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Isometric Immersions into Complex Projective Space

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In the theory of isometric immersions of submanifolds there are fundamental results of John Nash for the C^∞ case and Burstin-Cartan-Janet-Schläfly for the analytic case (also see Robert Greene [G] for the case of local isometric immersions). However, these theorems require a large codimension and are of practically no help in considering concrete questions in low codimensions.

An obvious way of producing large varieties of isometrically immersed homogeneous submanifolds is to take the orbits of Lie group actions. In low codimensions the following result should often be true:

Let a compact, connected Lie group G act on the connected Riemannian manifold N with principal orbit type $M = G/H$. Then, among all the G-homogeneous metrics on G/H the only ones which allow an isometric immersions into N are the ones which are already realized as the orbit metrics of this action.

Obviously this is true for the geodesic spheres $S^{n-1}(r)$ of $\mathbb{R}^n$ under the standard action of $SO(n)$. With a little work it is also easy to prove for the larger classes of metrics invariant under the unitary or symplectic groups. A somewhat more challenging example is to prove this for the second Stiefel manifold $SO(n)/SO(n-2)$ of $\mathbb{R}^{2n}$ under the diagonal embedding $SO(n) \to SO(n) \times SO(n)$ acting on $\mathbb{R}^{2n}$.

In this paper we obtain far reaching generalizations of these results. First, we wish to consider isometric immersions into other homogeneous spaces than Euclidean (and spherical or hyperbolic) spaces. Secondly, we wish to include all dimensions. It is clear that the low dimensions are going to be considerably harder, since the main technique for using the Gauss equation fails in the lowest dimensions. We work out here the concrete case of isometric immersions of spheres $S^{2n-1} = U(n)/U(n-1)$

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into $N = \mathbb{C}P(n)$. The techniques developed here should be quite useful in many other homogeneous spaces, also. In particular, quite analogous results hold true for complex hyperbolic space and quaternionic projective and hyperbolic spaces, and will be published soon.

We also note that our results have a bearing on the fundamental theorem for hypersurfaces. This theorem states that for Euclidean (or spherical/hyperbolic) spaces N, any codimension one manifold M with given candidates for the first and second fundamental forms allow a local isometric immersion into N if and only if those forms satisfy the Gauss and Codazzi equations; moreover, this immersion is unique up to deformation by an isometry of N. Now for isometric embeddings of S^3 into $\mathbb{C}P(2)$ we get far more solutions to the Gauss equation than those which actually come from submanifolds. Most of those solutions are ruled out by the Codazzi equations, but some are not. Hence, the fundamental theorem cannot be generalized that simply to $\mathbb{C}P(n)$. One also need a condition that the curvature tensor of the total space N is parallel. Also see Eschenburg-Tribuzy [ET] about this question.

§1 These theorems follow from a careful study of the Gauss equation: $\langle \hat{R}(X \wedge Y)Z,\ W\rangle = B(X,W)B(Y,Z) - B(X,Z)B(Y,W)$ where $\hat{R}$ is the curvature operator of the submanifold M and B is the second fundamental form (for the case of a Euclidean surrounding space). In the case of a non-Euclidean surrounding space, however, the Gauss equation reads: $\langle R^t(X \wedge Y)Z, W\rangle = B(X,Z)B(Y,W) - B(X,W)B(Y,Z)$ where $R^t = \bar{R}^t - \hat{R}$, $\bar{R}^t$ is the part of the curvature operator $\bar{R}$ of N tangential to M. This will vary with M's position in N, so in this case the left hand side of the above equation is also unknown when M and its first and second fundamental forms are specified. We are not aware of much study of this situation in the literature; however, we present here the complete solution of the probably most basic question in isometric immersions in the case of non-classical geometries.

Let $N = \mathbb{C}P(n)$ with the metric normalized such that the sectional curvatures are in $[1,4]$. Let the metric of the geodesic sphere $S^{2n-1}(r) = U(n)/U(n-1)$ be γ_r, $r \in (0, \frac{\pi}{2})$ defined as follows: The isotropy representation of $U(n-1)$ on $T_pS^{2n-1}(r) = \mathbb{R} \oplus \mathbb{R}^{2n-2} = \mathbb{R} \oplus \mathbb{C}^{n-1}$ is given, as on p. 12, by $\theta \oplus \mu_{n-1}$ where θ is the trivial representation and μ_{n-1} is the standard representation. Consequently, the inner product at p of any $U(n)$-invariant metric is given by two real, positive scalars a and b, which represent the stretching factors from the standard metric on $\mathbb{R}^{2n-1} = \mathbb{R} \oplus \mathbb{R}^{2n-2}$ for the two summands (by Schur's lemma). For the Berger metric sphere $S^{2n-1}(r)$ we have $a = \sin r \cos r$ and $b = \sin r$; hence $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$. We now call $S^{2n-1}(r)$ with this metric $S^{2n-1}_{a,b}$.

Proposition 1. $S^{2n-1}_{a,b} = U(n)/U(n-1)$ *is isometric to a geodesic sphere* $S^{2n-1}(r)$ *in* $\mathbb{C}P(n)$ *iff* $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$, *in* $\mathbb{R}^{2n}$ *iff* $\frac{1}{b^2} - \frac{a^2}{b^4} = 0$ *and in complex hyperbolic space* $\mathbb{C}H(n)$ *with metric normalized such that sectional curvatures lie in* $[-4,-1]$ *iff* $\frac{1}{b^2} - \frac{a^2}{b^4} = -1$.

Proof. Let $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$, then $0 < b < 1$, and we set $b = \sin r$, $r \in (0, \frac{\pi}{2})$, the result then follows from the above observation. In the second case we have that $a = b$; hence this is the Euclidean case. In the third case we have $\frac{a^2}{b^4} > 1$; hence, setting $\frac{a}{b^2} = \coth r$; then $b = \sinh r$, i.e. $a = \sinh r \cosh r$. This is exactly the geodesic spheres in $\mathbb{C}H(n)$ with metric normalized as above. q.e.d.

Remark. If $\frac{1}{b^2} - \frac{a^2}{b^4} = t > 0$, this corresponds to a geodesic sphere in a $\mathbb{C}P(n)$ with a homothetic metric, similarly for $\frac{1}{b^2} - \frac{a^2}{b^4} = -t < 0$, this corresponds to a geodesic sphere in a $\mathbb{C}H(n)$ with a homothetic metric.

We now have an interpretation of all $U(n)$-invariant metrics on S^{2n-1}. We note that in $\mathbb{C}P(n)$ and $\mathbb{C}H(n)$ the geodesic spheres $S^{2n-1}(r)$ for different r determine distinct homothety classes of metrics, whereas in $\mathbb{R}^{2n}$ all such metrics are homothetic.

Our main result, here proved for $\mathbb{C}P(n)$, is the following:

Theorem 2. *The Berger metrics* γ_r *are the only* $U(n)$*-invariant metrics on* S^{2n-1} *which allow an isometric immersion into* $\mathbb{C}P(n)$.

Remark. We only need to prove: $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$. Obviously, homothetic metrics admit an isometric immersion into $\mathbb{C}P(n)$ with a homothetic metric. Thus, the set of $U(n)$-homogeneous metrics admitting an isometric immersion into a given $\mathbb{C}P(n)$, is a one-dimensional subset of the two-dimensinal variety of all $U(n)$-homogeneous metrics.

§2 Let V be an inner product space. Let $\mathcal{R}$ be the linear space of all curvature-like 4-tensors on V; i.e. $R \in \mathcal{R}$ iff $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$; $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$ (it then follows that $R(X,Y,Z,W) = R(Z,W,X,Y)$). Let B be a symmetric 2-tensor on V, then the 4-tensor $B \wedge B$ on V defined by $B \wedge B(X,Y,Z,W) = B(X,Z)B(Y,W) - B(X,W)B(Y,Z)$ is a curvature-like tensor. Also, note that $B \wedge B(X,Y,Z,W) = \langle \hat{B} \wedge \hat{B}(X \wedge Y), Z \wedge W \rangle$, with $\hat{B}$ defined as follows:

Let $\hat{B}$ be the symmetric linear operator on V defined by $B(X,Y) = \langle \hat{B}(X), Y \rangle$ (the shape operator corresponding to B). Let the 2-tensor $\hat{B} \wedge \hat{B}$ be defined by $\hat{B} \wedge \hat{B}(X \wedge Y) = \hat{B}(X) \wedge \hat{B}(Y)$. The inner product on $\bigwedge^2(V)$ is defined in the usual way.

Now, for a point $p \in S^{2n-1}_{a,b} \subset \mathbb{C}P(n)$ let $\bar{R}^t$ be the orthogonal projection of the curvature operator of $\mathbb{C}P(n)$ restricted to $T_pS^{2n-1}_{a,b}$ and let $\hat{R}$ be the curvature operator of $S^{2n-1}_{a,b}$. The Gauss equation, $R^t(X,Y,Z,W) = \bar{R}^t(X,Y,Z,W) - \hat{R}(X,Y,Z,W) = B(X,Z)B(Y,W) - B(X,W)B(Y,Z)$, may then be written $R^t(X \wedge Y) = (\hat{B} \wedge \hat{B})(X \wedge Y)$, where B is the second fundamental form. The fact that $R^t = \hat{B} \wedge \hat{B}$ has surprisingly strong consequences. It is already sufficient to prove Theorem 2 for $n \geq 4$, which we do in §4. This is based on the following result:

Proposition 3 (see also Agaoka [A]). *$R^t(X \wedge Y) \wedge R^t(X \wedge Z) = 0$ for all X, Y, $Z \in T_p(S^{2n-1}_{a,b})$.*

Proof. $R^t(X \wedge Y) \wedge R^t(X \wedge Z) = (\hat{B} \wedge \hat{B})(X \wedge Y) \wedge (\hat{B} \wedge \hat{B})(X \wedge Z) = \hat{B}(X) \wedge \hat{B}(Y) \wedge \hat{B}(X) \wedge \hat{B}(Z) = 0.$ q.e.d.

Remark. For one case of $n = 3$ we also need to use the Codazzi-Mainardi equation, which is done in §5. For $n = 2$, $R^t(X \wedge Y) \wedge R^t(X \wedge Z)$ will always be a 4-vector in $T_pS^3_{a,b}$, hence it is automatically zero and gives no information. This is by far the most difficult case, and the proof is given in §6–§8.

§3 For $\mathbb{C}P(n)$ we have the standard results:

$$
\begin{aligned}
&\bar{R}(e_i \wedge e_j) = -e_i \wedge e_j - Je_i \wedge Je_j \\
&\bar{R}(e_i \wedge Je_j) = -e_i \wedge Je_j + Je_i \wedge e_j \qquad (i \neq j) \\
&\bar{R}(Je_i \wedge Je_j) = -Je_i \wedge Je_j - e_i \wedge e_j \\
&\bar{R}(e_i \wedge Je_i) = -2(e_0 \wedge Je_0 + e_1 \wedge Je_1 + \cdots \\
&\qquad\qquad \cdots + e_{n-1} \wedge Je_{n-1}) - 2(e_i \wedge Je_i)
\end{aligned} \tag{3.1}
$$

Here J denotes the almost complex structure of $\mathbb{C}P(n)$, $e_0, e_1, \ldots, e_{n-1}$ is a complex basis for $T_p\,\mathbb{C}P(n)$. Also we use the convention $\bar{R}(X \wedge Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$.

Also $S^{2n-1}_{a,b} = U(n)/U(n-1)$, from here it has an almost cocomplex structure J', i.e. J' is defined on the $\mathbb{C}^{n-1}$ part in $T_pS^{2n-1}_{a,b} = \mathbb{R} \oplus \mathbb{C}^{n-1}$ by the action of i sitting diagonally in the isotropy group $U(n-1)$ at p. Notice, however, that this a priori is not related to the almost cocomplex structure inherited from the almost complex structure of $\mathbb{C}P(n)$ when S^{2n-1} is isometrically immersed. In fact, a challenge will be to demonstrate that these two structures on S^{2n-1} coincide. (see e.g. Theorem 4 in §4 and the proof of Theorem 5 for showing that $Je_2 = J'e_2$ $(= J'Y_2)$ in that case.)

We now compute the curvature operator for $S^{2n-1}_{a,b}$.

We have: $G = U(n)$, $H = U(n-1) \subset U(n)$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with

$$\mathfrak{p} = \left\{ \left(\begin{array}{c|c} 0 & d \\ \hline -{}^t\bar{d} & ie \end{array} \right) \middle| \, d \in \mathbb{C}^{n-1} \quad \text{(column matrix)} \right\}.$$

Here ${}^t d$ means the transpose of d. Clearly the pair $(\mathfrak{g}, \mathfrak{h})$ is reductive.

We have (see [KNII]): Let X, Y etc. also denote the Killing fields determined by $X, Y \in T_p(S^{2n-1}_{a,b})$. Then

$$\text{(3.2)} \qquad \hat{R}(X \wedge Y) = [\Lambda_{\mathfrak{p}}(X), \Lambda_{\mathfrak{p}}(Y)] - \Lambda_{\mathfrak{p}}([X,Y]_{\mathfrak{p}}) - \operatorname{ad}([X,Y]_{\mathfrak{h}}).$$

Here $\Lambda_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{gl}(\mathfrak{p})$ is a map defined by $\Lambda_{\mathfrak{p}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y)$ where U is the symmetric bilinear map from $\mathfrak{p} \times \mathfrak{p}$ to $\mathfrak{p}$ determined by $2\langle U(X,Y), Z\rangle = \langle [Z,X]_{\mathfrak{p}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{p}}\rangle$ (see p. 201 of [KNII]). Also, here $[\ ,\ ]$ denotes the extension of the Lie bracket $[\ ,\ ]_{\mathfrak{p}}$ to $\bigwedge^2(T_p S^{2n-1}_{a,b})$ given by $[X \wedge Y, Z \wedge W] = \langle X, Z\rangle Y \wedge W + \langle Y, W\rangle X \wedge Z - \langle Y, Z\rangle X \wedge W - \langle X, W\rangle Y \wedge Z$. In fact, we often consider the element $X \wedge Y$ as a linear endomorphism of $\mathfrak{p}$ by the formula $(X \wedge Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X$. This bracket just coincides with the usual bracket of two linear maps $X \wedge Y$, $Z \wedge W : \mathfrak{p} \to \mathfrak{p}$.

Let

$$Y_0 = \frac{1}{a}\begin{pmatrix} 0 \cdots 0 & 0 \\ \vdots & \vdots \\ 0 \cdots 0 & 0 \\ 0 \cdots 0 & i \end{pmatrix}, \quad Y_j = \frac{1}{b}\begin{pmatrix} 0 \ \cdots \ 0 & E_j \\ 0 \ \cdots \ 0 & \\ -{}^tE_j & 0 \end{pmatrix}, \quad J'Y_j = \frac{1}{b}\begin{pmatrix} 0 \ \cdots \ 0 & iE_j \\ 0 \ \cdots \ 0 & \\ i\,{}^tE_j & 0 \end{pmatrix}$$

be an orthonormal basis for $\mathfrak{p}$. $(j \neq 0)$. $E_j = {}^t(0, \ldots, 1, \ldots 0)$ is the j-th basis vector, and i is the imaginary unit. We compute:

$$[Y_0, Y_j] = Y_0 Y_j - Y_j Y_0 = \frac{1}{ab}\begin{pmatrix} 0 \ \cdots \ 0 & -iE_j \\ 0 \ \cdots \ 0 & \\ -i\,{}^tE_j & 0 \end{pmatrix} = -\frac{1}{a} J'Y_j.$$

Similarly, $[Y_0, J'Y_j] = \frac{1}{a}Y_j$. Now, let j, $k \neq 0$. We then have: $[Y_j, Y_k] = Y_j Y_k - Y_k Y_j = \frac{1}{b^2}(-E_{jk} + E_{kj})$ where E_{jk} is the matrix whose only non-zero component is $(E_{jk})_{jk} = 1$. Similarly: $[Y_j, J'Y_k] = Y_j J'Y_k - J'Y_k Y_j = \frac{1}{b^2}(iE_{jk} + iE_{kj}) - \frac{2a}{b^2}\delta_{jk}Y_0$ and: $[J'Y_j, J'Y_k] = J'Y_j J'Y_k - J'Y_k J'Y_j = \frac{1}{b^2}(-E_{jk} + E_{kj})$.

Hence we have

$$[Y_0, Y_j]_{\mathfrak{p}} = -\frac{1}{a}J'Y_j, \quad [Y_0, Y_j]_{\mathfrak{h}} = 0$$
$$[Y_0, J'Y_j]_{\mathfrak{p}} = \frac{1}{a}Y_j, \quad [Y_0, J'Y_j]_{\mathfrak{h}} = 0$$
$$[Y_j, Y_k]_{\mathfrak{p}} = 0, \quad [Y_j, Y_k]_{\mathfrak{h}} = \frac{1}{b^2}(-E_{jk} + E_{kj})$$
$$[Y_j, J'Y_k]_{\mathfrak{p}} = -\frac{2a}{b^2}\delta_{jk}Y_0, \quad [Y_j, J'Y_k]_{\mathfrak{h}} = \frac{i}{b^2}(E_{jk} + E_{kj})$$
$$[J'Y_j, J'Y_k]_{\mathfrak{p}} = 0, \quad [J'Y_j, J'Y_k]_{\mathfrak{h}} = \frac{1}{b^2}(-E_{jk} + E_{kj}).$$

Here j, $k \neq 0$.

We have $2\langle U(Y_0, Y_0), Z\rangle = \langle [Z, Y_0]_{\mathfrak{p}}, Y_0\rangle + \langle Y_0, [Z, Y_0]_{\mathfrak{p}}\rangle$. For $Z = Y_0$ this is 0, for $Z = Y_j$ it is $\frac{2}{a}\langle J'Y_j, Y_0\rangle = 0$, and for $Z = J'Y_j$ it is $-\frac{2}{a}\langle Y_j, Y_0\rangle = 0$. It follows that $U(Y_0, Y_0) = 0$.

We have $2\langle U(Y_0, Y_j), Z\rangle = \langle [Z, Y_0]_{\mathfrak{p}}, Y_j\rangle + \langle Y_0, [Z, Y_j]_{\mathfrak{p}}\rangle$. For $Z = Y_0$ and $Z = Y_k$ this is easily seen to be zero, but for $Z = J'Y_k$ we get $2\langle U(Y_0, Y_j), J'Y_k\rangle = -\frac{1}{a}\langle Y_k, Y_j\rangle + \langle Y_0, \frac{2a}{b^2}\delta_{jk}Y_0\rangle = \delta_{jk}(\frac{2a}{b^2} - \frac{1}{a})$. Hence $U(Y_0, Y_j) = (\frac{a}{b^2} - \frac{1}{2a})J'Y_j$. Similarly $U(Y_0, J'Y_j) = (\frac{1}{2a} - \frac{a}{b^2})Y_j$. Furthermore, for j, $k \neq 0$ we have $U(Y_j, Y_k) = U(Y_j, J'Y_k) = U(J'Y_j, J'Y_k) = 0$.

From [KNII] we know that the covariant derivative $\nabla_X Y = \Lambda_{\mathfrak{p}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y)$. Hence we compute $\nabla_{Y_0}Y_0 = \frac{1}{2}[Y_0, Y_0]_{\mathfrak{p}} + U(Y_0, Y_0) = 0$. Furthermore: $\nabla_{Y_0}Y_j = \frac{1}{2}[Y_0, Y_j]_{\mathfrak{p}} + U(Y_0, Y_j) = -\frac{1}{2a}J'Y_j + (\frac{a}{b^2} - \frac{1}{2a})J'Y_j = (\frac{a}{b^2} - \frac{1}{a})J'Y_j$. Similar computations give $\nabla_{Y_0}J'Y_j = (\frac{1}{a} - \frac{a}{b^2})Y_j$, $\nabla_{Y_j}Y_0 = \frac{a}{b^2}J'Y_j$, and $\nabla_{J'Y_j}Y_0 = -\frac{a}{b^2}Y_j$.

For j, $k \neq 0$ we now have: $\nabla_{Y_j}Y_k = \frac{1}{2}[Y_j, Y_k]_{\mathfrak{p}} + U(Y_j, Y_k) = 0$, $\nabla_{Y_j}J'Y_k = \frac{1}{2}[Y_j, J'Y_k]_{\mathfrak{p}} + U(Y_j, J'Y_k) = -\frac{a}{b^2}\delta_{jk}Y_0$, $\nabla_{J'Y_j}Y_k = \frac{1}{2}[J'Y_j, Y_k]_{\mathfrak{p}} + U(J'Y_j, Y_k) = \frac{a}{b^2}\delta_{jk}Y_0$, and $\nabla_{J'Y_j}J'Y_k = 0$. Hence we have:

$$\nabla_{Y_0} = \Lambda_{\mathfrak{p}}(Y_0) = \left(\frac{a}{b^2} - \frac{1}{a}\right)\sum_{j=1}^{n-1} Y_j \wedge J'Y_j$$

For $j \neq 0$:

$$\nabla_{Y_j} = \Lambda_{\mathfrak{p}}(Y_j) = \frac{a}{b^2}Y_0 \wedge J'Y_j$$
$$\nabla_{J'Y_j} = \Lambda_{\mathfrak{p}}(J'Y_j) = -\frac{a}{b^2}Y_0 \wedge Y_j$$

Furthermore,

$$\Lambda_{\mathfrak{p}}([Y_0, Y_j]_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}\left(-\frac{1}{a}J'Y_j\right) = \frac{1}{b^2}Y_0 \wedge Y_j$$

$$\Lambda_{\mathfrak{p}}([Y_0, J'Y_j]_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}\left(\frac{1}{a}Y_j\right) = \frac{1}{b^2}Y_0 \wedge J'Y_j$$

$$\Lambda_{\mathfrak{p}}([Y_j, J'Y_k]_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}\left(-\frac{2a}{b^2}\delta_{jk}Y_0\right) = 2\delta_{jk}\left(\frac{1}{b^2} - \frac{a^2}{b^4}\right)\sum_{l=1}^{n-1} Y_l \wedge J'Y_l.$$

We also have: $[\Lambda_{\mathfrak{p}}(Y_0), \Lambda_{\mathfrak{p}}(Y_j)] = [(\frac{a}{b^2}-\frac{1}{a})\sum_{k=1}^{n-1} Y_k \wedge J'Y_k, \frac{a}{b^2}Y_0 \wedge J'Y_j] = (\frac{a^2}{b^4} - \frac{1}{b^2})\sum_{k=1}^{n-1}[Y_k \wedge J'Y_k, Y_0 \wedge J'Y_j] = (\frac{a^2}{b^4} - \frac{1}{b^2})Y_j \wedge Y_0$ and similarly: $[\Lambda_{\mathfrak{p}}(Y_0), \Lambda_{\mathfrak{p}}(J'Y_j)] = (\frac{a^2}{b^4} - \frac{1}{b^2})J'Y_j \wedge Y_0$,

$$[\Lambda_{\mathfrak{p}}(Y_j), \Lambda_{\mathfrak{p}}(Y_k)] = \frac{a^2}{b^4}J'Y_j \wedge J'Y_k$$

$$[\Lambda_{\mathfrak{p}}(Y_j), \Lambda_{\mathfrak{p}}(J'Y_k)] = \frac{a^2}{b^4}Y_k \wedge J'Y_j$$

$$[\Lambda_{\mathfrak{p}}(J'Y_j), \Lambda_{\mathfrak{p}}(J'Y_k)] = \frac{a^2}{b^4}Y_j \wedge Y_k.$$

Finally: $\mathrm{ad}([Y_0, Y_j]_{\mathfrak{h}}) = \mathrm{ad}([Y_0, J'Y_j]_{\mathfrak{h}}) = 0$,

$$\mathrm{ad}([Y_j, Y_k]_{\mathfrak{h}}) = \frac{1}{b^2}\,\mathrm{ad}(-E_{jk} + E_{kj}) = \frac{1}{b^2}Y_j \wedge Y_k + \frac{1}{b^2}J'Y_j \wedge J'Y_k$$

$$\mathrm{ad}([Y_j, J'Y_k]_{\mathfrak{h}}) = \frac{i}{b^2}\,\mathrm{ad}(E_{jk} + E_{kj}) = \frac{1}{b^2}Y_j \wedge J'Y_k + \frac{1}{b^2}Y_k \wedge J'Y_j$$

(notice that $\frac{i}{b^2}\,\mathrm{ad}(E_{jk} + E_{kj})Y_j = \frac{i}{b^2}Y_k = \frac{1}{b^2}J'Y_k$, etc.)

$$\mathrm{ad}([J'Y_j, J'Y_k]_{\mathfrak{h}}) = \frac{1}{b^2}\,\mathrm{ad}(-E_{jk} + E_{kj}) = \frac{1}{b^2}Y_j \wedge Y_k + \frac{1}{b^2}J'Y_j \wedge J'Y_k.$$

We are now ready to compute the curvature operators $\hat{R}(X \wedge Y)$ according to the formula (3.2) $\hat{R}(X\wedge Y) = [\Lambda_{\mathfrak{p}}(X), \Lambda_{\mathfrak{p}}(Y)] - \Lambda_{\mathfrak{p}}([X, Y]_{\mathfrak{p}}) - \mathrm{ad}([X, Y]_{\mathfrak{h}})$.

The results are:

$$\text{(3.3)} \qquad \hat{R}(Y_0 \wedge Y_j) = \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)Y_j \wedge Y_0 - \Lambda_{\mathfrak{p}}\left(-\frac{1}{a}J'Y_j\right) - 0$$

$$= \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)Y_j \wedge Y_0 + \frac{1}{a}\left(-\frac{a}{b^2}Y_0 \wedge Y_j\right)$$

$$= -\frac{a^2}{b^4}Y_0 \wedge Y_j.$$

Similarly: $\hat{R}(Y_0 \wedge J'Y_j) = -\frac{a^2}{b^4} Y_0 \wedge J'Y_j$, $\hat{R}(Y_j \wedge Y_k) = (\frac{a^2}{b^4} - \frac{1}{b^2}) J'Y_j \wedge J'Y_k - \frac{1}{b^2} Y_j \wedge Y_k$. For $j,\, k \neq 0$ we have:

$$\begin{aligned}\hat{R}(Y_j \wedge J'Y_k) &= \frac{a^2}{b^4} Y_k \wedge J'Y_j - \Lambda_{\mathfrak{p}}\left(-\frac{2a}{b^2}\delta_{jk} Y_0\right)\\ &\quad -\frac{1}{b^2}(Y_j \wedge J'Y_k + Y_k \wedge J'Y_j)\\ &= \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_k \wedge J'Y_j\\ &\quad -\frac{1}{b^2} Y_j \wedge J'Y_k + \delta_{jk}\left(2\frac{a^2}{b^4} - \frac{2}{b^2}\right)\sum_{l=1}^{n-1} Y_l \wedge J'Y_l\\ \hat{R}(J'Y_j \wedge J'Y_k) &= \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_j \wedge Y_k - \frac{1}{b^2} J'Y_j \wedge J'Y_k.\end{aligned}$$

§4 Let $S^{2n-1}_{a,b} = M$ be locally isometrically immersed into $\mathbb{C}P(n)$ around the point p. Let N be the unit normal to M at p, then $JN \in T_pM$ and we put $e_0 = JN$. When M is the geodesic sphere, e_0 is the structural vector Y_0 of M at p. We first show that this is always the case (for $n \geq 3$).

Theorem 4. *For $n \geq 3$ we may choose $Y_0 = e_0$.*

Proof. We set $Y_0 = \cos\varphi\, e_0 + \sin\varphi\, Je_1$ and $Y_1 = e_1$ (normal to e_0 and Je_1, hence to Y_0). We usually replace vectors Y_i, $J'Y_j$, e_k etc. to another one by applying the isotropy action of $S^{2n-1}_{a,b}$ or $\mathbb{C}P(n)$. Note that by this modification, the expression of connections, curvatures and brackets are unaltered. We have:

$$\begin{aligned}R^t(Y_0 \wedge Y_1) &= \bar{R}^t(\cos\varphi\, e_0 \wedge e_1 + \sin\varphi\, Je_1 \wedge e_1) - \hat{R}(Y_0 \wedge Y_1)\\ &= -\cos\varphi\, e_0 \wedge e_1 + \sin\varphi(4e_1 \wedge Je_1 + 2e_2 \wedge Je_2 + \cdots\\ &\quad \cdots + 2e_{n-1} \wedge Je_{n-1}) + \frac{a^2}{b^4} Y_0 \wedge Y_1\\ &= \left(\frac{a^2}{b^4} - 1\right)\cos\varphi\, e_0 \wedge e_1\\ &\quad + \sin\varphi\left[\left(4 - \frac{a^2}{b^4}\right) e_1 \wedge Je_1 + 2e_2 \wedge Je_2 + \cdots\right.\\ &\quad \left.\cdots + 2e_{n-1} \wedge Je_{n-1}\right]\end{aligned}$$

according to (3.1) and (3.3). We have: $R^t(Y_0 \wedge Y_1) \wedge R^t(Y_0 \wedge Y_1) = 4(\frac{a^2}{b^4} - 1)\sin\varphi\cos\varphi\, e_0 \wedge e_1 \wedge e_2 \wedge Je_2 + 4(4 - \frac{a^2}{b^4})\sin^2\varphi\, e_1 \wedge Je_1 \wedge e_2 \wedge$

$Je_2 + 8\sin^2\varphi\, e_2 \wedge Je_2 \wedge e_3 \wedge Je_3 + \cdots \equiv 0$ according to Proposition 4. In case $n \geq 4$ it follows immediately from the $e_2 \wedge Je_2 \wedge e_3 \wedge Je_3$ term that $\sin\varphi = 0$. For the case $n = 3$ we have from the two first terms: $(\frac{a^2}{b^4} - 1)\sin\varphi\cos\varphi = (4 - \frac{a^2}{b^4})\sin^2\varphi = 0$. Hence either (A): $\sin^2\varphi = 0$ or (B): $\frac{a^2}{b^4} = 4$, $\cos\varphi = 0$. In case (A) we are done, in case (B) we may choose $Y_0 = Je_1$. Let $Y_2 = e_2$ where $J'Y_1 = \cos\psi\, e_0 + \sin\psi\, Je_2$, ($Y_2$ is orthonormal to Y_0, Y_1 and $J'Y_1$). Then: $R^t(Y_0 \wedge Y_2) = \bar{R}^t(Je_1 \wedge e_2) - \hat{R}(Y_0 \wedge Y_2) = -Je_1 \wedge e_2 + e_1 \wedge Je_2 + \frac{a^2}{b^4} Y_0 \wedge Y_2 = (\frac{a^2}{b^4} - 1)Je_1 \wedge e_2 + e_1 \wedge Je_2 = 3Je_1 \wedge e_2 + e_1 \wedge Je_2$. But then $R^t(Y_0 \wedge Y_2) \wedge R^t(Y_0 \wedge Y_2) = 6Je_1 \wedge e_2 \wedge e_1 \wedge Je_2 \neq 0$, which contradicts Proposition 4. Hence (B) is impossible. q.e.d.

We now wish to prove our main result (Theorem 2) $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$; this means that the metric of M is the metric of $S^{2n-1}(r)$ for some $r \in (0, \frac{\pi}{2})$ (Proposition 1). We start with the easier case, $n \geq 4$.

Theorem 5. *For $n \geq 4$ we have $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$.*

Proof. We have $Y_0 = e_0$, $Y_1 = e_1$, let $J'Y_1 = \cos\psi\, Je_1 + \sin\psi\, e_2$, $Y_2 = -\sin\psi\, Je_1 + \cos\psi\, e_2$ (orthonormal to Y_0, Y_1 and $J'Y_1$) and $J'Y_2 = \cos\xi\, Je_2 + \sin\xi\, e_3$. We have $R^t(Y_0 \wedge Y_1) = (\frac{a^2}{b^4} - 1)e_0 \wedge e_1$ as above.

$$\begin{aligned}
R^t(Y_1 \wedge Y_2) &= \bar{R}^t(e_1 \wedge (-\sin\psi\, Je_1 + \cos\psi\, e_2)) - \hat{R}(Y_1 \wedge Y_2)\\
&= \sin\psi(4e_1 \wedge Je_1 + 2e_2 \wedge Je_2 + \cdots + 2e_{n-1} \wedge Je_{n-1}) - \cos\psi\, e_1 \wedge e_2\\
&\quad - \cos\psi\, Je_1 \wedge Je_2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) J'Y_1 \wedge J'Y_2 + \frac{1}{b^2} Y_1 \wedge Y_2\\
&= \cdots - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)[\cos\psi\cos\xi\, Je_1 \wedge Je_2 + \cos\psi\sin\xi\, Je_1 \wedge e_3\\
&\quad + \sin\psi\cos\xi\, e_2 \wedge Je_2 + \sin\psi\sin\xi\, e_2 \wedge e_3]\\
&\quad - \frac{1}{b^2}\sin\psi\, e_1 \wedge Je_1 + \frac{1}{b^2}\cos\psi\, e_1 \wedge e_2\\
&= \left(\frac{1}{b^2} - 1\right)\cos\psi\, e_1 \wedge e_2 - \left[\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi + 1\right]\cos\psi\, Je_1 \wedge Je_2\\
&\quad - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\sin\psi\sin\xi\, e_2 \wedge e_3 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\psi\sin\xi\, Je_1 \wedge e_3\\
&\quad + \left(4 - \frac{1}{b^2}\right)\sin\psi\, e_1 \wedge Je_1\\
&\quad + \left[2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right]\sin\psi\, e_2 \wedge Je_2 + 2\sin\psi \sum_{k=3}^{n-1} e_k \wedge Je_k.
\end{aligned}$$

We have

$$R^t(Y_0 \wedge Y_1) \wedge R^t(Y_1 \wedge Y_2) = \left(\frac{a^2}{b^4} - 1\right) e_0 \wedge e_1 \wedge R^t(Y_1 \wedge Y_2)$$
$$= -\left(\frac{a^2}{b^4} - 1\right)\left[1 + \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right] \cos\psi \, e_0 \wedge e_1 \wedge Je_1 \wedge Je_2$$
$$+ \left(\frac{a^2}{b^4} - 1\right)\left(\frac{1}{b^2} - \frac{a^2}{b^4}\right) \sin\psi \sin\xi \, e_0 \wedge e_1 \wedge e_2 \wedge e_3$$
$$- \left(\frac{a^2}{b^4} - 1\right)\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) \cos\psi \sin\xi \, e_0 \wedge e_1 \wedge Je_1 \wedge e_3$$
$$+ \left(\frac{a^2}{b^4} - 1\right)\left[2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right] \sin\psi \, e_0 \wedge e_1 \wedge e_2 \wedge Je_2$$
$$+ 2\left(\frac{a^2}{b^4} - 1\right) \sin\psi \, e_0 \wedge e_1 \wedge \sum_{k=3}^{n-1} e_k \wedge Je_k = 0.$$

There are two possibilities:

(A) $\frac{a^2}{b^4} = 1$

(B) $\frac{a^2}{b^4} \neq 1$, $\sin\psi = 0$, $(\frac{1}{b^2} - \frac{a^2}{b^4})\cos\xi = 1$, $\sin\xi = 0$

$((\frac{1}{b^2} - \frac{a^2}{b^4})$ must be different from zero).

In case (B) we then have $\cos\xi = \pm 1$, $\frac{1}{b^2} - \frac{a^2}{b^4} = \cos\xi = \pm 1$, and we only need to show that $\cos\xi = -1$ is impossible.

Consider now case (B). We have $R^t(Y_1 \wedge Y_2) = (\frac{1}{b^2} - 1)\cos\psi \, e_1 \wedge e_2$.

$$R^t(Y_1 \wedge J'Y_1) = \bar{R}^t(e_1 \wedge \cos\psi \, Je_1) - \hat{R}(Y_1 \wedge J'Y_1)$$
$$= -\cos\psi(4e_1 \wedge Je_1 + 2e_2 \wedge Je_2 + \cdots + 2e_{n-1} \wedge Je_{n-1})$$
$$- \left(3\frac{a^2}{b^4} - \frac{4}{b^2}\right) Y_1 \wedge J'Y_1 - 2\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) \sum_{j=2}^{n-1} Y_j \wedge J'Y_j$$
$$= -\left(4 + 3\frac{a^2}{b^4} - \frac{4}{b^2}\right) \cos\psi \, e_1 \wedge Je_1$$
$$-2\left[1 + \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right] \cos\psi \, e_2 \wedge Je_2 - 2\cos\psi \sum_{j=3}^{n-1} e_j \wedge Je_j$$
$$-2\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) \sum_{j=3}^{n-1} Y_j \wedge J'Y_j$$
$$= -\left(4 - 3\cos\xi - \frac{1}{b^2}\right) \cos\psi \, e_1 \wedge Je_1$$

$$-2\cos\psi\sum_{j=3}^{n-1} e_j\wedge Je_j + 2\cos\xi\sum_{j=3}^{n-1} Y_j\wedge J'Y_j$$

Hence $R^t(Y_0\wedge Y_1)\wedge R^t(Y_1\wedge J'Y_1) = -2(\frac{a^2}{b^4}-1)\cos\psi\, e_0\wedge e_1\wedge\sum_{j=3}^{n-1} e_j\wedge Je_j + 2(\frac{a^2}{b^4}-1)\cos\xi e_0\wedge e_1\wedge\sum_{j=3}^{n-1} Y_j\wedge J'Y_j = 0$. Hence, for $\frac{a^2}{b^4}\neq 1$ we have:

$$\cos\psi\sum_{j=3}^{n-1} e_j\wedge Je_j = \cos\xi\sum_{j=3}^{n-1} Y_j\wedge J'Y_j. \tag{4.1}$$

Similarly

$$\begin{aligned}
R^t(Y_0\wedge Y_2) &= \bar{R}^t(e_0\wedge\cos\psi\, e_2) - \hat{R}(Y_0\wedge Y_2)\\
&= -\cos\psi\, e_0\wedge e_2 + \frac{a^2}{b^4}Y_0\wedge Y_2 = \left(\frac{a^2}{b^4}-1\right)\cos\psi\, e_0\wedge e_2\\
R^t(Y_2\wedge J'Y_2) &= \bar{R}^t(\cos\psi\, e_2\wedge\cos\xi\, Je_2) - \hat{R}(Y_2\wedge J'Y_2)\\
&= -\cos\psi\cos\xi(2e_1\wedge Je_1 + 4e_2\wedge Je_2 + \cdots + 2e_{n-1}\wedge Je_{n-1})\\
&\quad -\left(3\frac{a^2}{b^4}-\frac{4}{b^2}\right)Y_2\wedge J'Y_2 - 2\left(\frac{a^2}{b^4}-\frac{1}{b^2}\right)Y_1\wedge J'Y_1\\
&\quad -2\left(\frac{a^2}{b^4}-\frac{1}{b^2}\right)\sum_{j=3}^{n-1} Y_j\wedge J'Y_j\\
&= -2\cos\psi\left(\cos\xi + \left(\frac{a^2}{b^4}-\frac{1}{b^2}\right)\right)e_1\wedge Je_1\\
&\quad -\cos\psi\cos\xi\left(4+3\frac{a^2}{b^4}-\frac{4}{b^2}\right)e_2\wedge Je_2\\
&\quad -2\cos\psi\cos\xi\sum_{j=3}^{n-1} e_j\wedge Je_j - 2\left(\frac{a^2}{b^4}-\frac{1}{b^2}\right)\sum_{j=3}^{n-1} Y_j\wedge J'Y_j\\
&= -\cos\psi\cos\xi\left(4-3\cos\xi-\frac{1}{b^2}\right)e_2\wedge Je_2\\
&\quad -2\cos\psi\cos\xi\sum_{j=3}^{n-1} e_j\wedge Je_j + 2\cos\xi\sum_{j=3}^{n-1} Y_j\wedge J'Y_j.
\end{aligned}$$

Hence

$$\begin{aligned}
&R^t(Y_0\wedge Y_2)\wedge R^t(Y_2\wedge J'Y_2)\\
&= -2\left(\frac{a^2}{b^4}-1\right)\cos\xi\, e_0\wedge e_2\wedge\left[\sum_{j=3}^{n-1} e_j\wedge Je_j - \cos\psi\sum_{j=3}^{n-1} Y_j\wedge J'Y_j\right]
\end{aligned}$$

$\equiv 0.$

Hence, for $\frac{a^2}{b^4} \neq 1$ we have

$$\sum_{j=3}^{n-1} e_j \wedge Je_j = \cos\psi \sum_{j=3}^{n-1} Y_j \wedge J'Y_j.$$

Comparing with (4.1) this gives $\cos\xi = 1$ in case (B).

Now consider (A), $\frac{a^2}{b^4} = 1$. We have

$$\begin{aligned}
R^t(Y_1 \wedge J'Y_2) &= \bar{R}^t(e_1 \wedge (\cos\xi\, Je_2 + \sin\xi\, e_3)) - \hat{R}(Y_1 \wedge J'Y_2)\\
&= -\cos\xi\, e_1 \wedge Je_2 + \cos\xi\, Je_1 \wedge e_2 - \sin\xi\, e_1 \wedge e_3 - \sin\xi\, Je_1 \wedge Je_3\\
&\quad - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_2 \wedge J'Y_1 + \frac{1}{b^2} Y_1 \wedge J'Y_2\\
&= -\cos\xi\, e_1 \wedge Je_2 + \cos\xi\, Je_1 \wedge e_2 - \sin\xi\, e_1 \wedge e_3 - \sin\xi\, Je_1 \wedge Je_3\\
&\quad - \left(1 - \frac{1}{b^2}\right) e_2 \wedge Je_1 + \frac{1}{b^2} e_1 \wedge \cos\xi\, Je_2 + \frac{1}{b^2} e_1 \wedge \sin\xi\, e_3\\
&= \left(\frac{1}{b^2} - 1\right) \cos\xi\, e_1 \wedge Je_2 + \left(\cos\xi + 1 - \frac{1}{b^2}\right) Je_1 \wedge e_2\\
&\quad + \left(\frac{1}{b^2} - 1\right) \sin\xi\, e_1 \wedge e_3 - \sin\xi\, Je_1 \wedge Je_3.
\end{aligned}$$

$$\begin{aligned}
&R^t(Y_1 \wedge J'Y_2) \wedge R^t(Y_1 \wedge J'Y_2)\\
&= 2\left(\frac{1}{b^2} - 1\right)\left(\cos\xi + 1 - \frac{1}{b^2}\right) \cos\xi e_1 \wedge Je_2 \wedge Je_1 \wedge e_2\\
&\quad -2\left(\frac{1}{b^2} - 1\right) \sin\xi \cos\xi\, e_1 \wedge Je_2 \wedge Je_1 \wedge Je_3\\
&\quad +2\left(\cos\xi + 1 - \frac{1}{b^2}\right)\left(\frac{1}{b^2} - 1\right) \sin\xi\, Je_1 \wedge e_2 \wedge e_1 \wedge e_3\\
&\quad -2\left(\frac{1}{b^2} - 1\right) \sin^2\xi\, e_1 \wedge e_3 \wedge Je_1 \wedge Je_3 \equiv 0.
\end{aligned}$$

From the last term we see: either (C) $b^2 = 1$ or (D) $\sin\xi = 0$. If $b^2 \neq 1$, $\sin\xi = 0$ and

$$\begin{aligned}
&R^t(Y_1 \wedge J'Y_2) \wedge R^t(Y_1 \wedge J'Y_2)\\
&\quad = 2\left(\frac{1}{b^2} - 1\right)\left(\cos\xi + 1 - \frac{1}{b^2}\right) \cos\xi\, e_1 \wedge Je_2 \wedge Je_1 \wedge e_2,
\end{aligned}$$

hence $1 - \frac{1}{b^2} = -\cos\xi$. This does not hold for $\cos\xi = -1$, hence $\cos\xi = 1$ and $1 - \frac{1}{b^2} = \frac{a^2}{b^4} - \frac{1}{b^2} = -1$ ($b^2 = \frac{1}{2}$). This finishes (D). Assume $b^2 = 1$,

$\frac{a^2}{b^4} = 1$, i.e. $a = b = 1$. Then

$$\begin{aligned} &R^t(Y_1 \wedge Y_2) \\ &= -\cos\psi\, Je_1 \wedge Je_2 + 3\sin\psi\, e_1 \wedge Je_1 + 2\sin\psi\, e_2 \wedge Je_2 \\ &\quad + 2\sin\psi \sum_{k=3}^{n-1} e_k \wedge Je_k. \end{aligned}$$

Setting $R^t(Y_1 \wedge Y_2) \wedge R^t(Y_1 \wedge Y_2) = 0$ we get $\sin\psi = 0$. From the formula at the end of §3: $\hat{R}(Y_1 \wedge J'Y_1) = -Y_1 \wedge J'Y_1$, i.e. $R^t(Y_1 \wedge J'Y_1) = \bar{R}^t(e_1 \wedge \cos\psi\, Je_1) + Y_1 \wedge J'Y_1 = -\cos\psi(4e_1 \wedge Je_1 + 2e_2 \wedge Je_2 + \cdots + 2e_{n-1} \wedge Je_{n-1}) + Y_1 \wedge J'Y_1 = -3\cos\psi\, e_1 \wedge Je_1 - 2\cos\psi \sum_{k=2}^{n-1} e_k \wedge Je_k$. $R^t(Y_1 \wedge J'Y_1) \wedge R^t(Y_1 \wedge J'Y_1) = 12e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 + \cdots + \neq 0$, which excludes this case. q.e.d.

§5 We now deal with the case $n = 3$. We know already (Theorem 5 and the proof of Theorem 6) that $Y_0 = e_0$, $Y_1 = e_1$, $J'Y_1 = \cos\psi\, Je_1 + \sin\psi\, e_2$, $Y_2 = -\sin\psi Je_1 + \cos\psi\, e_2$, $J'Y_2 = \cos\xi\, Je_2$ $(= \pm Je_2)$ (see the beginning of the proof of Theorem 6, and note that we have no e_3 here). The formulas (3.1) and (3.2) specify to:

$$\begin{aligned} &\bar{R}(e_j \wedge e_k) = -e_j \wedge e_k - Je_j \wedge Je_k = \bar{R}(Je_j \wedge Je_k) \\ &\bar{R}(e_j \wedge Je_k) = -e_j \wedge Je_k + Je_j \wedge e_k - 2\delta_{jk} \sum_{l=0}^{2} e_l \wedge Je_l \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\hat{R}(Y_0 \wedge Y_j) = -\frac{a^2}{b^4} Y_0 \wedge Y_j \\ &\hat{R}(Y_0 \wedge J'Y_j) = -\frac{a^2}{b^4} Y_0 \wedge J'Y_j \\ &\hat{R}(Y_j \wedge Y_k) = \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) J'Y_j \wedge J'Y_k - \frac{1}{b^2} Y_j \wedge Y_k \quad (j,\ k \neq 0) \\ &\hat{R}(J'Y_j \wedge J'Y_k) = \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_j \wedge Y_k - \frac{1}{b^2} J'Y_j \wedge J'Y_k \\ &\hat{R}(Y_j \wedge J'Y_k) = \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_k \wedge J'Y_j - \frac{1}{b^2} Y_j \wedge J'Y_k \\ &\quad +2\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) \delta_{jk}(Y_1 \wedge J'Y_1 + Y_2 \wedge J'Y_2). \end{aligned} \tag{5.2}$$

We now calculate $R(Y_i \wedge Y_j)$ and $R(Y_i \wedge J'Y_j)$ where $R(X \wedge Y) = \bar{R}(X \wedge Y) - \hat{R}(X \wedge Y)$. Although we only use the tangential part

$R^t = \bar{R}^t - \hat{R}$ to begin with, note that in the Codazzi-Mainardi equation

$$\begin{aligned}\langle \bar{R}(X\wedge Y)Z, -Je_0\rangle &= \langle R(X\wedge Y)Z, -Je_0\rangle \\ &= Y(B(X,Z)) - B(X, \nabla_Y Z) \\ &\quad - X(B(Y,Z)) + B(Y, \nabla_X Z) + B([X,Y], Z)\end{aligned}$$

we may use either $\bar{R}(X\wedge Y)$ or $R(X\wedge Y)$, since the normal $(-Je_0)$ components coincide for $\bar{R}$ and R. Also, it is nice to have $R(X\wedge Y)Z$ in the $(Y_j, J'Y_k)$ basis (instead of only in the (e_j, Je_k) basis).

Now,

(5.3)

$$\begin{aligned}R(Y_0\wedge Y_1) &= \bar{R}(e_0\wedge e_1) - \hat{R}(Y_0\wedge Y_1) \\ &= -e_0\wedge e_1 - Je_0\wedge Je_1 + \frac{a^2}{b^4} e_0\wedge e_1 \\ &= \left(\frac{a^2}{b^4} - 1\right) e_0\wedge e_1 - Je_0\wedge Je_1\end{aligned}$$

(5.4)

$$\begin{aligned}R(Y_1\wedge Y_2) &= \bar{R}(e_1\wedge(-\sin\psi\, Je_1 + \cos\psi\, e_2)) - \hat{R}(Y_1\wedge Y_2) \\ &= \sin\psi(2e_0\wedge Je_0 + 4e_1\wedge Je_1 + 2e_2\wedge Je_2) - \cos\psi\, e_1\wedge e_2 \\ &\quad - \cos\psi\, Je_1\wedge Je_2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) J'Y_1\wedge J'Y_2 + \frac{1}{b^2} Y_1\wedge Y_2 \\ &= \cdots - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)(\cos\psi\, Je_1 + \sin\psi\, e_2)\wedge \cos\xi\, Je_2 \\ &\quad + \frac{1}{b^2} e_1\wedge(-\sin\psi\, Je_1 + \cos\psi\, e_2) \\ &= 2\sin\psi\, e_0\wedge Je_0 + \left(4 - \frac{1}{b^2}\right)\sin\psi\, e_1\wedge Je_1 \\ &\quad + \left(2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right)\sin\psi\, e_2\wedge Je_2 + \left(\frac{1}{b^2} - 1\right)\cos\psi\, e_1\wedge e_2 \\ &\quad - \left(1 + \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right)\cos\psi\, Je_1\wedge Je_2.\end{aligned}$$

$$\begin{aligned}&R^t(Y_0\wedge Y_1)\wedge R^t(Y_1\wedge Y_2) \\ &= \left(\frac{a^2}{b^4} - 1\right)\left[\left(2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right)\sin\psi\, e_0\wedge e_1\wedge e_2\wedge Je_2\right. \\ &\quad \left. - \left(1 + \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right)\cos\psi\, e_0\wedge e_1\wedge Je_1\wedge Je_2\right] \equiv 0.\end{aligned}$$

Hence, either (A) $\frac{a^2}{b^4} = 1$ or

$$\left(2 - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right)\sin\psi = \left(1 + \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\right)\cos\psi = 0.$$

In the second case $\sin\psi$ or $\cos\psi$ must $= 0$, and $(\frac{a^2}{b^4} - \frac{1}{b^2})\cos\xi = 2$ or -1. In addition to (A) we have the two possiblities: (B) $\cos\psi = 0$ and $(\frac{a^2}{b^4} - \frac{1}{b^2})\cos\xi = 2$ or (C) $\sin\psi = 0$ and $(\frac{a^2}{b^4} - \frac{1}{b^2})\cos\xi = -1$.

Now,

$$\begin{aligned}
R(Y_1 \wedge J'Y_2) &= \bar{R}(e_1 \wedge \cos\xi\, Je_2) - \hat{R}(Y_1 \wedge J'Y_2)\\
&= -\cos\xi\, e_1 \wedge Je_2 + \cos\xi\, Je_1 \wedge e_2\\
&\quad - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_2 \wedge J'Y_1 + \frac{1}{b^2} Y_1 \wedge J'Y_2\\
&= -\cos\xi\, e_1 \wedge Je_2 + \cos\xi\, Je_1 \wedge e_2\\
&\quad - \left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) e_2 \wedge Je_1 + \frac{1}{b^2}\cos\xi\, e_1 \wedge Je_2\\
&= \left(\frac{1}{b^2} - 1\right)\cos\xi\, e_1 \wedge Je_2 + \left(\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) + \cos\xi\right) Je_1 \wedge e_2.
\end{aligned}$$

$$\begin{aligned}
R^t(Y_1 \wedge J'Y_2) &\wedge R^t(Y_1 \wedge J'Y_2)\\
&= 2\left(\frac{1}{b^2} - 1\right)\left[\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) + \cos\xi\right]\cos\xi\, e_1 \wedge Je_2 \wedge Je_1 \wedge e_2.
\end{aligned}$$

Hence either $b^2 = 1$ or $(\frac{a^2}{b^4} - \frac{1}{b^2}) + \cos\xi = 0$.

First check (A): $\frac{a^2}{b^4} = 1$. If $b^2 = 1$, we have $a = b = 1$. In this case we have: $R^t(Y_1 \wedge Y_2) = 3\sin\psi\, e_1 \wedge Je_1 + 2\sin\psi\, e_2 \wedge Je_2 - \cos\psi\, Je_1 \wedge Je_2$ and $R^t(Y_1 \wedge Y_2) \wedge R^t(Y_1 \wedge Y_2) = 12\sin^2\psi\, e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 = 0$, hence $\sin\psi = 0$.

(5.5)

$$\begin{aligned}
&R(Y_1 \wedge J'Y_1)\\
&= \bar{R}(e_1 \wedge (\cos\psi\, Je_1 + \sin\psi\, e_2)) - \hat{R}(Y_1 \wedge J'Y_1)\\
&= -\cos\psi(2e_0 \wedge Je_0 + 4e_1 \wedge Je_1 + 2e_2 \wedge Je_2) - \sin\psi\, e_1 \wedge e_2\\
&\quad - \sin\psi\, Je_1 \wedge Je_2 - \left(3\frac{a^2}{b^4} - \frac{4}{b^2}\right) Y_1 \wedge J'Y_1 - 2\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right) Y_2 \wedge J'Y_2\\
&= \cdots - \left(3\frac{a^2}{b^4} - \frac{4}{b^2}\right) e_1 \wedge (\cos\psi\, Je_1 + \sin\psi\, e_2)\\
&\quad - 2\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\cos\xi\, (-\sin\psi\, Je_1 + \cos\psi\, e_2) \wedge Je_2
\end{aligned}$$

$$= -2\cos\psi\, e_0\wedge Je_0 - \left(4+3\frac{a^2}{b^4}-\frac{4}{b^2}\right)\cos\psi\, e_1\wedge Je_1$$
$$-2\left(1+\left(\frac{a^2}{b^4}-\frac{1}{b^2}\right)\cos\xi\right)\cos\psi\, e_2\wedge Je_2 - \left(1+3\frac{a^2}{b^4}-\frac{4}{b^2}\right)\sin\psi\, e_1\wedge e_2$$
$$+\left(2\left(\frac{a^2}{b^4}-\frac{1}{b^2}\right)\cos\xi - 1\right)\sin\psi\, Je_1\wedge Je_2$$

In the case under investigation ($a = b = 1$, $\sin\psi = 0$) this reduces to $R^t(Y_1\wedge J'Y_1) = -3\cos\psi\, e_1\wedge Je_1 - 2\cos\psi\, e_2\wedge Je_2$. Hence $R^t(Y_1\wedge J'Y_1)\wedge R^t(Y_1\wedge J'Y_1) = 12\cos^2\psi\, e_1\wedge Je_1\wedge e_2\wedge Je_2 = 0$, which is a contradiction since $\cos\psi = \pm 1$.

It follows that in case (A) we must have: $(\frac{a^2}{b^4}-\frac{1}{b^2}) = -\cos\xi = \mp 1$. $1-\frac{1}{b^2} = 1$ is impossible, hence we have: $1-\frac{1}{b^2} = -1$, i.e. $b^2 = \frac{1}{2}$, $a^2 = \frac{1}{4}$ and $\frac{1}{b^2}-\frac{a^2}{b^4} = 1$, which corresponds to the solution $S^5(\frac{\pi}{4})$.

Now we consider case (B): $(\frac{a^2}{b^4}-\frac{1}{b^2})\cos\xi = 2$, $\cos\psi = 0$. From $R^t(Y_1\wedge J'Y_2)\wedge R^t(Y_1\wedge J'Y_2) = 2(\frac{1}{b^2}-1)[(\frac{a^2}{b^4}-\frac{1}{b^2})\cos\xi + 1]e_1\wedge Je_2\wedge Je_1\wedge e_2 = 6(\frac{1}{b^2}-1)e_1\wedge Je_2\wedge Je_1\wedge e_2 = 0$, we conclude that $b=1$, $R^t(Y_1\wedge J'Y_2) = 3\cos\xi\, Je_1\wedge e_2$. $R^t(Y_0\wedge Y_1)\wedge R^t(Y_1\wedge J'Y_2) = 3(\frac{a^2}{b^4}-1)\cos\xi\, e_0\wedge e_1\wedge Je_1\wedge e_2 \neq 0$ (in case (B) we may assume that $\frac{a^2}{b^4}\neq 1$). Hence this contradicts Proposition 4 and (B) cannot occur.

We now consider case (C): $\sin\psi = 0$ and $(\frac{a^2}{b^4}-\frac{1}{b^2})\cos\xi = -1$. If $\cos\xi = 1$, this says $\frac{1}{b^2}-\frac{a^2}{b^4} = 1$, which is exactly what we wish to prove. Hence it is sufficient to show that $\cos\xi = -1$ is impossible.

Assume $\cos\xi = -1$, we now have: $Y_0 = e_0$, $Y_1 = e_1$, $J'Y_1 = \cos\psi\, Je_1$, $Y_2 = \cos\psi\, e_2$, $J'Y_2 = -Je_2$, where $\cos\psi = \pm 1$, $\frac{a^2}{b^4}-\frac{1}{b^2} = 1$. Now:

$$R(Y_0\wedge Y_1) = \frac{1}{b^2}e_0\wedge e_1 - Je_0\wedge Je_1$$
$$R(Y_0\wedge J'Y_1) = \bar{R}(e_0\wedge\cos\psi\, Je_1) - \hat{R}(Y_0\wedge J'Y_1)$$
$$= -\cos\psi\, e_0\wedge Je_1 + \cos\psi\, Je_0\wedge e_1 + \frac{a^2}{b^4}Y_0\wedge J'Y_1$$
$$= \left(\frac{a^2}{b^4}-1\right)\cos\psi\, e_0\wedge Je_1 + \cos\psi\, Je_0\wedge e_1$$
$$= \frac{1}{b^2}\cos\psi\, e_0\wedge Je_1 + \cos\psi\, Je_0\wedge e_1$$
$$R(Y_1\wedge J'Y_1) = -2\cos\psi\, e_0\wedge Je_0 - \left(7-\frac{1}{b^2}\right)\cos\psi\, e_1\wedge Je_1$$

(from (5.5)).

From Gauss' equation we compute:

G1. $\langle R^t(Y_0 \wedge Y_1)Y_0, Y_1\rangle = b_{00}b_{11} - b_{01}^2 = \dfrac{1}{b^2}$

$(b_{ij} = B(Y_i, Y_j),\ b_{i\bar{j}} = B(Y_i, J'Y_j),\ b_{\bar{i}\bar{j}} = B(J'Y_i, J'Y_j)).$

G2. $\langle R^t(Y_0 \wedge Y_1)Y_0, J'Y_1\rangle = b_{00}b_{1\bar{1}} - b_{01}b_{0\bar{1}} = 0$

G3. $\langle R^t(Y_0 \wedge Y_1)Y_1, J'Y_1\rangle = b_{01}b_{1\bar{1}} - b_{0\bar{1}}b_{11} = 0$

G4. $\langle R^t(Y_0 \wedge J'Y_1)Y_0, J'Y_1\rangle = b_{00}b_{\bar{1}\bar{1}} - b_{0\bar{1}}^2$

$= \dfrac{1}{b^2}\cos\psi\langle(e_0 \wedge Je_1)e_0, \cos\psi\, Je_1\rangle = \dfrac{1}{b^2}$

G5. $\langle R^t(Y_0 \wedge J'Y_1)Y_1, J'Y_1\rangle = b_{01}b_{\bar{1}\bar{1}} - b_{0\bar{1}}b_{1\bar{1}}$

$= \dfrac{1}{b^2}\cos\psi\langle(e_0 \wedge Je_1)e_1, \cos\psi Je_1\rangle = 0$

G6. $\langle R(Y_1 \wedge J'Y_1)Y_1, J'Y_1\rangle = b_{11}b_{\bar{1}\bar{1}} - b_{1\bar{1}}^2$

$= \left\langle -\left(7 - \dfrac{1}{b^2}\right)\cos\psi(e_1 \wedge Je_1)e_1, \cos\psi\, Je_1\right\rangle = \dfrac{1}{b^2} - 7.$

We have: $0 = \begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{00} & b_{01} & b_{0\bar{1}} \\ b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \end{vmatrix} = (b_{01}b_{\bar{1}\bar{1}} - b_{0\bar{1}}b_{1\bar{1}})b_{00} - (b_{00}b_{\bar{1}\bar{1}} - b_{0\bar{1}}^2)b_{01} +$ $(b_{00}b_{1\bar{1}} - b_{01}b_{0\bar{1}})b_{0\bar{1}} = -\frac{1}{b^2}b_{01}$ according to G2, G4 and G5, hence $b_{01} = 0$.

Similarly: $0 = \begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \\ b_{00} & b_{01} & b_{0\bar{1}} \end{vmatrix} = -\frac{1}{b^2}b_{0\bar{1}}$ (after developing the first line), hence $b_{0\bar{1}} = 0$

$0 = \begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \end{vmatrix} = \frac{1}{b^2}b_{1\bar{1}}$, hence $b_{1\bar{1}} = 0$.

The remaining equations are now: $b_{00}b_{11} = b_{00}b_{\bar{1}\bar{1}} = \frac{1}{b^2}$, $b_{11}b_{\bar{1}\bar{1}} = \frac{1}{b^2} - 7$. Hence $b_{11} = b_{\bar{1}\bar{1}} = (\frac{1}{b^2} - 7)^{1/2}$, $b_{00} = \frac{1}{b^2}(\frac{1}{b^2} - 7)^{-1/2}$.

To deduce a contradiction from this we need to apply the Codazzi-Mainardi equations. We first observe that (C) must hold at all points, hence $\sin\psi \equiv 0$ and $\cos\xi \equiv -1$. Hence the above values for b_{ij} must also be globally true. In particular, since b_{ij} are constants, $Y_k(b_{ij}) = J'Y_k(b_{ij}) = 0$, $i, j \in \{0, 1, \bar{1}\}$. The Codazzi-Mainardi equation states: $\langle R(X \wedge Y)Z, -Je_0\rangle = Y(B(X, Z)) - B(X, \nabla_Y Z) - X(B(Y, Z)) + B(Y, \nabla_X Z) + B([X, Y], Z)$, where B is the second fundamental form w.r.t. the normal $-Je_0$ of M at p; i.e. $\bar{\nabla}_X Y = \nabla_X Y - B(X, Y)(-Je_0) = \nabla_X Y + B(X, Y)Je_0$. We set $X = Y_0$, $Y = Y_1$, $Z = J'Y_1$ and get: $\langle R(Y_0 \wedge Y_1)J'Y_1, -Je_0\rangle = \cos\psi\langle\bar{R}(e_0 \wedge e_1)Je_1, -Je_0\rangle = \cos\psi\langle(-e_0 \wedge e_1 - Je_0 \wedge Je_1)Je_1, -Je_0\rangle = \cos\psi\langle Je_0, -Je_0\rangle = -\cos\psi = Y_1(b_{0\bar{1}}) - B(Y_0, \nabla_{Y_1}J'Y_1) - Y_0(b_{1\bar{1}}) + B(Y_1, \nabla_{Y_0}J'Y_1) + B([Y_0, Y_1], J'Y_1) = 0 + \frac{a}{b^2}b_{00} + (\frac{1}{a} - \frac{a}{b^2})b_{11} - \frac{1}{a}b_{\bar{1}\bar{1}} = \mp 1$. Here, $\frac{a}{b^2} = \frac{\sqrt{1+b^2}}{b}$ and

$b_{00} - b_{11} = \frac{1}{b^2}(\frac{1}{b^2} - 7)^{-1/2} - (\frac{1}{b^2} - 7)^{1/2} = 7b(1 - 7b^2)^{-1/2}$. So $\langle R(Y_0 \wedge Y_1)J'Y_1, -Je_0\rangle = 7(1+b^2)^{1/2}(1-7b^2)^{-1/2} = \mp 1$. $49(1+b^2) = 1 - 7b^2$. This is a contradition. q.e.d.

§6 We now embark upon the study of homogeneous 3-spheres isometrically immersed into $\mathbb{C}P(2)$. The first point is that in this case it is important to note that, as we explained above, the isometry group is $U(2)$ rather than $SU(2)$ (which acts simply transitively on S^3). On the contrary, in higher dimensions, we may consider S^{2n-1} as $SU(n)/SU(n-1)$, since the isotropy group for $SU(n)$ is $SU(n-1)$, with isotropy representation equal to $\mu_{n-1} \oplus \theta$ on $\mathbb{R}^{2n-2} \oplus \mathbb{R}$. Consequently, for $n > 2$ an invariant inner product is the standard inner product stretched by a factor a on $\mathbb{R}$ and by a factor b on $\mathbb{R}^{2n-2}$. For $n = 2$, $SU(1) \cong \{1\}$, and we do not have this limitation. Let $\mathbb{C}P(2)$ be coordinatized around $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ by $\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}$, z_1, $z_2 \in \mathbb{C}$. Let $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, and consider the geodesic $\exp(tX)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin t \\ 0 \\ \cos t \end{bmatrix} = \begin{bmatrix} \tan t \\ 0 \\ 1 \end{bmatrix}$. Similarly, with $E_0' = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$, $E_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ and $J'E_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}$ we have: $(\exp tE_0')\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i\tan t \\ 0 \\ 1 \end{bmatrix}$, $(\exp tE_1')\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \tan t \\ 1 \end{bmatrix}$, and $(\exp tJ'E_1')\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -i\tan t \\ 1 \end{bmatrix}$ (corresponding to tangent vectors in local coordinates: $\begin{pmatrix} -i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \end{pmatrix}$).

We prove that E_0', E_1', $J'E_1'$ correspond to $E_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $J'E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ respectively under the map $J(r)\colon \mathfrak{u}(2) \to T_{\exp rX}S^3(r)$ defined by $J(r)(Z) =$ the Killing field of Z at the point $\exp(rX)p$, (identified with $S^3(r)$, $r \in (0, \frac{\pi}{2})$), p is the base point, (see Proposition 3 in Tomter [To]). To do this it is sufficient to see that $[X, E_0] = E_0'$, $[X, E_1] = E_1'$ and $[X, J'E_1] = J'E_1'$. $[X, E_0] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = E_0'$. Similarly $[X, E_1] = E_1'$, $[X, J'E_1] = J'E_1'$. To check the isotropy subgroup of $U(2)$ on $\begin{bmatrix} r \\ 0 \\ 1 \end{bmatrix}$ we need only check the isotropy of the standard representation of $U(2)$ on $\begin{pmatrix} r \\ 0 \end{pmatrix}$, and that is $\begin{pmatrix} 1 & 0 \\ 0 & e^{it} \end{pmatrix} \cong U(1)$. This acts trivially on $\begin{pmatrix} it \\ 0 \end{pmatrix}$ and by the standard representation on $\begin{pmatrix} 0 \\ z \end{pmatrix} \cong \mathbb{C}$. Hence, by Schur's lemma an invariant inner product is given by the standard inner product on $\mathbb{R}^3 = Sp(E_0, E_1, J'E_1)$, stretched by a factor a on $\mathbb{R}E_0$ and a factor b on $\mathbb{R}^2 = Sp(E_1, J'E_1)$. We denote the sphere with this metric $S^3_{a,b}$. Since $S^3_{a,b}$ is smoothly diffeomorphic to $SU(2)$, E_0, E_1 and $J'E_1$ now constitute a basis of left invariant vector fields, and we wish to take advantage of that. We have $[E_0, E_1] = 2J'E_1$, $[E_0, J'E_1] = -2E_1$, $[E_1, J'E_1] = 2E_0$. Here E_0 defines the distinguished direction. Let $Y_0 = \frac{E_0}{a}$, $Y_1 = \frac{E_1}{b}$,

$J'Y_1 = \frac{J'E_1}{b}$ be unit vectors. Then $[Y_0, Y_1] = \frac{2}{a}J'Y_1$, $[Y_0, J'Y_1] = -\frac{2}{a}Y_1$, $[Y_1, J'Y_1] = \frac{2a}{b^2}Y_0$. Koszul's formula says that $2\langle\nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle[X,Y],Z\rangle - \langle[Y,Z],X\rangle + \langle[Z,X],Y\rangle$. By left invariance the three first terms vanish, hence $2\langle\nabla_{Y_0}Y_1, Y_0\rangle = \langle[Y_0,Y_1],Y_0\rangle - \langle[Y_1,Y_0],Y_0\rangle + \langle[Y_0,Y_0],Y_1\rangle = 2\langle\frac{2}{a}J'Y_1, Y_0\rangle = 0$. Similarly $2\langle\nabla_{Y_0}Y_1, Y_1\rangle = 0$ and $2\langle\nabla_{Y_0}Y_1, J'Y_1\rangle = 2(\frac{2}{a} - \frac{a}{b^2})$. Then $\nabla_{Y_0}Y_1 = (\frac{2}{a} - \frac{a}{b^2})J'Y_1$. By corresponding computations $\nabla_{Y_0}J'Y_1 = (\frac{a}{b^2} - \frac{2}{a})Y_1$, $\nabla_{Y_1}Y_0 = -\frac{a}{b^2}J'Y_1$, $\nabla_{Y_1}J'Y_1 = \frac{a}{b^2}Y_0$, $\nabla_{J'Y_1}Y_0 = \frac{a}{b^2}Y_1$, $\nabla_{J'Y_1}Y_1 = -\frac{a}{b^2}Y_0$ and $\nabla_{Y_0}Y_0 = \nabla_{Y_1}Y_1 = \nabla_{J'Y_1}J'Y_1 = 0$. By left invariance these formulas hold at all $p \in S^3_{a,b}$. Then we get for the curvature tensors: $\hat{R}(Y_0 \wedge Y_1)Y_0 = \nabla_{Y_0}\nabla_{Y_1}Y_0 - \nabla_{Y_1}\nabla_{Y_0}Y_0 - \nabla_{[Y_0,Y_1]}Y_0 = \nabla_{Y_0}(-\frac{a}{b^2}J'Y_1) - 0 - \frac{2}{a}\nabla_{J'Y_1}Y_0 = -\frac{a}{b^2}(\frac{a}{b^2} - \frac{2}{a})Y_1 - \frac{2}{a}(\frac{a}{b^2}Y_1) = -\frac{a^2}{b^4}Y_1$. By corresponding computations $\hat{R}(Y_0 \wedge Y_1)Y_1 = \frac{a^2}{b^4}Y_0$ and $\hat{R}(Y_0 \wedge Y_1)J'Y_1 = 0$. It follows that $\hat{R}(Y_0 \wedge Y_1) = -\frac{a^2}{b^4}Y_0 \wedge Y_1$. Similarly $\hat{R}(Y_0 \wedge J'Y_1) = -\frac{a^2}{b^4}Y_0 \wedge J'Y_1$ and $\hat{R}(Y_1 \wedge J'Y_1) = (3\frac{a^2}{b^4} - \frac{4}{b^2})Y_1 \wedge J'Y_1$. We also have:

$$\bar{R}(e_0 \wedge e_1) = -e_0 \wedge e_1 - Je_0 \wedge Je_1$$
$$\bar{R}(e_0 \wedge Je_1) = -e_0 \wedge Je_1 + Je_0 \wedge e_1$$
$$\bar{R}(e_1 \wedge Je_1) = -2e_0 \wedge Je_0 - 4e_1 \wedge Je_1$$

Now choose $-Je_0$ to be the unit normal to $S^3_{a,b} \subset \mathbb{C}P(2)$, then $J(-Je_0) = e_0$ is a tangent vector field to $S^3_{a,b}$. (e_1, Je_1) is an orthonormal frame field in the orthogonal complement of the line bundle defined by e_0 on $S^3_{a,b}$. We define it such that $Y_0 = \cos\varphi\, e_0 + \sin\varphi\, Je_1$.

Let $J'Y_1 = -\sin\varphi\, e_0 + \cos\varphi\, Je_1$ (orthonormal to Y_0), then $Y_1 = \pm e_1$. We compute: $R(Y_0 \wedge Y_1) = \bar{R}((\cos\varphi\, e_0 + \sin\varphi\, Je_1) \wedge (\pm e_1)) - \hat{R}(Y_0 \wedge Y_1) = \mp\cos\varphi\, e_0 \wedge e_1 \mp \cos\varphi\, Je_0 \wedge Je_1 \pm \sin\varphi(2e_0 \wedge Je_0 + 4e_1 \wedge Je_1) + \frac{a^2}{b^4}Y_0 \wedge Y_1$. We have: $e_0 = \cos\varphi\, Y_0 - \sin\varphi\, J'Y_1$, $e_1 = \pm Y_1$, $Je_1 = \sin\varphi\, Y_0 + \cos\varphi\, J'Y_1$. Substituting this we get: $R(Y_0 \wedge Y_1) = -\cos^2\varphi\, Y_0 \wedge Y_1 \mp \cos\varphi\, Je_0 \wedge (\sin\varphi\, Y_0 + \cos\varphi\, J'Y_1) \pm 2\sin\varphi(\cos\varphi\, Y_0 - \sin\varphi\, J'Y_1) \wedge Je_0 + 4\sin\varphi\, Y_1 \wedge (\sin\varphi\, Y_0 + \cos\varphi\, J'Y_1) - \sin\varphi\cos\varphi\, Y_1 \wedge J'Y_1 + \frac{a^2}{b^4}Y_0 \wedge Y_1 = (\frac{a^2}{b^4} - 1 - 3\sin^2\varphi)Y_0 \wedge Y_1 + 3\sin\varphi\cos\varphi\, Y_1 \wedge J'Y_1 \mp Je_0 \wedge (3\sin\varphi\cos\varphi\, Y_0 + (1 - 3\sin^2\varphi)J'Y_1)$. By similar computations we get the result:

(6.1)

$$\begin{aligned} R(Y_0 \wedge Y_1) &= \left(\frac{a^2}{b^4} - 1 - 3\sin^2\varphi\right) Y_0 \wedge Y_1 + 3\sin\varphi\cos\varphi\, Y_1 \wedge J'Y_1 \\ &\quad \mp Je_0 \wedge (3\sin\varphi\cos\varphi\, Y_0 + (1 - 3\sin^2\varphi)J'Y_1) \\ R(Y_0 \wedge J'Y_1) &= \left(\frac{a^2}{b^4} - 1\right) Y_0 \wedge J'Y_1 \pm Je_0 \wedge Y_1 \end{aligned}$$

$$R(Y_1 \wedge J'Y_1) = 3\sin\varphi\cos\varphi\, Y_0 \wedge Y_1 \\ - \left(4 + 3\frac{a^2}{b^4} - \frac{4}{b^2} - 3\sin^2\varphi\right) Y_1 \wedge J'Y_1 \\ \mp Je_0 \wedge ((-2 + 3\sin^2\varphi)Y_0 + 3\sin\varphi\cos\varphi\, J'Y_1).$$

We are now in a position to consider the Gauss equations:

G1. $\langle R^t(Y_0 \wedge Y_1)Y_0, Y_1\rangle = \dfrac{a^2}{b^4} - 1 - 3\sin^2\varphi = b_{00}b_{11} - b_{01}^2$

G2. $\langle R^t(Y_0 \wedge Y_1)Y_0, J'Y_1\rangle = 0 = b_{00}b_{1\bar{1}} - b_{01}b_{0\bar{1}}$

G3. $\langle R^t(Y_0 \wedge Y_1)Y_1, J'Y_1\rangle = 3\sin\varphi\cos\varphi = b_{01}b_{1\bar{1}} - b_{11}b_{0\bar{1}}$

G4. $\langle R^t(Y_0 \wedge J'Y_1)Y_0, J'Y_1\rangle = \dfrac{a^2}{b^4} - 1 = b_{00}b_{\bar{1}\bar{1}} - b_{0\bar{1}}^2$

G5. $\langle R^t(Y_0 \wedge J'Y_1)Y_1, J'Y_1\rangle = 0 = b_{01}b_{\bar{1}\bar{1}} - b_{0\bar{1}}b_{1\bar{1}}$

G6. $\langle R^t(Y_1 \wedge J'Y_1)Y_1, J'Y_1\rangle = \dfrac{4}{b^2} - 3\dfrac{a^2}{b^4} - 4 + 3\sin^2\varphi = b_{11}b_{\bar{1}\bar{1}} - b_{1\bar{1}}^2.$

Now consider: $\begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{00} & b_{01} & b_{0\bar{1}} \\ b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \end{vmatrix} = 0 = (b_{01}b_{\bar{1}\bar{1}} - b_{0\bar{1}}b_{1\bar{1}})b_{00} - (b_{00}b_{\bar{1}\bar{1}} - b_{0\bar{1}}^2)b_{01} + (b_{00}b_{1\bar{1}} - b_{01}b_{0\bar{1}})b_{0\bar{1}} = 0 + (1 - \frac{a^2}{b^4})b_{01} + 0$ according to G5, G4 and G2.

D1: Hence $(1 - \frac{a^2}{b^4})b_{01} = 0$ (and for $a^2 \neq b^4 : b_{01} = 0$).

Similarly by considering $\begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \\ b_{00} & b_{01} & b_{0\bar{1}} \end{vmatrix} = 0$ we obtain:

D2: $3\sin\varphi\cos\varphi\, b_{00} + (\frac{a^2}{b^4} - 1 - 3\sin^2\varphi)b_{0\bar{1}} = 0.$

$\begin{vmatrix} b_{01} & b_{11} & b_{1\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \\ b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \end{vmatrix} = 0$ gives:

D3: $(\frac{4}{b^2} - 3\frac{a^2}{b^4} - 4 + 3\sin^2\varphi)b_{01} + 3\sin\varphi\cos\varphi\, b_{1\bar{1}} = 0$

$\begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \end{vmatrix} = 0$ gives:

D4: $3\sin\varphi\cos\varphi\, b_{01} + (\frac{a^2}{b^4} - 1 - 3\sin^2\varphi)b_{1\bar{1}} = 0$

$\begin{vmatrix} b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \\ b_{01} & b_{11} & b_{1\bar{1}} \\ b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \end{vmatrix} = 0$ gives:

D5: $(\frac{4}{b^2} - 3\frac{a^2}{b^4} - 4 + 3\sin^2\varphi)b_{0\bar{1}} + 3\sin\varphi\cos\varphi\, b_{\bar{1}\bar{1}} = 0$

and $\begin{vmatrix} b_{00} & b_{01} & b_{0\bar{1}} \\ b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \\ b_{0\bar{1}} & b_{1\bar{1}} & b_{\bar{1}\bar{1}} \end{vmatrix} = 0$ gives:

D6: $(1 - \frac{a^2}{b^4})b_{1\bar{1}} = 0.$

We first deal with the case $a^2 \neq b^4$. Then, by D1 and D6: $b_{01} = b_{1\bar{1}} = 0$. Furthermore

D7: $\det B = \begin{vmatrix} b_{00} & 0 & b_{0\bar{1}} \\ 0 & b_{11} & 0 \\ b_{0\bar{1}} & 0 & b_{\bar{1}\bar{1}} \end{vmatrix} = (\frac{4}{b^2} - 3\frac{a^2}{b^4} - 4 + 3\sin^2\varphi)b_{00} + 3\sin\varphi\cos\varphi\, b_{0\bar{1}} = (\frac{a^2}{b^4} - 1)b_{11} = 3\sin\varphi\cos\varphi\, b_{0\bar{1}} + (\frac{a^2}{b^4} - 1 - 3\sin^2\varphi)b_{\bar{1}\bar{1}}$.

Proposition 6. *For $a^2 \neq b^4$ we have $\det B \neq 0$ on an open dense subset of $S^3_{a,b}$.*

Proof. Assume $\det B = 0$ on an open, nonempty subset. If $\det B = 0$, it follows from D7 that $b_{11} = 0$, from G3 that $\sin\varphi\cos\varphi = 0$, and from G1 that $\frac{a^2}{b^4} - 1 - 3\sin^2\varphi = 0$, hence $\sin\varphi \neq 0$ and $\cos\varphi = 0$. We then have $\frac{a^2}{b^4} = 4$. From G6 we get: $0 = \frac{4}{b^2} - 12 - 4 + 3$, i.e. $\frac{4}{b^2} = 13$, $b^2 = \frac{4}{13}$, $a^2 = \frac{64}{169}$. We wish to apply the Codazzi-Mainardi equations, these say that $\langle R(X \wedge Y)Z, -Je_0\rangle = Y(B(X,Z)) - B(X, \nabla_Y Z) - X(B(Y,Z)) + B(Y, \nabla_X Z) + B([X,Y],Z)$.

There are 9 basic cases of this equation. We substitute $X = Y_0$, $Y = Y_1$, $Z = Y_0$ to get: $\langle R(Y_0 \wedge Y_1)Y_0, -Je_0\rangle = \pm\langle R(\sin\varphi\, Je_1 \wedge e_1)\sin\varphi\, Je_1, -Je_0\rangle = \pm\langle(2e_0 \wedge Je_0 + 4e_1 \wedge Je_1)Je_1, -Je_0\rangle = 0 = Y_1(b_{00}) - B(Y_0, \nabla_{Y_1}Y_0) - Y_0(b_{01}) + B(Y_1, \nabla_{Y_0}Y_0) + B([Y_0,Y_1],Y_0) = Y_1(b_{00}) - B(Y_0, -\frac{a}{b^2}J'Y_1) - 0 + 0 + B(\frac{2}{a}J'Y_1, Y_0) = Y_1(b_{00}) + \frac{a}{b^2}b_{0\bar{1}} + \frac{2}{a}b_{0\bar{1}} = Y_1(b_{00}) + \frac{21}{4}b_{0\bar{1}}$. Hence $Y_1(b_{00}) = -\frac{21}{4}b_{0\bar{1}}$. This case of the Codazzi-Mainardi equation we denote by (C1).

Similarly, we have (C2) for $X = Y_0$, $Y = Y_1$, $Z = Y_1$ and (C3) for $X = Y_0$, $Y = Y_1$, $Z = J'Y_1$. Also:

(C4):	$X = Y_0$,	$Y = J'Y_1$,	$Z = Y_0$
(C5):	$X = Y_0$,	$Y = J'Y_1$,	$Z = Y_1$
(C6):	$X = Y_0$,	$Y = J'Y_1$,	$Z = J'Y_1$
(C7):	$X = Y_1$,	$Y = J'Y_1$,	$Z = Y_0$
(C8):	$X = Y_1$,	$Y = J'Y_1$,	$Z = Y_1$
(C9):	$X = Y_1$,	$Y = J'Y_1$,	$Z = J'Y_1$.

We have (by (6.1)): (C5): $\langle R(Y_0 \wedge J'Y_1)Y_1, -Je_0\rangle = \langle((\frac{a^2}{b^4} - 1)Y_0 \wedge J'Y_1 \pm Je_0 \wedge Y_1)Y_1, -Je_0\rangle = \langle\pm(Je_0 \wedge Y_1)Y_1, -Je_0\rangle = \pm 1 = J'Y_1(b_{01}) - B(Y_0, \nabla_{J'Y_1}Y_1) - Y_0(b_{1\bar{1}}) + B(J'Y_1, \nabla_{Y_0}Y_1) + B([Y_0, J'Y_1], Y_1) = 0 - B(Y_0, -\frac{a}{b^2}Y_0) - 0 + B(J'Y_1, (\frac{2}{a} - \frac{a}{b^2})J'Y_1) + B(-\frac{2}{a}Y_1, Y_1) = 2b_{00} + \frac{5}{4}b_{\bar{1}\bar{1}} - \frac{13}{4}b_{11} = 2b_{00} + \frac{5}{4}b_{\bar{1}\bar{1}}$.

Finally, by (6.1) and $\cos\varphi = 0$ we have (C9):

$$\langle R(Y_1 \wedge J'Y_1)J'Y_1, -Je_0\rangle$$
$$= \langle\mp(Je_0 \wedge ((-2 + 3\sin^2\varphi)Y_0 + 3\sin\varphi\cos\varphi\, J'Y_1))J'Y_1, -Je_0\rangle = 0$$

$$
\begin{aligned}
&= J'Y_1(b_{1\bar{1}}) - B(Y_1, \nabla_{J'Y_1} J'Y_1) - Y_1(b_{\bar{1}\bar{1}}) + B(J'Y_1, \nabla_{Y_1} J'Y_1) \\
&\quad + B([Y_1, J'Y_1], J'Y_1) \\
&= 0 - 0 - Y_1(b_{\bar{1}\bar{1}}) + B\left(J'Y_1, \frac{a}{b^2} Y_0\right) + B\left(\frac{2a}{b^2} Y_0, J'Y_1\right) \\
&= -Y_1(b_{\bar{1}\bar{1}}) + 6b_{0\bar{1}},
\end{aligned}
$$

i.e. $Y_1(b_{\bar{1}\bar{1}}) = 6b_{0\bar{1}}$. Now, we differentiate $2b_{00} + \frac{5}{4}b_{\bar{1}\bar{1}} = \pm 1$ along the vector field Y_1 and obtain: $0 = 2Y_1(b_{00}) + \frac{5}{4}Y(b_{\bar{1}\bar{1}}) = -\frac{21}{2}b_{0\bar{1}} + \frac{15}{2}b_{0\bar{1}} = -\frac{6}{2}b_{0\bar{1}}$, hence $b_{0\bar{1}} = 0$.

It only remains to determine b_{00} and $b_{\bar{1}\bar{1}}$. By (6.1) and $\cos\varphi = 0$ (C3) gives

$$
\begin{aligned}
&\langle R(Y_0 \wedge Y_1) J'Y_1, -Je_0 \rangle \\
&= \mp \langle -(Je_0 \wedge 2J'Y_1) J'Y_1, -Je_0 \rangle = \pm 2 \\
&= Y_1(b_{0\bar{1}}) - B(Y_0, \nabla_{Y_1} J'Y_1) - Y_0(b_{1\bar{1}}) + B(Y_1, \nabla_{Y_0} J'Y_1) \\
&\quad + B([Y_0, Y_1], J'Y_1) \\
&= 0 - B\left(Y_0, \frac{a}{b^2} Y_0\right) - 0 + B\left(Y_1, \left(\frac{a}{b^2} - \frac{2}{a}\right) Y_1\right) + B\left(\frac{2}{a} J'Y_1, J'Y_1\right) \\
&= -2b_{00} + \frac{13}{4} b_{\bar{1}\bar{1}}.
\end{aligned}
$$

Hence $\frac{18}{4}b_{\bar{1}\bar{1}} = \pm 1 \pm 2 = \pm 3$, $b_{\bar{1}\bar{1}} = \pm\frac{2}{3}$ from (C3) and (C5). Hence, from (C9), $b_{00} = -\frac{5}{8}b_{\bar{1}\bar{1}} = \mp\frac{5}{12}$. Hence $b_{00}b_{\bar{1}\bar{1}} = -\frac{5}{18}$, but this is a contradiction, since by (G4) $b_{00}b_{\bar{1}\bar{1}} = 3$. Hence: $\det B \equiv 0$ on an open set is impossible when $a^2 \neq b^4$. q.e.d.

We continue the computation on the open set $\{p \in S^3_{a,b} | \det B(p) \neq 0\}$.

$$
\begin{aligned}
(\det B)^3 &= (\det B) \begin{vmatrix} b_{11}b_{\bar{1}\bar{1}} - b_{1\bar{1}}^2 & b_{0\bar{1}}b_{1\bar{1}} - b_{01}b_{\bar{1}\bar{1}} & b_{01}b_{1\bar{1}} - b_{0\bar{1}}b_{11} \\ b_{0\bar{1}}b_{1\bar{1}} - b_{01}b_{\bar{1}\bar{1}} & b_{00}b_{\bar{1}\bar{1}} - b_{0\bar{1}}^2 & b_{01}b_{0\bar{1}} - b_{00}b_{1\bar{1}} \\ b_{01}b_{1\bar{1}} - b_{0\bar{1}}b_{11} & b_{01}b_{0\bar{1}} - b_{00}b_{1\bar{1}} & b_{00}b_{11} - b_{01}^2 \end{vmatrix} \\
&= (\det B) \begin{vmatrix} \frac{4}{b^2} - 3\frac{a^2}{b^4} + 3\sin^2\varphi - 4 & 0 & 3\sin\varphi\cos\varphi \\ 0 & \frac{a^2}{b^4} - 1 & 0 \\ 3\sin\varphi\cos\varphi & 0 & \frac{a^2}{b^4} - 1 - 3\sin^2\varphi \end{vmatrix} \\
&= (\det B)\left(\frac{a^2}{b^4} - 1\right)\left[\left(\frac{4}{b^2} - 3\frac{a^2}{b^4} + 3\sin^2\varphi - 4\right)\left(\frac{a^2}{b^4} - 1 - 3\sin^2\varphi\right)\right. \\
&\qquad \left. - 9\sin^2\varphi\cos^2\varphi\right] \\
&= \left(\frac{a^2}{b^4} - 1\right)\left[4\frac{a^2}{b^6} - \frac{4}{b^2} - \frac{12}{b^2}\sin^2\varphi - 3\frac{a^4}{b^8} + 3\frac{a^2}{b^4} + 9\frac{a^2}{b^4}\sin^2\varphi\right.
\end{aligned}
$$

$$+3\frac{a^2}{b^4}\sin^2\varphi - 3\sin^2\varphi - 9\sin^4\varphi - 4\frac{a^2}{b^4} + 4 + 12\sin^2\varphi$$
$$\left. - 9\sin^2\varphi + 9\sin^4\varphi\right] (\det B)$$
$$= \left(\frac{a^2}{b^4} - 1\right)\left[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4\right.$$
$$\left. + 12\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\sin^2\varphi\right] (\det B).$$

We cancel $\det B$ and obtain $(\det B)^2 = (\frac{a^2}{b^4} - 1)[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4 + 12(\frac{a^2}{b^4} - \frac{1}{b^2})\sin^2\varphi]$. Now we substitute $(\frac{a^2}{b^4} - 1 - 3\sin^2\varphi)b_{0\bar{1}} = -3\sin\varphi\cos\varphi\, b_{00}$ (D2) into D7 to obtain:

$$\text{(6.2)}\quad \left(\frac{a^2}{b^4} - 1 - 3\sin^2\varphi\right)\det B$$
$$= \left(\frac{a^2}{b^4} - 1 - 3\sin^2\varphi\right)\left(\frac{4}{b^2} - 3\frac{a^2}{b^4} - 4 + 3\sin^2\varphi\right) b_{00}$$
$$+3\sin\varphi\cos\varphi(-3\sin\varphi\cos\varphi)b_{00}$$
$$= \left[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4 + 12\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\sin^2\varphi\right] b_{00}.$$

Similarly, by substituting for $b_{0\bar{1}}$ from D5 into D7 we obtain:

$$\text{(6.3)}\quad \left(\frac{4}{b^2} - 3\frac{a^2}{b^4} - 4 + 3\sin^2\varphi\right)\det B$$
$$= \left[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4 + 12\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\sin^2\varphi\right] b_{\bar{1}\bar{1}},$$

and we already have

$$\text{(6.4)}\qquad \det B = \left(\frac{a^2}{b^4} - 1\right) b_{11}.$$

Proposition 7. *We have* $\sin\varphi$ *is constant on* $S^3_{a,b}$ *(i.e.* φ *is constant) in the case* $a^2 \neq b^4$.

Proof. Consider the Codazzi-Mainardi equation

$$\langle \bar{R}(Y_0 \wedge J'Y_1)Y_1, -Je_0\rangle = \pm\langle \bar{R}(e_0 \wedge Je_1)e_1, -Je_0\rangle$$
$$= \pm\langle(-e_0 \wedge Je_1 + Je_0 \wedge e_1)e_1, -Je_0\rangle = \pm 1$$
$$= J'Y_1(b_{01}) - B(Y_0, \nabla_{J'Y_1}Y_1) - Y_0(b_{1\bar{1}})$$
$$+ B(J'Y_1, \nabla_{Y_0}Y_1) + B([Y_0, J'Y_1], Y_1).$$

Since $b_{01} = b_{1\bar{1}} \equiv 0$ we have $J'Y_1(b_{01}) = Y_0(b_{1\bar{1}}) = 0$ and (C5) $\frac{a}{b^2}b_{00} - \frac{2}{a}b_{11} + (\frac{2}{a} - \frac{a}{b^2})b_{\bar{1}\bar{1}} = \pm 1$. Multiply both sides by $(\frac{a^2}{b^4} - 1)[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4 + 12(\frac{a^2}{b^4} - \frac{1}{b^2})\sin^2\varphi]$. We obtain:

$$\begin{aligned}
&\frac{a}{b^2}\left(\frac{a^2}{b^4} - 1\right)\left(\frac{a^2}{b^4} - 1 - 3\sin^2\varphi\right)\det B \\
&\quad - \frac{2}{a}\left[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4 + 12\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\sin^2\varphi\right]\det B \\
&\quad + \left(\frac{2}{a} - \frac{a}{b^2}\right)\left(\frac{a^2}{b^4} - 1\right)\left(\frac{4}{b^2} - 3\frac{a^2}{b^4} - 4 + 3\sin^2\varphi\right)\det B \\
&= \pm\left(\frac{a^2}{b^4} - 1\right)\left[4\frac{a^2}{b^6} - 3\frac{a^4}{b^8} - \frac{a^2}{b^4} - \frac{4}{b^2} + 4 + 12\left(\frac{a^2}{b^4} - \frac{1}{b^2}\right)\sin^2\varphi\right].
\end{aligned}$$

We wish to keep track of the terms involving $\sin\varphi$. On the RHS (right hand side) this is simply $\pm 12(\frac{a^2}{b^4} - 1)(\frac{a^2}{b^4} - \frac{1}{b^2})\sin^2\varphi$. On the LHS we have $(R_1(a,b)\sin^2\varphi + R_2(a,b))\det B$ where $R_1(a,b)$ and $R_2(a,b)$ are rational functions in a and b. Assume first that $\frac{a^2}{b^4} \neq \frac{1}{b^2}$. Then $\sin^2\varphi$ does occur with nonzero coefficient both in RHS and in $(\det B)^2$. Assume first that $R_1(a,b) \neq 0$. Then, on squaring both sides of the equation, we get on the LHS: $12R_1(a,b)^2(\frac{a^2}{b^4} - 1)(\frac{a^2}{b^4} - \frac{1}{b^2})\sin^6\varphi$, hence the coefficient of $\sin^6\varphi$ is nonzero. On the RHS we only have terms of $\sin^4\varphi$ and $\sin^2\varphi$. We now have a polynomial of degree 3 in $\sin^2\varphi$ equal to zero. We conclude that $\sin\varphi$ is constant on the open, dense subset $\{p \in S^3_{a,b} \mid \det B(p) \neq 0\}$ and hence on $S^3_{a,b}$. Next assume that $R_1(a,b) = 0$. Then, on the RHS we have $144(\frac{a^2}{b^4} - 1)^2(\frac{a^2}{b^4} - \frac{1}{b^2})^2\sin^4\varphi +$ lower order terms, on the LHS we only have $\sin^2\varphi$-terms. Hence, since the coefficient of $\sin^4\varphi$ is nonzero, we again have that $\sin\varphi$ is constant. This finishes the case $\frac{a^2}{b^4} \neq \frac{1}{b^2}$.

Now assume $\frac{a^2}{b^4} = \frac{1}{b^2}$, i.e. $a = b$. There are no $\sin^2\varphi$-terms in RHS or in $(\det B)^2$. We compute $R_1(a,b) = \frac{1}{b}(\frac{1}{b^2} - 1)(-3) + 3\frac{1}{b}(\frac{1}{b^2} - 1) = 0$, hence there are no φ-terms at all in this case. For the remaining terms we obtain:

$$\begin{aligned}
&\left[\frac{1}{b}\left(\frac{1}{b^2} - 1\right)^2 - \frac{2}{b}\left(\frac{1}{b^4} - \frac{5}{b^2} + 4\right) + \frac{1}{b}\left(\frac{1}{b^2} - 1\right)\left(\frac{1}{b^2} - 4\right)\right]\det B \\
&\quad = \pm\left(\frac{1}{b^2} - 1\right)\left(\frac{1}{b^4} - \frac{5}{b^2} + 4\right).
\end{aligned}$$

$\frac{3}{b}(\frac{1}{b^2} - 1)\det B = \pm(\frac{1}{b^2} - 1)(\frac{1}{b^4} - \frac{5}{b^2} + 4)$. Since $\frac{a^2}{b^4} = 1$ for $b = 1$, this is impossible, hence we get $\frac{3}{b}\sqrt{(\frac{1}{b^2} - 1)(\frac{1}{b^4} - \frac{5}{b^2} + 4)} = \pm(\frac{1}{b^4} - \frac{5}{b^2} + 4)$. By

Proposition 7 $\frac{1}{b^4}-\frac{5}{b^2}+4 \neq 0$, hence $\frac{9}{b^2}(\frac{1}{b^2}-1)(\frac{1}{b^4}-\frac{5}{b^2}+4) = (\frac{1}{b^4}-\frac{5}{b^2}+4)^2$ and $\frac{9}{b^2}(\frac{1}{b^2}-1) = \frac{1}{b^4}-\frac{5}{b^2}+4 = (\frac{1}{b^2}-1)(\frac{1}{b^2}-4)$. So $\frac{9}{b^2} = \frac{1}{b^2}-4$, i.e. $\frac{8}{b^2} = -4$ but this is a contradiction. Hence this case cannot occur. q.e.d.

Proposition 8. *In the case* $a^2 \neq b^4$ *we have* $\sin\varphi\cos\varphi = 0$.

Lemma. *If* φ *is constant on* $S^3_{a,b}$*, it follows that* b_{ij} *is constant,* i, $j = 0, 1, \bar{1}$.

proof of Lemma. First, according to the explanation after the proof of Proposition 7, we have $(\det B)^2 = (\frac{a^2}{b^4}-1)\,[4\frac{a^2}{b^6}-3\frac{a^4}{b^8}-\frac{a^2}{b^4}-\frac{4}{b^2}+4+12(\frac{a^2}{b^4}-\frac{1}{b^2})\sin^2\varphi]$ on $\{p \in S^3_{a,b} \mid \det B(p) \neq 0\}$, hence $\det B$ is constant there, and consequently everywhere. From D7 it follows that b_{11} is constant and non-zero; hence, by G3 it follows that $b_{0\bar{1}}$ is constant. Now, if $\sin\varphi\cos\varphi \neq 0$, it follows from D2 and D5 that b_{00} and $b_{\bar{1}\bar{1}}$ also are constants. If $\sin\varphi\cos\varphi = 0$ we use D7. Since $\{p \in S^3_{a,b} \mid \det B(p) \neq 0\}$ is open dense in $S^3_{a,b}$, we obtain the desired result. q.e.d.

Proof. By Proposition 8 φ is constant. By (C1)

$$\begin{aligned}
&\langle \bar{R}(Y_0 \wedge Y_1)Y_0, -Je_0\rangle \\
&\quad= \pm\,\langle \bar{R}((\cos\varphi\, e_0 + \sin\varphi Je_1)\wedge e_1)(\cos\varphi\, e_0 + \sin\varphi\, Je_1), -Je_0\rangle \\
&\quad= \pm\,\langle(-\cos\varphi\, e_0 \wedge e_1 - \cos\varphi\, Je_0 \wedge Je_1 + 2\sin\varphi\, e_0 \wedge Je_0 \\
&\qquad + 4\sin\varphi\, e_1 \wedge Je_1)(\cos\varphi e_0 + \sin\varphi\, Je_1), -Je_0\rangle \\
&\quad= \pm\,\langle 2\sin\varphi\cos\varphi\, Je_0 + \sin\varphi\cos\varphi Je_0, -Je_0\rangle = \mp 3\sin\varphi\cos\varphi \\
&\quad= Y_1(b_{00}) - B(Y_0, \nabla_{Y_1}Y_0) - Y_0(b_{01}) \\
&\qquad + B(Y_1, \nabla_{Y_0}Y_0) + B([Y_0, Y_1], Y_0) \\
&\quad= \frac{a}{b^2}b_{0\bar{1}} + \frac{2}{a}b_{0\bar{1}} = \left(\frac{a}{b^2}+\frac{2}{a}\right)b_{0\bar{1}}.
\end{aligned}$$

Similarly, (C9)

$$\begin{aligned}
&\langle \bar{R}(Y_1 \wedge J'Y_1)J'Y_1, -Je_0\rangle \\
&\quad= \pm\,\langle \bar{R}(e_1 \wedge (-\sin\varphi\, e_0 + \cos\varphi\, Je_1))(-\sin\varphi\, e_0 + \cos\varphi\, Je_1), -Je_0\rangle \\
&\quad= \pm\,\langle(-\sin\varphi\, e_0 \wedge e_1 - \sin\varphi\, Je_0 \wedge Je_1 - 2\cos\varphi\, e_0 \wedge Je_0 \\
&\qquad - 4\cos\varphi\, e_1 \wedge Je_1)(-\sin\varphi\, e_0 + \cos\varphi\, Je_1), -Je_0\rangle \\
&\quad= \pm\,\langle\sin\varphi\cos\varphi\, Je_0 + 2\sin\varphi\cos\varphi\, Je_0, -Je_0\rangle \\
&\quad= \mp\, 3\sin\varphi\cos\varphi = J'Y_1(b_{1\bar{1}}) - B(Y_1, \nabla_{J'Y_1}J'Y_1) - Y_1(b_{\bar{1}\bar{1}}) \\
&\qquad + B(J'Y_1, \nabla_{Y_1}J'Y_1) + B([Y_1, J'Y_1], J'Y_1) \\
&\quad= \frac{a}{b^2}b_{0\bar{1}} + \frac{2a}{b^2}b_{0\bar{1}} = \frac{3a}{b^2}b_{0\bar{1}}.
\end{aligned}$$

We subtract C1 from C9 and get: $(\frac{2a}{b^2} - \frac{2}{a})b_{0\bar{1}} = 0$. Hence $b_{0\bar{1}} = 0$ unless $\frac{2a}{b^2} = \frac{2}{a}$, i.e. $a = b$. But $a = b$ was proved not to occur at the end of the proof of Proposition 8. Hence $b_{0\bar{1}} = 0$ and $\sin\varphi\cos\varphi = 0$. q.e.d.

§7 We continue with the case $a^2 \neq b^4$, and we only need to check $\sin\varphi = 0$ and $\cos\varphi = 0$.

Theorem 9. *For $a^2 \neq b^4$ and $\sin\varphi = 0$; we have: $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$.*

Proof. By D1, D2 and D6 we have $b_{01} = b_{0\bar{1}} = b_{1\bar{1}} \equiv 0$. Furthermore we may choose $Y_0 = e_0$, $Y_1 = \pm e_1$, $J'Y_1 = Je_1$. From G1 and G4 we have: $\frac{a^2}{b^4} - 1 = b_{00}b_{11} = b_{00}b_{\bar{1}\bar{1}}$. From this $b_{11} = b_{\bar{1}\bar{1}}$.

We now apply an efficient method. Both the curvature tensor $\bar{R}$ and the complex structure J of $\mathbb{C}P(2)$ are parallel. Hence $\bar{\nabla}_X J = 0$ for any X. Consider for example $\bar{\nabla}_{Y_1} J'Y_1 = \nabla_{Y_1} J'Y_1 + b_{1\bar{1}} Je_0 = \frac{a}{b^2} Y_0 + 0 = \frac{a}{b^2} Y_0$. On the other hand, writing C for contraction, we have: $\bar{\nabla}_{Y_1} J'Y_1 = \bar{\nabla}_{Y_1} Je_1 = \bar{\nabla}_{Y_1}(C(J \otimes e_1)) = C(\bar{\nabla}_{Y_1}(J \otimes e_1)) = C(\bar{\nabla}_{Y_1} J \otimes e_1 + J \otimes \bar{\nabla}_{Y_1} e_1) = 0 + J(\bar{\nabla}_{Y_1} e_1) = \pm J(\bar{\nabla}_{Y_1} Y_1) = \pm J(\nabla_{Y_1} Y_1 + b_{11} Je_0) = 0 \pm b_{11}(J^2 e_0) = \mp b_{11} e_0 = \mp b_{11} Y_0$. So $b_{11} = b_{\bar{1}\bar{1}} = \mp \frac{a}{b^2}$. Substituting into G6 we obtain: $\frac{a^2}{b^4} = \frac{4}{b^2} - 3\frac{a^2}{b^4} - 4$. Hence $4(\frac{1}{b^2} - \frac{a^2}{b^4}) = 4$, and the result follows. q.e.d.

Proposition 10. *For $a^2 \neq b^4$ we cannot have $\cos\varphi = 0$ except possibly for $a = \sqrt{3}$, $b = 1$.*

Proof. In this case we have: $Y_0 = Je_1$, $Y_1 = \pm e_1$, $J'Y_1 = -e_0$, as well as $b_{01} = b_{1\bar{1}} = 0$. It is easy to see that $b_{0\bar{1}} = 0$ also: From D7 ($\det B = (\frac{a^2}{b^4} - 1)b_{11}$) and Proposition 7, we have that $b_{11} \neq 0$ on the open dense subset of $S^3_{a,b}$. Then, from G3, we immediately have $b_{0\bar{1}} = 0$ on $S^3_{a,b}$ because $\cos\varphi = 0$ and $b_{01} = 0$.

We have $\bar{\nabla}_{Y_0} Y_0 = \nabla_{Y_0} Y_0 + b_{00} Je_0 = b_{00} Je_0$, and also: $\bar{\nabla}_{Y_0} Y_0 = \bar{\nabla}_{Y_0} C(J \otimes e_1) = C\bar{\nabla}_{Y_0}(J \otimes e_1) = \pm C(J \otimes \bar{\nabla}_{Y_0} Y_1) = \pm(\frac{a}{b^2} - \frac{2}{a}) Je_0$. So $b_{00} = \pm(\frac{a}{b^2} - \frac{2}{a})$. Similarly $\bar{\nabla}_{Y_1} Y_0 = \nabla_{Y_1} Y_0 + b_{01} Je_0 = -\frac{a}{b^2} J'Y_1 = \frac{a}{b^2} e_0$ and also $\bar{\nabla}_{Y_1} Y_0 = \bar{\nabla}_{Y_1}(C(J \otimes e_1)) = \pm C(J \otimes \bar{\nabla}_{Y_1} Y_1) = \pm J(\nabla_{Y_1} Y_1 + b_{11} Je_0) = \mp b_{11} e_0$. Hence $b_{11} = \mp \frac{a}{b^2}$. From G1 we have $b_{00}b_{11} = \frac{a^2}{b^4} - 4 = -\frac{a}{b^2}(\frac{a}{b^2} - \frac{2}{a})$, so $\frac{a^2}{b^4} = \frac{1}{b^2} + 2$, $a^2 = b^2 + 2b^4$. We have from G4 and G6:

$$b_{11}^2 = \frac{a^2}{b^4} = \frac{b_{00}b_{11}b_{11}b_{\bar{1}\bar{1}}}{b_{00}b_{\bar{1}\bar{1}}} = \left(\frac{a^2}{b^4} - 4\right)\left(\frac{4}{b^2} - \frac{3a^2}{b^4} - 1\right)\left(\frac{a^2}{b^4} - 1\right)^{-1}.$$

So $\frac{a^2}{b^4}(\frac{a^2}{b^4} - 1) = (\frac{a^2}{b^4} - 4)(\frac{4}{b^2} - \frac{3a^2}{b^4} - 1)$. From this we get $\frac{a^4}{b^8} + \frac{4}{b^2} = \frac{a^2}{b^6} + \frac{3a^2}{b^4} + 1$, and substituting $a^2 = b^2 + 2b^4$ we get $b = 1$, $a = \sqrt{3}$. q.e.d.

It is easily seen that this solution, with $b_{00} = \pm\frac{1}{\sqrt{3}}$, $b_{11} = \mp\sqrt{3}$, $b_{\bar{1}\bar{1}} = \pm 2\sqrt{3}$ satisfies all Gauss equations, Codazzi-Mainardi equations, and also the requirement that J be parallel. Since the principal curvatures b_{ij} are constant, $S^3_{a,b}$ is a Hopf hypersurface. Kimura [K] proved that any such hypersurface is an open subset of a submanifold which is homogeneous under a subgroup of the isometry group $U(3)$ of $\mathbb{C}P(2)$. Such submanifolds are classified in Takagi [TaI]. For the case $n = 2$ we only have the geodesic spheres and tubes over an $\mathbb{R}P(2)$ (or a complex quadric). We show that one of those tubes is similar, but not equal to our possibility.

We find the factors a and b by computing the lengths of Killing-Jacobi fields along the geodesic $\exp(tX)p$, starting at a point $p \in S^3_{a,b}$ (recall $X = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$). Consider the field $J(t) = f(t)e_0$.

Since the sectional curvature of $Sp(e_0, Je_0)$ is 4, we have, by the Jacobi equation $\frac{D^2J}{dt^2} = -4J(t)$. Hence $f''(t) = -4f(t)$. Since $\mathbb{R}P(2)$ is real, its tangent space must equal $Sp(e_0, e_1)$, hence the Jacobi field starts out at its maximum length for $t = 0$. So $f(t) = \cos 2t$. By a similar argument the field $K(t) = g(t)e_1$ must satisfy $g''(t) = -g(t)$. Here we assumed e_1 was chosen such that $T_p\,\mathbb{R}P(2) = Sp(e_0, e_1)$, (unproblematic when $Y_0 = e_0$). Hence $g(t) = \cos t$. Finally, for $L(t) = h(t)Je_1$ we also get $h''(t) = -h(t)$, this time, however, Je_1 is orthogonal to $T_p\,\mathbb{R}P(2)$, and $h(t) = \sin t$. For precisely one $t \in (0, \frac{\pi}{4})$ two of these are equal: $\cos(2 \cdot \frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$. By the formulas of Cecil and Ryan [CR, p. 494] the principal curvatures are $2\tan 2t$, $\tan t$, $-\cot t$. For $t = \frac{\pi}{6}$ we get $2\sqrt{3}$, $\frac{1}{\sqrt{3}}$, and $-\sqrt{3}$, coninciding with the values (albeit in a different order) we had computed for $S^3_{\sqrt{3},1}$. From Takagi's list [TaII] it follows easily that for $n = 2$ only geodesic spheres and real projective planes can occur, hence, by Kimura's theorem, this is the only possibility. It is not an S^3 with the given $U(2)$-action.

§8 We finish with discussing the case $a^2 = b^4$. We have

D2: $$3\sin\varphi\cos\varphi\, b_{00} - 3\sin^2\varphi\, b_{0\bar{1}} = 0$$

D7a: $$\left(\frac{4}{b^2} - 7 + 3\sin^2\varphi\right) b_{00} + 3\sin\varphi\cos\varphi\, b_{0\bar{1}} = 0$$

D3 and D4 are linear equations exactly like this with b_{01} substituted for b_{00} and $b_{1\bar{1}}$ for $b_{0\bar{1}}$. D5 and D7b are again the same with this time $b_{0\bar{1}}$ substituted for b_{00} and $b_{\bar{1}\bar{1}}$ for $b_{0\bar{1}}$. The determinant of this system equals $12(\frac{1}{b^2} - 1)\sin^2\varphi$. If this is not zero, we must have $b_{00} = b_{0\bar{1}} = 0$ and similarly $b_{01} = b_{1\bar{1}} = 0$ and $b_{0\bar{1}} = b_{\bar{1}\bar{1}} = 0$. From G1 it follows that $\sin\varphi = 0$, which contradicts $12(\frac{1}{b^2} - 1)\sin^2\varphi \neq 0$. Hence this determinant must be equal to zero.

Now either: (a) $b = 1$ or (b) $\sin\varphi \equiv 0$.

First consider $b = 1$ $(= a)$. We may assume that there exists a point $p \in S^3_{1,1}$ such that $\sin\varphi(p) \neq 0$. These points constitute an open set of $S^3_{1,1}$, and on this open set, from D2 we have $b_{0\bar{1}} = \cot\varphi\, b_{00}$, similarly $b_{1\bar{1}} = \cot\varphi\, b_{01}$ and $b_{\bar{1}\bar{1}} = \cot\varphi\, b_{0\bar{1}} = \cot^2\varphi\, b_{00}$. In addition to this G1 gives $b_{00}b_{11} - b_{01}^2 = -3\sin^2\varphi$ (the rest of G2–G6 are easily seen to depend on these 4 equations).

We compute:

$$\begin{aligned}
\bar{\nabla}_{Y_0} Je_0 &= J[\bar{\nabla}_{Y_0}(\cos\varphi\, Y_0 - \sin\varphi\, J'Y_1)] \\
&= J[-\sin\varphi\, Y_0(\varphi)Y_0 + \cos\varphi\, b_{00} Je_0 - \cos\varphi\, Y_0(\varphi)J'Y_1 \\
&\qquad - \sin\varphi \nabla_{Y_0} J'Y_1 - \sin\varphi\, b_{0\bar{1}} Je_0] \\
&= J[-Y_0(\varphi)Je_1 + \sin\varphi\, Y_1] + (\sin\varphi\, b_{0\bar{1}} - \cos\varphi\, b_{00})e_0 \\
&= Y_0(\varphi)e_1 \pm \sin\varphi\, Je_1 + (\sin\varphi\, b_{0\bar{1}} - \cos\varphi\, b_{00})e_0.
\end{aligned}$$

On the other hand $\langle \bar{\nabla}_{Y_0} Je_0, Y_0\rangle = -\langle Je_0, \bar{\nabla}_{Y_0} Y_0\rangle = -\langle Je_0, b_{00}Je_0\rangle = -b_{00}$. Similarly $\langle \bar{\nabla}_{Y_0} Je_0, Y_1\rangle = -\langle Je_0, \bar{\nabla}_{Y_0} Y_1\rangle = -b_{01}$ and $\langle \bar{\nabla}_{Y_0} Je_0, J'Y_1\rangle = -\langle Je_0, \bar{\nabla}_{Y_0} J'Y_1\rangle = -b_{0\bar{1}}$. Hence $\bar{\nabla}_{Y_0} Je_0 = -b_{00}Y_0 - b_{01}Y_1 - b_{0\bar{1}}J'Y_1 = (-\cos\varphi\, b_{00} + \sin\varphi\, b_{0\bar{1}})e_0 \mp b_{01}e_1 - (b_{00}\sin\varphi + b_{0\bar{1}}\cos\varphi)Je_1$. Hence we have: $b_{00}\sin\varphi + b_{0\bar{1}}\cos\varphi = \mp\sin\varphi$. Substituting $b_{0\bar{1}} = \cot\varphi\, b_{00}$ into this we get: $b_{00}\sin^2\varphi + b_{00}\cos^2\varphi = b_{00} = \mp\sin^2\varphi$. Also $b_{0\bar{1}} = \mp\sin\varphi\cos\varphi$ and $b_{\bar{1}\bar{1}} = \cot^2\varphi\, b_{00} = \mp\cos^2\varphi$.

Now we also have:

$$\begin{aligned}
&\bar{\nabla}_{Y_1} Je_0 \\
&= J[\nabla_{Y_1}(\cos\varphi\, Y_0 - \sin\varphi\, J'Y_1) + \cos\varphi\, b_{01}Je_0 - \sin\varphi\, b_{1\bar{1}}Je_0] \\
&= J[-\sin\varphi\, Y_1(\varphi)Y_0 + \cos\varphi\, \nabla_{Y_1}Y_0 - \cos\varphi\, Y_1(\varphi)J'Y_1 - \sin\varphi \nabla_{Y_1} J'Y_1] \\
&\qquad + \sin\varphi\, b_{1\bar{1}}e_0 - \cos\varphi\, b_{01}e_0 \\
&= J[-Y_1(\varphi)Je_1 - \cos\varphi\, J'Y_1 - \sin\varphi\, Y_0] + \sin\varphi\, b_{1\bar{1}}e_0 - \cos\varphi\, b_{01}e_0 \\
&= Y_1(\varphi)e_1 - J(Je_1) + (\sin\varphi\, b_{1\bar{1}} - \cos\varphi\, b_{01})e_0 \\
&= Y_1(\varphi)e_1 + e_1 + (\sin\varphi\, b_{1\bar{1}} - \cos\varphi\, b_{01})e_0.
\end{aligned}$$

Also $\langle \bar{\nabla}_{Y_1} Je_0, Y_0\rangle = -\langle Je_0, \bar{\nabla}_{Y_1} Y_0\rangle = -b_{01}$, $\langle \bar{\nabla}_{Y_1} Je_0, Y_1\rangle = -\langle Je_0, \bar{\nabla}_{Y_1} Y_1\rangle = -b_{11}$, and $\langle \bar{\nabla}_{Y_1} Je_0, J'Y_1\rangle = -\langle Je_0, \bar{\nabla}_{Y_1} J'Y_1\rangle = -b_{1\bar{1}}$. Hence: $\bar{\nabla}_{Y_1} Je_0 = -b_{01}Y_0 - b_{11}Y_1 - b_{1\bar{1}}J'Y_1 = (b_{1\bar{1}}\sin\varphi - b_{01}\cos\varphi)e_0 \mp b_{11}e_1 - (b_{01}\sin\varphi + b_{1\bar{1}}\cos\varphi)Je_1$. So we have: $b_{01}\sin\varphi + b_{1\bar{1}}\cos\varphi = 0$. We also have from D4: $b_{01}\cos\varphi - b_{1\bar{1}}\sin\varphi = 0$ and the determinant of this homogeneous system is 1. The only solution is $b_{01} = b_{1\bar{1}} = 0$. From G1 we have $b_{00}b_{11} = \mp \sin^2 \varphi b_{11} = -3 \sin^2 \varphi$, hence $b_{11} = \pm 3$.

Consider the Codazzi-Mainardi equation (C5): From (6.1):

$$\begin{aligned}
&\langle R(Y_0 \wedge J'Y_1)Y_1, -Je_0\rangle \\
&= \left\langle \left(\left(\frac{a^2}{b^4} - 1 \right) Y_0 \wedge J'Y_1 \pm Je_0 \wedge Y_1 \right) Y_1, -Je_0 \right\rangle \\
&= \langle 0 \mp \langle Y_1, Y_1 \rangle Je_0, -Je_0 \rangle \\
&= \pm 1 = J'Y_1(b_{01}) - B(Y_0, \nabla_{J'Y_1} Y_1) - Y_0(b_{1\bar{1}}) \\
&\quad + B(J'Y_1, \nabla_{Y_0} Y_1) + B([Y_0, J'Y_1], Y_1) = b_{00} + b_{\bar{1}\bar{1}} - 2b_{11} \\
&= \mp \sin^2 \varphi \mp \cos^2 \varphi - 2(\pm 3)) = \mp 7.
\end{aligned}$$

But this is a contradiction, hence the case a) does not occur.

Next, consider (b): $\sin \varphi \equiv 0$. Then we may choose $Y_0 = e_0$, $Y_1 = \pm e_1$, $J'Y_1 = Je_1$. We have:

$\bar{\nabla}_{Y_1} Y_1 = \nabla_{Y_1} Y_1 + b_{11} Je_0 = b_{11} Je_0$. Also: $\bar{\nabla}_{Y_1} Y_1 = -\bar{\nabla}_{Y_1} C(J \otimes JY_1) = \mp \bar{\nabla}_{Y_1} C(J \otimes Je_1) = \mp \bar{\nabla}_{Y_1} C(J \otimes J'Y_1) = \mp J(\nabla_{Y_1} J'Y_1 + b_{1\bar{1}} Je_0) = \mp J(\frac{a}{b^2} Y_0 + b_{1\bar{1}} Je_0) = \pm(-\frac{a}{b^2} Je_0 + b_{1\bar{1}} e_0)$; then $b_{1\bar{1}} = 0$ and $b_{11} = \mp 1$. Similarly:

$$\begin{aligned}
\bar{\nabla}_{J'Y_1} J'Y_1 &= 0 + b_{\bar{1}\bar{1}} Je_0 = b_{\bar{1}\bar{1}} Je_0 = -\bar{\nabla}_{J'Y_1} C(J \otimes J(Je_1)) \\
&= \pm C(J \otimes \bar{\nabla}_{J'Y_1} Y_1) = \pm J(\nabla_{J'Y_1} Y_1 + b_{1\bar{1}} Je_0) \\
&= \mp \left(\frac{a}{b^2} Je_0 + b_{1\bar{1}} e_0 \right).
\end{aligned}$$

Hence $b_{\bar{1}\bar{1}} = \mp 1$. Furthermore:

$$\begin{aligned}
\bar{\nabla}_{Y_0} J'Y_1 &= \nabla_{Y_0} J'Y_1 + b_{0\bar{1}} Je_0 = \left(\frac{a}{b^2} - \frac{2}{a} \right) Y_1 + b_{0\bar{1}} Je_0 \\
&= \pm \left(1 - \frac{2}{a} \right) e_1 + b_{0\bar{1}} Je_0 = -\bar{\nabla}_{Y_0} C(J \otimes J(Je_1)) \\
&= \pm C(J \otimes \bar{\nabla}_{Y_0}(Y_1)) = \pm J(\bar{\nabla}_{Y_0} Y_1) \\
&= \pm J(\nabla_{Y_0} Y_1 + b_{01} Je_0) = \pm \left(\frac{2}{a} - 1 \right) J(Je_1) \mp b_{01} e_0 \\
&= \pm \left(1 - \frac{2}{a} \right) e_1 \mp b_{01} e_0,
\end{aligned}$$

i.e. $b_{01} = b_{0\bar{1}} = 0$. Hence we have now proved that $b_{01} = b_{0\bar{1}} = b_{1\bar{1}} = 0$, $b_{11} = b_{\bar{1}\bar{1}} = \mp 1$. It follows from G4 that $b_{00} = 0$ also. From G6: $\frac{4}{b^2} - 7 = 1$, $b^2 = \frac{1}{2}$, $a^2 = \frac{1}{4}$. Hence we have proved:

Theorem 11. *For $a^2 = b^4$ the only possibility for $S^3_{a,b}$ is for $a = \frac{1}{2}$, $b = \frac{1}{\sqrt{2}}$. Hence $\frac{1}{b^2} - \frac{a^2}{b^4} = 1$.*

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