Omega_M(\nu^2)$.

As a corollary of the theorem, we have

Corollary 3.16.
(i) *The restriction* $\partial|_{W_M}$ *is a Fedosov connection.*
(ii) *The curvature of the connection* ∂ *is given by the adjoint of a 2-form which is a curvature form of Fedosov connection.*
(iii) *The Poincaré-Cartan class is equal to the cohomology class of Fedosov connection*; $[\Omega_M(\nu^2)] = c(W_M)$.

The statements (ii) and (iii) are direct consequences of the above theorem and (i). A proof of (i) will be given in the next section.

§4. Weyl charts and Classical charts

Recall that a Weyl manifold is obtained by gluing locally trivial bundles $\{W_{U_\alpha}\}$ with Weyl diffeomorphisms $\Phi_{\alpha\beta}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ and local trivializations $\Phi_\alpha\colon W_{V_\alpha} \to W_{U_\alpha}$ such that $\Phi_\beta\Phi_\alpha^{-1} = \Phi_{\alpha\beta}$ are called local Weyl charts.

Let $\Psi_\alpha\colon C_{V_\alpha} \to C_{U_\alpha}$ be a locally trivialization of C_M given in §3.1 such that the restriction $\Phi_\alpha = \Psi_\alpha|_{W_{U_\alpha}}$ is a local Weyl chart. For simplicity, we also call $\Psi_\alpha\colon C_{V_\alpha} \to C_{U_\alpha}$ a local **Weyl chart** of C_M.

In this section, we introduce another system of local trivializations of C_M, called *classical charts* of C_M. We also obtain an expression of the connection ∂ with respect to classical charts, which shows the restriction to a Weyl manifold $\partial|_{W_M}$ gives Fedosov connection explicitly.

4.1. Basic Lemma

The essential part of the contruction of classical charts depends on the following lemma for a ν-automorphism of the contact algebra C.

Before stating the basic lemma, we remark first the following. For a ν-automorphism $\Psi\colon C \to C$, there exist $t(\nu^2) \in \mathbf{R}[[\nu^2]]$ and $T \in W_3^{\circ+}$ such that

$$\Psi(\tau) = \tau + T + t(\nu^2)$$

due to Proposition 2.18. We also have the converse direction.

Lemma 4.1. *For* $T \in W_3^{\circ+}$, $t(\nu^2) \in \mathbf{R}[[\nu^2]]$, *there exist uniquely* $F \in W_3^{\circ+}$ *and* $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ *such that*

$$\exp \mathrm{ad}\Big(\frac{1}{\nu}F + \frac{1}{\nu}c(\nu^2)\Big)(\tau) = \tau + T + t(\nu^2).$$

Proof. Notice $\left[\frac{1}{\nu}F_{(3)}, \tau\right] = -F_{(3)}$ for an element $F_{(3)}$ of homogeneous degree 3. Write $T = T_{(3)} + O(4)$ where $T_{(3)}$ is the terms of homogenous degree 3. Then putting $F_{(3)} = T_{(3)}$ we have

$$\exp \operatorname{ad}\Big(\frac{1}{\nu}F_{(3)}\Big)(\tau + T + t(\nu^2)) = \tau + T_4 + t'(\nu^2),$$

where $T_4 \in W_4^{\circ+}$, $t'(\nu^2) \in \mathbf{R}[\nu^2]]$. Similarly we put $T_4 = T_{(4)} + O(5)$ where $T_{(4)}$ is the term of homogeneous degree 4. Then taking a certain $F_{(4)} \in W_4^{\circ+}$ of homogeneous degree 4, we can eliminate $T_{(4)}$ by means of $\exp \operatorname{ad}\big(\frac{1}{\nu}F_{(4)}\big)(\tau + T_4 + t'(\nu^2)) = \tau + T_5 + t''(\nu^2)$ where $T_5 \in W_5^{\circ+}$, $t''(\nu^2) \in \mathbf{R}[\nu^2]]$. Repeating this procedure with the Campbell-Hausdorff formula gives

$$\exp \operatorname{ad}\Big(-\frac{1}{\nu}F\Big)(\tau + T + t(\nu^2)) = \tau + \tilde{t}(\nu^2)$$

for certain $F \in W_3^{\circ+}$ and $\tilde{t}(\nu^2) \in \mathbf{R}[[\nu^2]]$. By the similar argument as in the proof of Proposition 2.18, there exists $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ such that $\exp \operatorname{ad}\big(-\frac{1}{\nu}c(\nu^2)\big)(\tau + \tilde{t}(\nu^2)) = \tau$. Hence $\exp \operatorname{ad}\big(\frac{1}{\nu}F + \frac{1}{\nu}c(\nu^2)\big)$ is the desired ν-automorphism. The uniqueness is obtained by the similar argument in the proof of Proposition 2.18. Q.E.D.

4.2. Section $\widehat{\tau} \in \Gamma(C_M)$ and classical charts

Regarding τ as a constant section for each C_{U_λ}, we set a global section of C_M by

$$\widehat{\tau} = \sum_\lambda \chi_\lambda \Psi_\lambda^* \tau \tag{4.1}$$

where $\{\chi_\lambda\}_\lambda$ is a partition of unity. On each local Weyl chart, we set the local expression as

$$\widehat{\tau}_\mu = \Psi_\mu^{-1*}\widehat{\tau} = \sum_\lambda \varphi_\mu^{-1*}\chi_\lambda \Psi_\mu^{-1*}\Psi_\lambda^*\tau \in \Gamma(C_{U_\mu}). \tag{4.2}$$

Since $\Psi_\mu^{-1*}\Psi_\lambda^*\tau = \Psi_{\mu\lambda}^*\tau$, as we see at the begining of the previous subsection there exist $t_\mu(\nu^2) = \sum_{k=0}^\infty t_\mu^{(2k)}\nu^{2k} \in \mathfrak{a}_{\nu^2}(U_\mu)$ and $T_\mu \in \Gamma(U_\mu \times W_3^{\circ+})$ such that $\widehat{\tau}_\mu = \tau + t_\mu(\nu^2) + T_\mu$. Hence, by Lemma 4.1 there exists an algebra bundle isomorphism $\psi_\mu \colon C_{U_\mu} \to C_{U_\mu}$ such that $\psi_\mu^*\tau = \widehat{\tau}_\mu$. Thus, we have a local trivialization

$$\widetilde{\Psi}_\mu = \psi_\mu \circ \Psi_\mu \colon C_{V_\mu} \to C_{U_\mu} \tag{4.3}$$

giving a contact algebra isomorphism at each fiber and satisfying $\widetilde{\Psi}^*\tau = \widehat{\tau}$. We remark here the algebra bundle isomorphism ψ_μ does not give a Weyl diffeomorphism when restricted to Weyl algebra bundle W_{U_μ} in general. Thus a local trivialization $\widetilde{\Psi}_\mu \colon C_{V_\mu} \to C_{U_\mu}$ is not necessarily a Weyl chart, and hence the algebra of local Weyl functions are not preserved by the transformation $\widetilde{\Psi}_\mu \circ \widetilde{\Psi}_\lambda^{-1}$.

Definition 4.2. The local trivialization $\widetilde{\Psi}_\mu \colon C_{V_\mu} \to C_{U_\mu}$ is referred to as a **classical chart**.

4.3. Expression of ∂ in the classical charts

In this section we will give the explicit form of $\widetilde{\Psi}_{\mu *}\partial$.
We denote transition functions between classical charts by

$$\widetilde{\Psi}_{\lambda\mu} = \widetilde{\Psi}_\mu \circ \widetilde{\Psi}_\lambda^{-1} \colon C_{U_{\lambda\mu}} \to C_{U_{\mu\lambda}}, \quad U_{\lambda\mu} = \varphi_\lambda(V_\lambda \cap V_\mu).$$

We consider the transformation rule for $\widetilde{\Psi}_{\lambda\mu}^*\tau$ and $\widetilde{\Psi}_{\lambda\mu}^* Z^i$, $i = 1, \dots, 2n$. By the identities (4.2) and (4.3), the constant section $\tau \in \Gamma(C_{U_\lambda})$ is transfered to $\widehat{\tau}$, $\widetilde{\Psi}_\lambda^*\tau = \widehat{\tau}$ which yields $\widetilde{\Psi}_{\lambda\mu}^*\tau = \tau$. We put the expansion

$$\widetilde{\Psi}_{\lambda\mu}^* Z^i = \sum_j A^i_{\lambda\mu,j} Z^j + G_{(2)} + G_{(3)} + \cdots + G_{(k)} + \cdots,$$

where $A^i_{\lambda\mu,j} \in \mathfrak{a}_{\nu^2}(U_{\lambda\mu})$, $i, j = 1, \dots, 2n$ give a symplectic matirx at each point of $U_{\lambda\mu}$ and $G_{(k)} \in \Gamma(W^+_{U_{\lambda\mu}})$ is the terms of homogeneous degree k. Applying $\widetilde{\Psi}_{\lambda\mu}^*$ to the identities $[\tau, Z^i] = \nu Z^i$, $i = 1, \dots, 2n$ gives

$$\nu \sum_j A^i_{\lambda\mu,j} Z^j + \cdots + k\nu G_{(k)} + \cdots = \nu \sum_j A^i_{\lambda\mu,j} Z^j + \cdots + \nu G_{(k)} + \cdots$$

since $[\tau, G_{(k)}] = k\nu G_{(k)}$, which shows $G_{(k)} = 0$ for $k \geq 2$. Then we have a transformation formula of classical charts

$$\widetilde{\Psi}_{\lambda\mu}^*\nu = \nu, \quad \widetilde{\Psi}_{\lambda\mu}^* Z^i = \sum_j A^i_{\lambda\mu,j} Z^j, \quad \widetilde{\Psi}_{\lambda\mu}^*\tau = \tau. \tag{4.4}$$

In what follows, we see the functions $A^i_{\lambda\mu,j}$ is expressed by means of the symplectic transformation $\varphi_{\lambda\mu}$. Notice the Weyl chart transformation $\Psi_{\lambda\mu}$ and classical chart transformation $\widetilde{\Psi}_{\lambda\mu}$ have the relation

$$\widetilde{\Psi}_{\lambda\mu}\psi_\lambda = \psi_\mu \Psi_{\lambda\mu}. \tag{4.5}$$

We put the expansion

$$\psi_\mu^* Z^i = Z^i + \frac{1}{2}\sum_{j_1,j_2} B^i_{\mu,j_1j_2} Z^{j_1} Z^{j_2} + O(3) \tag{4.6}$$

where $B^i_{\mu,j_1j_2} \in C^\infty(U_\mu)$ and $O(3)$ indicates the terms of $\Gamma(U_{U_\mu} \times W_3^+)$. As for calculating $\Psi^*_{\lambda\mu} Z^i$ we remark $Z^i = z^{i\#}_\mu - z^i_\mu$ on U_μ and $\Psi^*_{\lambda\mu} z^{i\#}_\mu = \psi^{i\#}_{\lambda\mu} + \nu^2 g^{i\#}$ for some $g^{i\#} \in \mathcal{F}(W_{U_{\lambda\mu}})$. Then we have

$$\begin{aligned} \Psi^*_{\lambda\mu} Z^i &= \Psi^*_{\lambda\mu}(z^{i\#}_\mu - z^i_\mu) \\ &= \sum_j \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^j_\lambda} Z^j + \frac{1}{2}\sum_{j_1,j_2} \frac{\partial^2 \varphi^i_{\lambda\mu}}{\partial z^{j_1}_\lambda \partial z^{j_2}_\lambda} Z^{j_1} Z^{j_2} + O(3). \end{aligned} \tag{4.7}$$

Using the identities (4.6) and (4.7) for $\Psi^*_{\lambda\mu}\psi^*_\mu Z^i$ we have

$$\Psi^*_{\lambda\mu}\psi^*_\mu Z^i = \sum_j \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^j_\lambda} Z^j + \frac{1}{2}\sum_{j_1,j_2} C^i_{\lambda\mu,j_1j_2} Z^{j_1} Z^{j_2} + O(3) \tag{4.8}$$

where

$$C^i_{\lambda\mu,j_1j_2} = \frac{\partial^2 \varphi^i_{\lambda\mu}}{\partial z^{j_1}_\lambda \partial z^{j_2}_\lambda} + \sum_{lm} \varphi^*_{\lambda\mu} B^i_{\mu,lm} \frac{\partial \varphi^l_{\lambda\mu}}{\partial z^{j_1}_\lambda} \frac{\partial \varphi^m_{\lambda\mu}}{\partial z^{j_2}_\lambda}. \tag{4.9}$$

Also we calculate

$$\psi^*_\lambda \widetilde{\Psi}^*_{\lambda\mu} Z^i = \sum_l A^i_{\lambda\mu,l}\left(Z^l + \frac{1}{2}\sum_{j_1,j_2} B^l_{\lambda,j_1j_2} Z^{j_1} Z^{j_2}\right) + O(3), \tag{4.10}$$

which shows $A^i_{\lambda\mu,j} = \partial \varphi^i_{\lambda\mu}/\partial z^j_\lambda$.

Thus we have the transformation formula of classical charts:

Lemma 4.3.

$$\widetilde{\Psi}^*_{\lambda\mu}\nu = \nu, \quad \widetilde{\Psi}^*_{\lambda\mu} Z^i = \sum_j \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^j_\lambda} Z^j, \quad \widetilde{\Psi}^*_{\lambda\mu}\tau = \tau.$$

Remark 4.4. Substituting (4.8), (4.10) into (4.5) shows B_{λ,j_1j_2}'s are transformed as

$$\sum_{j=l}^{2n} \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^l_\lambda} B^l_{\lambda,j_1j_2} = \frac{\partial^2 \varphi^i_{\lambda\mu}}{\partial z^{j_1}_\lambda \partial z^{j_2}_\lambda} + \sum_{l,m=1}^{2n} \varphi^*_{\lambda\mu} B^i_{\mu,lm} \frac{\partial \varphi^l_{\lambda\mu}}{\partial z^{j_1}_\lambda} \frac{\partial \varphi^m_{\lambda\mu}}{\partial z^{j_2}_\lambda}.$$

4.4. Expression of ∂ in classical charts

In this subsection, we give a local expression of the connection with respect to the classical chart.

By the Definition 3.14, the local expression of the connection in the Weyl chart is given by (3.17), $\Psi_{\lambda *}\partial = \partial_\lambda = d - \delta_\lambda + \mathrm{ad}\left(\frac{1}{\nu}\kappa_\lambda(\nu^2)\right)$, where $\delta_\lambda = \mathrm{ad}\left(\frac{1}{\nu}\sum_{ij} dz^i_\lambda \omega_{ij} Z^j\right)$ and $\kappa_\lambda(\nu^2)$ is given in Lemma 3.10. According to the definition (4.3), the local expression in the classical chart is given by $\widetilde{\partial}_\lambda = \widetilde{\Psi}_{\lambda *}\partial = \psi_{\lambda *}\partial_\lambda = \psi_\lambda^{-1*}\partial_\lambda \psi_\lambda^*$. In what follows, we will calculate $\widetilde{\partial}_\lambda f$ for $f \in C^\infty(U_\lambda)$, $\widetilde{\partial}_\lambda \tau$, $\widetilde{\partial}_\lambda \nu$ and $\widetilde{\partial}_\lambda Z^i$ for $i = 1, 2, \ldots, 2n$, in order to determine the form of $\widetilde{\partial}_\lambda$.

Since f belongs to the center of $\Gamma(W_{U_\lambda})$, we have $\widetilde{\partial}_\lambda f = df$. It is obvious that $\widetilde{\partial}_\lambda \nu = 0$ by the same reason.

4.4.1. Calculation of $\widetilde{\partial}_\lambda Z^i$. In this subsection, we will determine the form $\widetilde{\partial}_\lambda$ up to the central component, which calculus is mainly given by $\widetilde{\partial}_\lambda Z^i$.

Now we put $\Gamma^i_{\lambda,jk} = B^i_{\lambda,jk}$. Then the identity in Remark 4.4 means $\{\Gamma^i_{\lambda,jk}\}$ defines a connection ∇. Substituting (4.6) into $[\psi_\lambda^* Z^i, \psi_\lambda^* Z^i] = \nu\Lambda^{ij}$ induces $\sum_l \Lambda^{il} B^j_{\lambda,lk} = \sum_l \Lambda^{jl} B^i_{\lambda,lk}$ which means ∇ is a symplectic connection on M. Hence we have

Lemma 4.5. *The expansion* (4.6) *gives*

$$\psi_\lambda^* Z^i = Z^i + \frac{1}{2}\sum_{j_1,j_2} \Gamma^i_{\lambda,j_1 j_2} Z^{j_1} Z^{j_2} + O(3), \tag{4.11}$$

where $\Gamma_{\mu,j_1 j_2}$ is the Chirstofell symbol of a symplectic connection ∇ with respect to a canonical coordinate $(z^1_\lambda, \ldots, z^{2n}_\lambda)$.

Notice $\widetilde{\partial}_\lambda Z^i = \psi_\lambda^{-1*}\partial_\lambda \psi_\lambda^* Z^i$. Then Lemma 4.5 induces

$$\widetilde{\partial}_\lambda Z^i = -dz^i_\lambda - \sum_{jk} \Gamma^i_{\lambda,jk}\, dz^j_\lambda Z^k + \frac{1}{2}\sum_{jj_1j_2} S^i_{\lambda,jj_1j_2}\, dz^j_\lambda Z^{j_1} Z^{j_2} + O(3) \tag{4.12}$$

where $S^i_{\lambda,jj_1j_2} = \partial\Gamma^i_{\lambda,j_1j_2}/\partial z^j_\lambda + \sum_m \Gamma^i_{\lambda,jm}\Gamma^m_{\lambda,j_1j_2}$.

Set $\widetilde{L}^i_{\lambda,2} = \frac{1}{2}\sum_{jj_1j_2} S^i_{\lambda,jj_1j_2}\, dz^j_\lambda Z^{j_1} Z^{j_2} + O(3)$ and apply $\widetilde{\partial}_\lambda$ to the identity $[Z^i, Z^j] = \nu\Lambda^{ij}$. Then, we have $[\widetilde{\partial}_\lambda Z^i, Z^j] + [Z^i, \widetilde{\partial}_\lambda Z^j] = 0$, which yields $[Z^i, \widetilde{L}^j_{\lambda,2}] = [Z^j, \widetilde{L}^i_{\lambda,2}]$. The Poincaré Lemma for formal power seires gives the unique section γ_λ in the space $\Gamma(\Lambda^1(U_\lambda)\otimes(W^{+\circ}_{3,U_\lambda}))$ such that

$$\widetilde{L}^i_{\lambda,2} = \frac{1}{\nu}\,\mathrm{ad}(\gamma_\lambda)(Z^i) \tag{4.13}$$

where $W^{+\circ}_{3,U_\lambda} = U_\lambda \times W^{+\circ}_3$.

The transformation rule in Lemma 4.3 shows the term $\sum_{ij} dz^i_\lambda \omega_{ij} Z^j$ given in classical charts defines a global section of C_M and hence we have a globally defined fiberwise derivation δ of C_M such that

$$\widetilde{\Psi}_{\lambda *}\delta = \widetilde{\delta}_\lambda = \frac{1}{\nu}\operatorname{ad}\Big(\sum_{ij} dz^i_\lambda \omega_{ij} Z^j\Big). \tag{4.14}$$

Notice $\widetilde{\delta}_\lambda Z^i = -dz^i_\lambda$ on C_{U_λ}. Also we can extend the classical connection ∇ as a globally defined derivation of $\Gamma(W_M)$ by

$$\widetilde{\Psi}_{\lambda *}\nabla Z^i = -\sum_m \Gamma^i_{\lambda,jk}\, dz^j_\lambda Z^k. \tag{4.15}$$

Thus, in terms of (4.14) and (4.15), the connection is expressed as

$$\widetilde{\Psi}_{\lambda *}\partial = \widetilde{\partial}_\lambda = \nabla - \widetilde{\delta}_\lambda + \frac{1}{\nu}\operatorname{ad}(\gamma_\lambda)$$

on the classical chart W_{U_λ}.

4.4.2. Proof of Corollary 3.16. Notice the transformation $\widetilde{\Psi}_{\lambda\mu *}\widetilde{\partial}_\lambda = \widetilde{\partial}_\mu$, $\widetilde{\Psi}_{\lambda\mu *}\widetilde{\delta}_\lambda = \widetilde{\delta}_\mu$. Since the section γ_λ is unique for each W_{U_λ}, we have the identity $\widetilde{\Psi}^*_{\lambda\mu}\gamma_\mu = \gamma_\lambda$. Thus, we have

Proposition 4.6. *There exists a section γ of W_M such that $\gamma = \widetilde{\Psi}^*_\lambda \gamma_\lambda$ and*

$$\partial|_{W_M} = \nabla - \delta + \frac{1}{\nu}\operatorname{ad}(\gamma),$$

where ∇, δ and γ_λ are given by (4.15), (4.14) *and* (4.13), *respectively.*

Now we are in a position to prove Corollary 3.16. By Theorem 3.15, $\partial^2 = 0$ shows $\partial|^2_{W_M} = 0$ where $\partial|_{W_M}$ is of the form in Proposition 4.6 and hence the restricted connection $\partial|_{W_M}$ is a Fedosov connection.

4.4.3. Central components of ∂. We proceed to determine the shape of ∂. First we remark on the classical chart C_{U_λ}

$$\widetilde{\Psi}_{\lambda *}\partial = \widetilde{\partial}_\lambda = \nabla - \widetilde{\delta}_\lambda + \operatorname{ad}\Big(\frac{1}{\nu}\gamma_\lambda\Big) + \operatorname{ad}\Big(\frac{1}{\nu}\widetilde{\sigma}_\lambda(\nu^2)\Big)$$

for certain central section $\widetilde{\sigma}_\lambda(\nu^2) \in \mathfrak{a}_{\nu^2}(U_\lambda)$. In fact, on classical chart C_{U_λ}, we consider the difference

$$\widetilde{\partial}^\circ_\lambda = \widetilde{\partial}_\lambda - \nabla + \widetilde{\delta}_\lambda - \operatorname{ad}\Big(\frac{1}{\nu}\gamma_\lambda\Big)$$

Since $\widetilde{\partial}_\lambda^\circ Z^i = 0$ for $i = 1, \dots, 2n$, applying $\widetilde{\partial}_\lambda^\circ$ to the identity $[\tau, Z^i] = \nu Z^i$ shows $\widetilde{\partial}_\lambda^\circ \tau$ is a central section. Thus, we see $\widetilde{\partial}_\lambda^\circ \tau = \rho_\lambda(\nu^2)$ for certain $\rho_\lambda(\nu^2) \in \mathfrak{a}_{\nu^2}(U_\lambda)$. Similary as in the proof of Lemma 3.10, we can take $\widetilde{\sigma}_\lambda(\nu^2) \in \mathfrak{a}_{\nu^2}(U_\lambda)$ such that

$$\mathrm{ad}\Big(\frac{1}{\nu}\widetilde{\sigma}_\lambda(\nu^2)\Big)\tau = \rho_\lambda(\nu^2).$$

Notice the connection is written as $\widetilde{\Psi}_{\lambda *}\nabla = d + \mathrm{ad}\big(\frac{1}{\nu}\Gamma_{\lambda,(2)}\big)$ where $\Gamma_{\lambda,(2)} = \frac{1}{2}\sum_m \omega_{jm}\Gamma^m_{\lambda,kl} Z^j Z^k dz^l_\lambda$. Thus, one can easily check $\nabla\tau = 0$. Then, we have

$$\widetilde{\partial}_\lambda \tau = -\sum_{ij} dz^i_\lambda \omega_{ij} Z^j + 2\gamma_\lambda - [\tau, \gamma_\lambda] + 2\widetilde{\sigma}_\lambda(\nu^2) - [\tau, \widetilde{\sigma}_\lambda(\nu^2)].$$

On the other hand, we calculate as

$$\begin{aligned}
\psi_\lambda^{*-1}\partial_\lambda \psi_\lambda^* \tau &= \psi_\lambda^{*-1}\partial_\lambda(\tau + t_\lambda(\nu^2) + T_\lambda) \\
&= \psi_\lambda^{*-1}\partial_\lambda \tau + dt_\lambda(\nu^2) + \psi_\lambda^{*-1}\partial_\lambda T_\lambda \\
&= -\sum_{ij} dz^i_\lambda \omega_{ij} \psi_\lambda^{*-1} Z^j + \widehat{\kappa}_\lambda(\nu^2) + dt_\lambda(\nu^2) + \psi_\lambda^{*-1}\partial_\lambda T_\lambda.
\end{aligned}$$

Comparing the central terms of the both equation we see

$$\begin{aligned}
&2\widetilde{\sigma}_\lambda(\nu^2) - [\tau, \widetilde{\sigma}_\lambda(\nu^2)] \\
&\quad = -\Big(\sum_{ij} dz^i_\lambda \omega_{ij} \psi_\lambda^{*-1} Z^j\Big)^\circ + \widehat{\kappa}_\lambda(\nu^2) + dt_\lambda(\nu^2) + (\psi_\lambda^{*-1}\partial_\lambda T_\lambda)^\circ
\end{aligned}$$

where N° means the central terms of N of $N \in \Gamma(C_{U_\lambda})$. Similary as in the proof of Lemma 3.10, the equation above gives the unique $\widetilde{\sigma}_\lambda(\nu^2)$.

References

[AG] V. I. Arnol'd and A. B. Givental, Symplectic Geometry; Dynamical Systems IV, Encyclopaedia of Mathematical Sciences, **4**, Springer-Verlag, 1990.

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowizc and D. Sternheimer, *Deformation theory and quantization*, Ann. Physics, **111** (1978), 61–111.

[F] B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom., **40** (1994), 213–238.

[K] M. Kontsevich, *Deformation quantization of Poisson manifolds* I, q-alg/9709040.

[DL] M. De Wilde and P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys., **7** (1983), 487–496.

[G] S. Gutt, *Cohomology associated with the Poisson Lie algebra*, Lett. Math. Phys., **3** (1979), 297–310.

[L] A. Lichnerowizc, *Déformaions d'algèbres associèes à une variété symplectique (les $*_\nu$-products)*, Ann. Inst. Fourier, Grenoble **32** (1982), 157–209.

[OMY1] H. Omori, Y. Maeda and A. Yoshioka, *Weyl manifolds and Deformation quantization*, Adv. Math., **85** (1991), 224–255.

[OMY2] H. Omori, Y. Maeda and A. Yoshioka, *Global calculus on Weyl manifolds*, Japanese. J. Math., **17** (1991), 57–82.

[OMMY1] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Poincaré-Cartan class and deformation quantization of Kähler manifolds*, Comm. Math. Phys., **194** (1998), 207–230.

[T] B. Tsygan, *Local Forumula for the index of a Fourier integral operator*, DQ/0004022.

[Y1] A. Yoshioka, *Contact structures on Weyl manifolds and Deformation quantization*, Proceedings of the conference on "Dynamical systems and differential geometry", RIMS Kokyuroku, **1119** (1999), 1–17, (in Japanese).

[Y2] A. Yoshioka, *Weyl manifold and quantized connection*, Proceedings of the Workshop "Towards 100 years after Sophus Lie" (Kazan, 1998), Lobachevskii J. Math., **4** (1999), 177–206, (electronic).

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