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Contact Transformations and Their Schwarzian Derivatives

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Abstract.

The second author introduced Schwarzian derivatives for 3-dimensional contact transformations in [Sat]. Our purposes of this paper are firstly to investigate the fundamental properties of the contact Schwarzian derivatives of 3-dimensional contact transformations that are satisfied by the usual Schwarzian derivatives, secondly to consider systems of linear PDE's with contact Schwarzian derivatives as coefficients and their integrability conditions, and finally to reconstruct the contact transformation from the solutions of the systems of linear PDE's. We obtain the necessary and sufficient condition for functions to be contact Schwarzian derivatives of a 3-dimensional contact transformation.

§1. Introduction

The classical theory of Schwarzian derivative plays an important role in the study of holomorphic equivalence of one-dimensional complex domains and especially in the Gauss-Schwarz theory of hypergeometric differential equation.

The Schwarzian derivative $S(f)$ of an analytic function f on $\mathbb{C}$ is defined by

$$S(f) = -\frac{1}{2}\Big(\frac{f''}{f'}\Big)' + \frac{1}{4}\Big(\frac{f''}{f'}\Big)^2.$$

Among many important properties of classical Schwarzian deirvative, we pay attention to the following three basic facts: (i) for an arbitrary function σ, the quotient $f = \varphi_1/\varphi_0$ of linearly independent solutions φ_0 and φ_1 of the differential equation

$$\varphi'' - \sigma\varphi = 0$$

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has the Schwarzian derivative $S(f)$ equal to σ, (ii) The Schwarzian derivative of a composition $f \circ g$ satisfies the formula

$$S(f \circ g) = S(g) + S(f)(g')^2,$$

which is nothing but the cocycle condition of continuous group cohomology (see, for example, [O-S2]), and (iii) the Schwarzian derivative $S(f)$ of a function f vanishes, if and only if f is a Möbius transformation;

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

where α, β, γ, and δ are complex constants.

The Schwarzian derivative of higher dimensional diffeomorphisms is studied by several authors, and linear partial differential equations whose coefficients are Schwarzian derivatives has been observed to give, as a special case, the Appell and Lauricella hypergeometric differential equations (see, for example, [Yos]).

The purpose of this paper is to develop the theory of Schwarzian derivative for contact transformations, especially study the properties that correspond to the above three facts (i), (ii) and (iii). Originally, the notion of Schwarzian derivative of contact transformation is introduced by the second author in [Sat], through the study of equivalence of third order ordinary differential equations.

The notion of equivalence of ordinary differential equations depends on the transformations employed to reduce a given equation to simpler one. In the most general setting, we deal with the problem through *contact transformations* $\phi\colon \mathbb{K}^3 \to \mathbb{K}^3$, where we regard, using the coordinate system (x, y, z), the three space $\mathbb{K}^3$ as a contact manifold with contact form $dy - z\,dx$. If we transform the simplest third order differential equation $d^3y/dx^3 = 0$ by a contact transformation $\phi\colon (x, y, z) \mapsto (X, Y, Z)$, then the resulting equation is

$$\frac{d^3y}{dx^3} = P + 3Q\Big(\frac{d^2y}{dx^2}\Big) + 3R\Big(\frac{d^2y}{dx^2}\Big)^2 + S\Big(\frac{d^2y}{dx^2}\Big)^3,$$

where P, Q, R and S are functions of $(x, y, z = dy/dx)$. Our definition of the contact Schwarzian derivative of ϕ is to be the quadruple

$$S(\varphi) = (P, Q, R, S)$$

that appears in the above transformed equation. The explicit formula of the Schwarzian derivative is given in Section 2. In this paper, we will not discuss on the equivalence problem of ordinary differential equations.

For the arguments from this point of view, the readers will refer the papers [O-S1] and [Sat] and the references therein.

For the calculus on the contact space $\mathbb{K}^3$ ($= \mathbb{C}^3$ or $\mathbb{R}^3$) with the contact form $dy - z\,dx$, it is convenient to use the vector fields

$$v_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y},$$

that satisfy the *Heisenberg relation* for Lie bracket; $v_3 = [v_2, v_1]$ and $[v_3, v_1] = [v_3, v_2] = 0$ (for useful formulae with v_1, v_2, v_3 on contact transformations, see Section 3).

In the classical theory of Schwarzian derivative, we could reconstruct an analytic function f, whose Schwarzian derivative is equal to a given function σ, from solutions of the linear ordinary differential equation as stated in the above fact (i). In order to reconstruct a contact transformation ϕ with Schwarzian derivative equal to a given quadruple (P, Q, R, S), we consider the following system of linear partial differential equations (PDE system, for short):

$$\begin{cases} {v_1}^2(\vartheta) = Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) = 2(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta) \\ {v_2}^2(\vartheta) = Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{cases}$$

where v_4 stands for

$$v_4 = v_2v_1 + v_1v_2 = 2\frac{\partial^2}{\partial x\partial z} + 2z\frac{\partial^2}{\partial y\partial z} + \frac{\partial}{\partial y},$$

and M_{11}, M_4 and M_{22} are certain functions of P, Q, R, S and their differentiations by the vector fields v_1 and v_2 (see the formula (2) in Section 4).

The dimension of solution space of this PDE system depends on the quadruple (P, Q, R, S), and attains the maximum equal to 4, if and only if (P, Q, R, S) satisfies the integrability condition (IC) (see Theorem 4.1). And also the solution space carries a natural linear symplectic structure as proved in Proposition 4.4, provided its dimension equals 4. We will prove that the contact transformation ϕ can be reconstructed by using a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space (Theorem 6.1);

$$\phi(x, y, z) = \left(\frac{\xi}{\vartheta}, \frac{1}{2}\left(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2}\right), \frac{\zeta}{\vartheta}\right).$$

The quadruple (P, Q, R, S) automatically satisfies the integrability condition (IC), if it is a Schwarzian derivative of contact transformation.

Therefore we find that the condition (IC) is necessary and sufficient for (P, Q, R, S) to be a Schwarzian deirvative of some contact transformation.

For the proof of Theorem 4.1, we introduce a set of operators V_1, V_2 and V_3 acting on a certain module over functions, and verify that it satisfies the Heisenberg relation; $V_3 = [V_2, V_1]$ and $[V_3, V_i] = 0$ $(i = 1, 2)$ (see Section 7). The verification is a series of quite long calculations, and we are compelled to use a computer software "Maple". We hope to find a simpler method for the proof.

The contact Schwarzian derivative satisfies a similar formula for compositions of maps as the classical Schwarzian derivative satisfies. The verification is easily performed by using the operators v_1 and v_2 (see Proposition 8.1).

In order to study contact transformations with vanishing Schwarzian derivative, we solve the above PDE system all of whose coefficients equal 0, and find that the functions 1, x, z and $2y - xz$ form a symplectic basis of the solution space. We notice that the embedding $\mathbb{K}^3 \to \mathbb{K}^4$ defined by

$$(x, y, z) \to (1, x, z, 2y - xz)$$

is compatible for both structures, contact and symplectic, on the source and the target. Especially, the pull back of Lagrangian planes are Legendrian curves. Through this embedding, we reduce the $Sp(2)$-action on $\mathbb{K}^4$ to that on $\mathbb{K}^3$, at least locally, and describe the exact set of contact transformations with vanishing Schwarzian derivative, in Theorem 9.5.

The paper is organized as follows. In Section 2, we give explicitly the formula of the contact Schwarzian derivative for three dimensional contact transformations. In Section 3, we prepare elementary calculations about contact transformations. In Section 4, we introduce a fundamental PDE system that has contact Schwarzian derivatives as its coefficients, and show that the solution space of this PDE system has a natural symplectic structure. In Section 5, we derive a PDE system whose solutions are coordinate functions of contact transformations, and a PDE system whose solutions are Jacobians of contact transformations. In Section 6, we show how to construct the contact transformation with a prescribed contact Schwarzian derivative from the solutions of the PDE systems introduced in Sections 4 and 5. The symplectic structure of the solution space of the PDE system introduced in Section 4 plays an essential role here. In Section 7, we show that the integrability conditions of the various PDE systems in this paper are equivalent in a certain sense, and prove Theorem 4.1 that concerns the integrability of the fundamental

PDE system. In Section 8, we prove that our contact Schwarzian derivative satisfies a connection formula with respect to compositions of contact transformations. Finally in Section 9, we show that the set of three dimensional contact transformations with vanishing contact Schwarzian derivatives form a Lie group locally isomorphic to $Sp(2, \mathbb{K})$.

§2. Contact Schwarzian derivative

Throughout the paper, we regard the affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate (x, y, z) as a contact manifold endowed with the contact form $\alpha = dy - z\,dx$. We use the following notations:

$$v_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y}, \quad v_4 = v_2 v_1 + v_1 v_2.$$

Notice that the vector fields v_1, v_2 and v_3 form a Heisenberg Lie algebra:

$$v_3 = [v_2, v_1], \quad \text{and} \quad [v_3, v_1] = [v_3, v_2] = 0,$$

and that the vector fields v_1 and v_2 span the contact distribution; $\alpha(v_1) = \alpha(v_2) = 0$.

A local diffeomorphism ϕ is said to be a *contact transformation*, if it preserves the contact distribution, or equivalently, it satisfies $\phi^*(\alpha) = \rho\alpha$ for some nonvanishing function ρ. For a contact transformation ϕ: $(x, y, z) \mapsto (X, Y, Z)$, we define the contact Schwarzian derivatives as follows: for i, j, $k = 1$, 2, set

$$s_{[ij,k]}(\phi) = v_i v_j(X) v_k(Z) - v_i v_j(Z) v_k(X),$$

and

$$S_{\{ijk\}}(\phi) = \frac{1}{3\Delta(\phi)}\big(s_{[ij,k]}(\phi) + s_{[jk,i]}(\phi) + s_{[ki,j]}(\phi)\big),$$

where $\Delta(\phi) = v_1(X)v_2(Z) - v_1(Z)v_2(X)$. We call the functions $S_{\{ijk\}}(\phi)$ the *contact Schwarzian derivatives* of the contact transformation ϕ. We denote the quadruple of functions by

$$S(\phi) = \big(S_{\{111\}}(\phi), S_{\{112\}}(\phi), S_{\{122\}}(\phi), S_{\{222\}}(\phi)\big),$$

which also we call the Schwarzian derivative of ϕ.

§3. Some remarks on contact transformations

The facts stated in the following lemmas will be used in later sections. The affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate

(x, y, z) is always regarded as a contact manifold endowed with the contact form $\alpha = dy - z\,dx$.

Lemma 3.1. *If a map $\phi\colon (x, y, z) \mapsto (X, Y, Z)$ is a contact transformation, then X, Y and Z satisfy the equations*

$$v_1(Y) = Zv_1(X), \quad \text{and} \quad v_2(Y) = Zv_2(X). \tag{1}$$

Proof. The pull back of the contact form is equal to

$$\begin{aligned}\phi^*(\alpha) &= dY - Z\,dX \\ &= \Big(\frac{\partial}{\partial x}Y - Z\frac{\partial}{\partial x}X\Big)dx + \Big(\frac{\partial}{\partial y}Y - Z\frac{\partial}{\partial y}X\Big)dy \\ &\quad + \Big(\frac{\partial}{\partial z}Y - Z\frac{\partial}{\partial z}X\Big)dz.\end{aligned}$$

Thus, if ϕ is a contact transformation, then it holds that

$$\frac{\partial}{\partial x}Y - Z\frac{\partial}{\partial x}X = -\rho z, \quad \frac{\partial}{\partial y}Y - Z\frac{\partial}{\partial y}X = \rho, \quad \frac{\partial}{\partial z}Y - Z\frac{\partial}{\partial z}X = 0,$$

and equivalently that $v_i(Y) = Zv_i(X)$, $(i = 1,\ 2)$. Q.E.D.

The equations (1) determine the function Y up to additive constant. These equations do not necessarily have a solution Y. The following lemma shows the integrability condition of these equations. The proof is given in the paper [Sat].

Lemma 3.2. *The necessary and sufficient condition for the equations (1) to have a solution Y is that the functions X and Z satisfy the relations*

$$v_i(\Delta) = v_i(X)v_3(Z) - v_3(X)v_i(Z), \quad (i = 1,\ 2),$$

where $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$.

We call $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$ the *contact Jacobian.* The relation between the contact Jacobian and the usual Jacobian of the map ϕ is given in the following lemma.

Lemma 3.3. *Let $\phi\colon (x, y, z) \mapsto (X, Y, Z)$ be a contact transformation, and ρ a scalar function that satisfies $\phi^*(\alpha) = \rho\alpha$. Then ρ is equal to the contact Jacobian $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$, and the usual Jacobian of ϕ is equal to*

$$\frac{\partial(X, Y, Z)}{\partial(x, y, z)} = \rho^2 = \Delta^2.$$

Proof. Differentiating the equations (1) by v_1 and v_2, we obtain

$$v_2 v_1(Y) = v_2(Z) v_1(X) + Z v_2 v_1(X),$$
$$v_1 v_2(Y) = v_1(Z) v_2(X) + Z v_1 v_2(X),$$

and subtracting them, obtain

$$\rho = \frac{\partial}{\partial y} Y - Z \frac{\partial}{\partial y} X = v_3(Y) - Z v_3(X) = v_2(Z) v_1(X) - v_1(Z) v_2(X),$$

which is equal to Δ.

By definition, the Jacobian satisfies

$$\phi^*(\alpha \wedge d\alpha) = \frac{\partial(X, Y, Z)}{\partial(x, y, z)} \alpha \wedge d\alpha.$$

Since ϕ is contact, we have

$$\phi^*(\alpha \wedge d\alpha) = \rho\alpha \wedge d(\rho\alpha) = \rho^2 \alpha \wedge d\alpha.$$

Therefore we get the required equality $\partial(X, Y, Z)/\partial(x, y, z) = \rho^2$. Q.E.D.

From these lemmas, it follows immediately that

Corollary 3.4. *A map ϕ: $(x, y, z) \mapsto (X, Y, Z)$ is a contact transformation, if and only if the contact Jacobian $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$ does not vanish anywhere, and X, Y and Z satisfy $v_i(Y) = Z v_i(X)$ $(i = 1, 2)$.*

In order to simplify the notation, we sometimes denote by (x^1, x^3, x^2) the coordinate functions of $\mathbb{K}^3$ in place of (x, y, z). In this notation, the contact form is written as $dy - z\,dx = dx^3 - x^2\,dx^1$.

Lemma 3.5. *For any contact transformation $(x^1, x^3, x^2) \mapsto (y^1, y^3, y^2)$ and any function $f(y^1, y^3, y^2)$, we have, for $i = 1$ and 2,*

$$v_i(f) = \sum_{j=1,2} v_i(y^j) u_j(f),$$

where we used the notation

$$u_1 = \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^3}, \quad u_2 = \frac{\partial}{\partial y^2}.$$

Proof. The differential of f with respect to v_1 is

$$\begin{aligned} v_1(f) &= \frac{\partial f}{\partial x^1} + x^2 \frac{\partial f}{\partial x^3} \\ &= \frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^1} + \frac{\partial f}{\partial y^3}\frac{\partial y^3}{\partial x^1} \\ &\quad + x^2 \left(\frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^3} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^3} + \frac{\partial f}{\partial y^3}\frac{\partial y^3}{\partial x^3} \right). \end{aligned}$$

Reminding that the map is contact; $dy^3 - y^2\, dy^1 = \rho(dx^3 - x^2\, dx^1)$, and thus

$$\frac{\partial y^3}{\partial x^1} - y^2 \frac{\partial y^1}{\partial x^1} = -\rho x^2, \quad \frac{\partial y^3}{\partial x^2} - y^2 \frac{\partial y^1}{\partial x^2} = 0, \quad \frac{\partial y^3}{\partial x^3} - y^2 \frac{\partial y^1}{\partial x^3} = \rho,$$

we get

$$\begin{aligned} v_1(f) &= \frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^1} + \frac{\partial f}{\partial y^3}\left(y^2 \frac{\partial y^1}{\partial x^1} - \rho x^2 \right) \\ &\quad + x^2 \left(\frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^3} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^3} + \frac{\partial f}{\partial y^3}\left(y^2 \frac{\partial y^1}{\partial x^3} + \rho \right) \right) \\ &= \frac{\partial f}{\partial y^1} v_1(y^1) + \frac{\partial f}{\partial y^2} v_1(y^2) + y^2 \frac{\partial f}{\partial y^3} v_1(y^1) \\ &= u_1(f) v_1(y^1) + u_2(f) v_1(y^2). \end{aligned}$$

We prove the equality for the differentiation by v_2 in the same way. Q.E.D.

Lemma 3.6. *For a compositions of contact transformations $\psi \circ \phi \colon x = (x^1, x^3, x^2) \mapsto y = (y^1, y^3, y^2) \mapsto z = (z^1, z^3, z^2)$, the contact Jacobians satisfy $\Delta(z, x) = \Delta(z, y)\Delta(y, x)$.*

Proof. Define functions ρ and σ by

$$\phi^*(dy^3 - y^2\, dy^1) = \rho(dx^3 - x^2\, dx^1), \text{ and } \psi^*(dz^3 - z^2\, dz^1) = \sigma(dy^3 - y^2\, dy^1)$$

then we have

$$(\psi \circ \phi)^*(dz^3 - z^2\, dz^1) = \sigma\rho(dx^3 - x^2\, dx^1).$$

The lemma follows immediately from Lemma 3.3. Q.E.D.

§4. Fundamental equation

In this section, we introduce a fundamental PDE system, which will play a central role in this paper. The coefficients P, Q, R and S of this PDE system will be replaced by contact Schwarzian derivatives in later sections.

Given functions $P = P(x,y,z)$, $Q = Q(x,y,z)$, $R = R(x,y,z)$ and $S = S(x,y,z)$, we define functions M_{11}, M_4 and M_{22} by

$$\begin{aligned} M_{11} &= -\frac{1}{4}\big(v_1(Q) - v_2(P) - 2Q^2 + 2PR\big) \\ M_4 &= -\frac{1}{4}\big(v_1(R) - v_2(Q) - QR + PS\big) \\ M_{22} &= -\frac{1}{4}\big(v_1(S) - v_2(R) - 2R^2 + 2QS\big). \end{aligned} \tag{2}$$

Theorem 4.1. *The necessary and sufficient condition for the linear PDE system*

$$\begin{cases} {v_1}^2(\vartheta) = Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) = 2\big(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta\big) \\ {v_2}^2(\vartheta) = Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{cases} \tag{Sp}$$

with unknown function ϑ to have 4-dimensional solution space is that the functions P, Q, R and S satisfy the set of relations

$$\begin{aligned} v_3(P) &= 2(v_1 - 2Q)(M_{11}) + 4PM_4 \\ 3v_3(Q) &= 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22} \\ 3v_3(R) &= 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11} \\ v_3(S) &= 2(v_2 + 2R)(M_{22}) - 4SM_4. \end{aligned} \tag{IC}$$

Moreover, if it is the case, the solution space is parameterized by initial values $\vartheta(p)$, $v_1(\vartheta)(p)$, $v_2(\vartheta)(p)$ and $v_3(\vartheta)(p)$ at an arbitrarily fixed point p.

The proof will be given in Section 7.

Remark 4.2. The above relations (IC) are automatically satisfied if the functions P, Q, R and S are contact Schwarzian derivatives, $S_{\{111\}}(\phi)$, $S_{\{112\}}(\phi)$, $S_{\{122\}}(\phi)$ and $S_{\{222\}}(\phi)$ respectively, of some contact transformation ϕ. This fact can be proved by a direct calculation. Other aspects of this can be found in the papers [Sat] and [S-Y].

Remark 4.3. It might be useful to express the functions M_{11}, M_4, M_{22} as follows: let w_1 and w_2 be operators defined by

$$w_1 = v_1 - \begin{pmatrix} Q & -P \\ R & -Q \end{pmatrix}, \quad w_2 = v_2 - \begin{pmatrix} R & -Q \\ S & -R \end{pmatrix}.$$

If we put $w_3 = [w_2, w_1]$, then we have

$$w_3 = v_3 - 4\begin{pmatrix} M_4 & -M_{11} \\ M_{22} & -M_4 \end{pmatrix}.$$

Furthermore, we compute the commutators $[w_3, w_1]$ and $[w_3, w_2]$ as follows:

$$[w_3, w_1] = \begin{pmatrix} -v_3(Q) + 4v_1(M_4) - 4RM_{11} + 4PM_{22} & v_3(P) - 4v_1(M_{11}) - 8PM_4 + 8QM_{11} \\ -v_3(R) + 4v_1(M_{22}) + 8QM_{22} - 8RM_4 & v_3(Q) - 4v_1(M_4) + 4RM_{11} - 4PM_{22} \end{pmatrix}$$

$$[w_3, w_2] = \begin{pmatrix} -v_3(R) + 4v_2(M_4) - 4SM_{11} + 4QM_{22} & v_3(Q) - 4v_2(M_{11}) - 8QM_4 + 8RM_{11} \\ -v_3(S) + 4v_2(M_{22}) + 8RM_{22} - 8SM_4 & v_3(R) - 4v_2(M_4) + 4SM_{11} - 4QM_{22} \end{pmatrix}$$

Supposing the PDE system (Sp) has a nonzero solution ϑ, we consider the following PDE system with unknown function ξ;

$$\text{(Sp-s)} \qquad \begin{cases} {v_1}^2(\xi) = Qv_1(\xi) - Pv_2(\xi) + M_{11}\xi \\ v_2v_1(\xi) = \left(R + \dfrac{v_2(\vartheta)}{\vartheta}\right) v_1(\xi) \\ \qquad\qquad - \left(Q + \dfrac{v_1(\vartheta)}{\vartheta}\right) v_2(\xi) + \left(M_4 + \dfrac{1}{2}\dfrac{v_3(\vartheta)}{\vartheta}\right)\xi \\ v_1v_2(\xi) = \left(R - \dfrac{v_2(\vartheta)}{\vartheta}\right) v_1(\xi) \\ \qquad\qquad - \left(Q - \dfrac{v_1(\vartheta)}{\vartheta}\right) v_2(\xi) + \left(M_4 - \dfrac{1}{2}\dfrac{v_3(\vartheta)}{\vartheta}\right)\xi \\ {v_2}^2(\xi) = Sv_1(\xi) - Rv_2(\xi) + M_{22}\xi. \end{cases}$$

We denote by $\mathcal{S}(P, Q, R, S)$ and $s(P, Q, R, S; \vartheta)$ the solution spaces of the PDE systems (Sp) and (Sp-s), respectively. Adding the second equation in (Sp-s) to the third, we get the second equation of (Sp). This implies that $s(P, Q, R, S; \vartheta)$ is a subspace of $\mathcal{S}(P, Q, R, S)$ for any ϑ.

As we will see in Section 6, the PDE systems (Sp) and (Sp-s) will play important roles to reconstruct contact transformations with prescribed contact Schwarzian derivatives. In the following propositions, we study the relation of the solution spaces $\mathcal{S}(P, Q, R, S)$ and $s(P, Q, R, S; \vartheta)$.

Proposition 4.4. *For any two solutions α and β of the PDE system* (Sp), *the function $I(\alpha,\beta)$ defined by*

$$(3)\qquad I(\alpha,\beta)=\frac{1}{2}\alpha v_3(\beta)-\frac{1}{2}v_3(\alpha)\beta+v_1(\alpha)v_2(\beta)-v_2(\alpha)v_1(\beta)$$

is constant on (x,y,z). Moreover this skew product $I(\alpha,\beta)$ is non-degenerate, and thus it defines a symplectic structure on the solution space $\mathcal{S}(P,Q,R,S)$, provided the dimension of $\mathcal{S}(P,Q,R,S)$ is equal to 4.

Proof. In order to prove the constantness of $I(\alpha,\beta)$, it is sufficient to verify the vanishing of the derivatives

$$v_i(I(\alpha,\beta))=0,\qquad (i=1,\ 2).$$

The derivative $v_1(I(\alpha,\beta))$ is calculated as follows: we use the notation $v_i(f)=f_i$ and $v_iv_j(f)=f_{ji}$. Remarking that $3v_1v_3=3v_3v_1=2v_2{v_1}^2-v_1v_4$, we get

$$\begin{aligned}
&v_1(\alpha\beta_3+2\alpha_1\beta_2)\\
&\quad=\alpha_1\beta_3+\alpha\beta_{31}+2\alpha_{11}\beta_2+2\alpha_1\beta_{21}\\
&\quad=\alpha_1\beta_4+2\alpha_{11}\beta_2+\frac{1}{3}(2(\beta_{11})_2-(\beta_4)_1)\alpha\\
&\quad=2(R\beta_1-Q\beta_2+M_4\beta)\alpha_1+2(Q\alpha_1-P\alpha_2+M_{11}\alpha)\beta_2\\
&\qquad\qquad+\frac{2}{3}\big((Q\beta_1-P\beta_2+M_{11})_2-(R\beta_1-Q\beta_2+M_4\beta)_1\big)\alpha\\
&\quad=2(R\alpha_1\beta_1+M_4\alpha_1\beta-P\alpha_2\beta_2+M_{11}\alpha\beta_2)\\
&\qquad+\frac{2}{3}\big(2Q(R\beta_1-Q\beta_2+M_4\beta)-P(S\beta_1-R\beta_2+M_{22}\beta)\\
&\qquad\qquad-R(Q\beta_1-P\beta_2+M_{11}\beta)+(Q_2-R_1-M_4\beta)\beta_1\\
&\qquad\qquad-(P_2-Q_1-M_{11})\beta_2+((M_{11})_2-(M_4)_1)\beta\big)\alpha\\
&\quad=2(M_4\alpha_1\beta+R\alpha_1\beta_1-P\alpha_2\beta_2)\\
&\qquad+\frac{2}{3}\big((2QM_4-PM_{22}-RM_{11}+(M_{11})_2-(M_4)_1\big)\alpha\beta\\
&\qquad\qquad+(Q_2-R_1+QR-PS-M_4)\alpha\beta_1\\
&\qquad\qquad-(P_2-Q_1+2Q^2-2PR-4M_{11})\alpha\beta_2\big)\\
&\quad=2(M_4(\alpha\beta_1+\alpha_1\beta)+R\alpha_1\beta_1-P\alpha_2\beta_2)\\
&\qquad\qquad+\frac{2}{3}\big(2QM_4-PM_{22}-RM_{11}+(M_{11})_2-(M_4)_1\big)\alpha\beta.
\end{aligned}$$

Since the last expression is symmetric on α and β, we have

$$v_1(I(\alpha,\beta))=\frac{1}{2}\alpha\beta_3+\alpha_1\beta_2-\Big(\frac{1}{2}\alpha_3\beta+\alpha_2\beta_1\Big)=0.$$

In a similar way, we get the equality $v_2(I(\alpha, \beta)) = 0$.

The non-degeneracy of I is understood from the fact that the solution space of (Sp) is parametrized by the initial values $\vartheta(p)$, $v_1(\vartheta)(p)$, $v_2(\vartheta)(p)$ and $v_3(\vartheta)(p)$, where p is a fixed point in the domain. Q.E.D.

Proposition 4.5. *Let $\vartheta \in \mathcal{S}(P, Q, R, S)$ be a non-zero solution. Then the solution space $s(P, Q, R, S; \vartheta)$ of the PDE system* (Sp-s) *is the skew orthogonal subspace of ϑ in the symplectic space $\mathcal{S}(P, Q, R, S)$;*

$$s(P, Q, R, S; \vartheta) = \{\alpha \in \mathcal{S}(P, Q, R, S) \mid I(\alpha, \vartheta) = 0\}.$$

Proof. Let $\xi \in s(P, Q, R, S; \vartheta)$ be fixed. Subtracting the third equation from the second of (Sp-s) that ξ satisfies, and multiplying it by ϑ, we get the equality $I(\xi, \vartheta) = 0$. And conversely, if a solution $\xi \in \mathcal{S}(P, Q, R, S)$ satisfies $I(\xi, \vartheta) = 0$, then it satisfies automatically the equations (Sp-s). This proves the proposition. Q.E.D.

§5. PDE systems related to contact transformation

If a transformation $\phi\colon (x, y, z) \mapsto (X, Y, Z)$ is contact, the coordinate functions X, Y and Z satisfy certain differential equations. In the following, we study relations of those equations, whose integrability conditions will be discussed in Section 7.

We start with the following PDE system with unknown function Ω and given functions P, Q, R, S and Δ

$$\text{(Ct-s)} \qquad \begin{cases} {v_1}^2(\Omega) = \left(Q + \dfrac{v_1(\Delta)}{\Delta}\right) v_1(\Omega) - P v_2(\Omega) \\ v_2 v_1(\Omega) = R v_1(\Omega) - \left(Q - \dfrac{v_1(\Delta)}{\Delta}\right) v_2(\Omega) \\ v_1 v_2(\Omega) = \left(R + \dfrac{v_2(\Delta)}{\Delta}\right) v_1(\Omega) - Q v_2(\Omega) \\ {v_2}^2(\Omega) = S v_1(\Omega) - \left(R - \dfrac{v_2(\Delta)}{\Delta}\right) v_2(\Omega). \end{cases}$$

Proposition 5.1. *If a transformation $\phi\colon (x, y, z) \mapsto (X, Y, Z)$ is contact, then X and Z are solutions of the PDE system* (Ct-s) *with $(P, Q, R, S) = S(\phi)$ and Δ any nonzero constant multiple of the contact Jacobian $\Delta(\phi)$.*

Conversely, if 1 *(a constant function), X and Z are mutually linearly independent solutions of* (Ct-s), *then there exists a function Y such that the map $\phi\colon (x, y, z) \mapsto (X, Y, Z)$ is a contact transformation whose* *contact Schwarzian derivative is equal to $S(\phi) = (P, Q, R, S)$, and whose contact Jacobian is equal to a constant multiple of Δ.*

Proof. If X and Z satisfy the equations (Ct-s), then we get

$$\begin{pmatrix} X_2 & X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_1 & X_2 & 0 \\ 0 & 0 & 0 & 0 & X_2 & 0 & 0 & X_1 \\ 0 & 0 & X_2 & X_1 & 0 & 0 & 0 & 0 \\ Z_2 & Z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_1 & Z_2 & 0 \\ 0 & 0 & 0 & 0 & Z_2 & 0 & 0 & Z_1 \\ 0 & 0 & Z_2 & Z_1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -P \\ Q + \Delta_1/\Delta \\ -R + \Delta_2/\Delta \\ S \\ -Q \\ R \\ -Q + \Delta_1/\Delta \\ R + \Delta_2/\Delta \end{pmatrix} = \begin{pmatrix} X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \\ Z_{11} \\ Z_{12} \\ Z_{21} \\ Z_{22} \end{pmatrix},$$

where $(\ )_i = v_i(\ )$ and $(\ )_{ij} = v_j v_i(\ )$. Put $D = v_1(X)v_2(Z) - v_2(X)v_1(Z)$. The determinant of the above 8×8-matrix equals $-D^4$. The linear independence of 1, X and Z implies that the determinant does not vanish. By multiplying the inverse matrix, and taking linear combinations, we get eight relations, four of which are exactly the definitions of contact Schwarzian derivative, and the others read

$$\begin{aligned} \frac{v_1(\Delta)}{\Delta} &= \frac{1}{D}\big(v_1{}^2(X)v_2(Z) + v_1(X)v_1v_2(Z) \\ &\qquad - v_1v_2(X)v_1(Z) - v_2(X)v_1{}^2(Z)\big) \\ \frac{v_2(\Delta)}{\Delta} &= \frac{1}{D}\big(v_2v_1(X)v_2(Z) + v_1(X)v_2{}^2(Z) \\ &\qquad - v_2{}^2(X)v_1(Z) - v_2(X)v_2v_1(Z)\big) \\ \frac{v_1(\Delta)}{\Delta} &= \frac{1}{D}\big(v_1(X)v_3(Z) - v_3(X)v_1(Z)\big) \\ \frac{v_2(\Delta)}{\Delta} &= \frac{1}{D}\big(v_2(X)v_3(Z) - v_3(X)v_2(Z)\big). \end{aligned}$$

The first two equations are equivalent to

$$\frac{v_1(\Delta)}{\Delta} = \frac{v_1(D)}{D}, \quad \frac{v_2(\Delta)}{\Delta} = \frac{v_2(D)}{D}.$$

These equations imply that Δ is a constant multiple of D. Furthermore, this fact allows us to write the last two equalities as

$$v_i(D) = v_i(X)v_3(Z) - v_3(X)v_i(Z), \quad (i = 1,\ 2).$$

These equations amount to the condition that guarantees the existence of the solution of the equation $v_i(Y) = Zv_i(X)$ $(i = 1, 2)$ (cf. Lemma 3.2).

Q.E.D.

Proposition 5.2. *If a transformation ϕ: $(x, y, z) \mapsto (X, Y, Z)$ is contact, then the functions X, Z and $W = 2Y - XZ$ satisfy*

$$\text{(Ct)} \qquad \begin{cases} {v_1}^2(\Omega) = \left(Q + \dfrac{v_1(\Delta)}{\Delta}\right) v_1(\Omega) - Pv_2(\Omega) \\ v_4(\Omega) = \left(2R + \dfrac{v_2(\Delta)}{\Delta}\right) v_1(\Omega) - \left(2Q - \dfrac{v_1(\Delta)}{\Delta}\right) v_2(\Omega) \\ {v_2}^2(\Omega) = Sv_1(\Omega) - \left(R - \dfrac{v_2(\Delta)}{\Delta}\right) v_2(\Omega), \end{cases}$$

provided $(P, Q, R, S) = S(\phi)$ and $\Delta = \Delta(\phi)$.

Proof. If $(x, y, z) \mapsto (X, Y, Z)$ is contact, then we have $v_i(Y) = Zv_i(X)$ $(i = 1,\ 2)$ (see Lemma 3.1), and therefore the function $W = 2Y - XZ$ satisfies

$$v_i(W) = Zv_i(X) - Xv_i(Z), \quad (i = 1,\ 2).$$

Differentiating it by v_j, we get

$$v_j v_i(W) = Zv_j v_i(X) - Xv_j v_i(Z) + \varepsilon_{ij}\Delta,$$

where Δ is the contact Jacobian, and $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\varepsilon_{12} = 1 = -\varepsilon_{21}$. Reminding that X and Z are solutions of (Ct-s), we find that the function W satisfies the equations

$$\begin{cases} {v_1}^2(W) = Qv_1(W) - Pv_2(W) + \dfrac{v_1(\Delta)}{\Delta} v_1(W) \\ v_2 v_1(W) = Rv_1(W) - Qv_2(W) + \dfrac{v_1(\Delta)}{\Delta} v_2(W) + \Delta \\ v_1 v_2(W) = Rv_1(W) - Qv_2(W) + \dfrac{v_2(\Delta)}{\Delta} v_1(W) - \Delta \\ {v_2}^2(W) = Sv_1(W) - Rv_2(W) + \dfrac{v_2(\Delta)}{\Delta} v_2(W). \end{cases}$$

By adding the second to the third expression, this system is reduced to the system (Ct). Therefore W satisfies the PDE system (Ct). Since the system (Ct-s) also reduces to (Ct), it follows, by Proposition 5.1, that X and Z also satisfy (Ct). Q.E.D.

If the functions $\{1, X, Z, W\}$ in the above proposition form a linear basis of the solution space, then the solution space of the system (Ct-s) is a subspace spanned by $\{1, X, Z\}$, and is characterized by the linear equation

$$\Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega) = 0. \tag{4}$$

Subtracting the third equation of (Ct-s) from the second, we get (4). The function $W = 2Y - XZ$ stays on the affine subspace parallel to this subspace, and characterized by

$$\Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega) - 2\Delta^2 = 0. \tag{5}$$

If a function Y corresponds to independent solutions X and Z of (Ct-s) so that $\phi = (X, Y, Z)$ is contact, then the function

$$\tilde{Y} = \frac{1}{2}\big((ad - bc)(2Y - XZ) + (aX + bZ)(cX + dZ)\big).$$

corresponds to the solutions $\tilde{X} = aX + bZ$ and $\tilde{Z} = cX + dZ$ for any constant a, b, c and d with $ad - bc \neq 0$. The resulting contact transformation $\tilde{\phi} = (\tilde{X}, \tilde{Y}, \tilde{Z})$ has the same contact Schwarzian derivatives $S(\tilde{\phi}) = S(\phi)$, and the contact Jacobian is equal to $\Delta(\tilde{\phi}) = (ad-bc)\Delta(\phi)$. Thus the PDE systems (Ct-s) and (Ct) remain the same for the contact transformation $\tilde{\phi}$.

We study equations that contact Jacobians $\Delta = \Delta(\phi)$ satisfy. Suppose (Ct-s) has two linearly independent non-constant solutions X and Z. They necessarily satisfy

$$0 = v_i v_3(\Omega) - v_3 v_i(\Omega), \quad (i = 1,\ 2).$$

Since $\Omega = X$ or Z is a solution of (Ct-s), the above is equivalent to

$$0 = F_i(\Delta)v_1(\Omega) + G_i(\Delta)v_2(\Omega),$$

where F_i and G_i are the operators given by

$$\begin{aligned}
F_2(\Delta) &= -2\Big(-{v_1}^2(\Delta) + Qv_1(\Delta) - Pv_2(\Delta) + \frac{1}{2}(v_1(\Delta))^2 - 2M_{11}\Big)\\
F_1(\Delta) &= G_2(\Delta)\\
&= -v_4(\Delta) + 2Rv_1(\Delta) - 2Qv_2(\Delta) + v_1(\Delta)v_2(\Delta) - 4M_4\\
G_1(\Delta) &= -2\Big(-{v_2}^2(\Delta) + Sv_1(\Delta) - Rv_2(\Delta) + \frac{1}{2}(v_2(\Delta))^2 - 2M_{22}\Big).
\end{aligned}$$

The independence of X and Z implies $F_i(\Delta) = G_i(\Delta) = 0$ $(i = 1,\ 2)$, which amounts to the PDE system

$$\begin{cases}
{v_1}^2(\Delta) = Qv_1(\Delta) - Pv_2(\Delta) + \frac{1}{2}(v_1(\Delta))^2 - 2M_{11}\\
v_4(\Delta) = 2Rv_1(\Delta) - 2Qv_2(\Delta) + v_1(\Delta)v_2(\Delta) - 4M_4\\
{v_2}^2(\Delta) = Sv_1(\Delta) - Rv_2(\Delta) + \frac{1}{2}(v_2(\Delta))^2 - 2M_{22}
\end{cases} \tag{6}$$

This PDE system is nonlinear. The change of variable $\vartheta = 1/\sqrt{\Delta}$ (taking one branch of square root) linearizes it, and the resulting PDE system is exactly the same as (Sp).

Proposition 5.3. *If Δ is a contact Jacobian, then $1/\sqrt{\Delta}$ is a solution of the PDE system* (Sp) *whose coefficients P, Q, R and S are equal to the contact Schwarzian derivatives.*

Remark 5.4. The PDE system (6) is a necessary condition for Δ to have a nonconstant solution of the PDE system (Ct-s), but not sufficient. If, in addition, P, Q, R and S satisfy the relations (IC), then the PDE system (Ct-s) has a nonconstant solution. This fact can be shown after a quite long calculation. In order to perform it, we used the computer program "Maple".

§6. Construction of contact transformation via solutions of PDE system

Let I be the skew product on the solution space $\mathcal{S}(P, Q, R, S)$ of the PDE system (Sp) defined in Proposition 4.4. We prove, in this section, the following

Theorem 6.1. *If a map ϕ: $(x, y, z) \mapsto (X, Y, Z)$ is contact, then there exists a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(S(\phi))$ of the PDE system* (Sp) *such that ϕ is given by*

$$(x, y, z) \mapsto \left(\frac{\xi}{\vartheta}, \frac{1}{2}\left(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2}\right), \frac{\zeta}{\vartheta}\right). \tag{7}$$

Conversely, given a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp), *the map ϕ defined by* (7) *is a contact transformation whose contact Schwarzian derivatives $S(\phi)$ is equal to (P, Q, R, S).*

Here if we say a linear basis $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is symplectic, we mean

$$\big(I(\xi_i, \xi_j)\big)_{i,j=0,3} = cJ, \quad \text{where } J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

and c is a nonzero constant. We denote by $Sp_2(\mathbb{K})$ and $CSp_2(\mathbb{K})$ the Lie groups defined by

$$\begin{aligned} Sp_2(\mathbb{K}) &= \{g \in GL(4; \mathbb{K}) \mid gJg^t = J\} \\ CSp_2(\mathbb{K}) &= \{gJg^t = cJ \text{ for some nonzero constant } c \in \mathbb{K}\}. \end{aligned}$$

In order to emphasize the PDE systems (Sp-s), (Ct-s) and (Ct) to contain the functions ϑ and Δ in their coefficients, we use the notations $(\text{Sp-s})^{\vartheta}$, $(\text{Ct-s})^{\Delta}$ and $(\text{Ct})^{\Delta}$. Their solution spaces are denoted by $s(P,Q,R,S;\vartheta)$, $t(P,Q,R,S;\Delta)$ and $\mathcal{T}(P,Q,R,S;\Delta)$, respectively;

$$\begin{aligned}
\mathcal{S}(P,Q,R,S) &= \text{the solution space of the PDE system (Sp)}\\
s(P,Q,R,S;\vartheta) &= \text{the solution space of the PDE system } (\text{Sp-s})^{\vartheta}\\
\mathcal{T}(P,Q,R,S;\Delta) &= \text{the solution space of the PDE system } (\text{Ct})^{\Delta}\\
t(P,Q,R,S;\Delta) &= \text{the solution space of the PDE system } (\text{Ct-s})^{\Delta}
\end{aligned}$$

We find a relation among those solution spaces in the following

Proposition 6.2. *For each nonzero solution ϑ of* (Sp), *the projectification $\xi \mapsto \xi/\vartheta$ gives a linear map between the solution spaces $\mathcal{S}(P,Q,R,S)$ and $\mathcal{T}(P,Q,R,S;\Delta)$, provided Δ is a constant multiple of ϑ^{-2}. The image of the subspace $s(P,Q,R,S;\vartheta)$ is contained in the subspace $t(P,Q,R,S;\Delta)$;*

$$\begin{array}{ccc}
\mathcal{S}(P,Q,R,S) & \longrightarrow & \mathcal{T}(P,Q,R,S;\Delta)\\
\cup & & \cup\\
s(P,Q,R,S;\vartheta) & \longrightarrow & t(P,Q,R,S;\Delta).
\end{array}$$

The largest dimension of the solution spaces $\mathcal{S}(P,Q,R,S)$ and $\mathcal{T}(P,Q,R,S;\Delta)$ are 4, and those of $s(P,Q,R,S;\theta)$ and $t(P,Q,R,S;\Delta)$ are 3. By Proposition 4.5, the codimension of $s(P,Q,R,S;\vartheta)$ in $\mathcal{S}(P,Q,R,S)$ is always 1, provided the PDE system (Sp) has at least one nonzero solution ϑ. Clearly the map $\xi \mapsto \xi/\vartheta$ is injective. Therefore, if the dimension of $\mathcal{S}(P,Q,R,S)$ is 4, that is, if the PDE system (Sp) is integrable, then all the other PDE systems are also integrable. We will manage the integrability of (Sp) in the next section.

Proof. Let a nonzero solution ϑ of (Sp) be fixed. Suppose ξ is a solution of (Sp). Then we have

$$\begin{aligned}
{v_1}^2\Big(\frac{\xi}{\vartheta}\Big) &= v_1\Big(\frac{1}{\vartheta^2}\big(v_1(\xi)\vartheta - \xi v_1(\vartheta)\big)\Big)\\
&= \frac{1}{\vartheta^4}\Big\{\big({v_1}^2(\xi)\vartheta - \xi {v_1}^2(\vartheta)\big)\vartheta^2 - \big(v_1(\xi)\vartheta - \xi v_1(\vartheta)\big)\big(2\vartheta v_1(\vartheta)\big)\Big\}\\
&= \frac{1}{\vartheta^3}\Big\{\big(Qv_1(\xi) - Pv_2(\xi) + M_{11}\xi\big)\vartheta^2\\
&\qquad - \xi\vartheta\big(Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta\big)\\
&\qquad - 2v_1(\xi)v_1(\vartheta)\vartheta + 2\xi v_1(\vartheta)^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= Q\frac{v_1(\xi)\vartheta - \xi v_1(\vartheta)}{\vartheta^2} - P\frac{v_2(\xi)\vartheta - \xi v_2(\vartheta)}{\vartheta^2} \\
&\quad -2\frac{v_1(\vartheta)}{\vartheta}\frac{v_1(\xi)\vartheta - \xi v_1(\vartheta)}{\vartheta^2} \\
&= \left(Q - 2\frac{v_1(\vartheta)}{\vartheta}\right) v_1\left(\frac{\xi}{\vartheta}\right) - Pv_2\left(\frac{\xi}{\vartheta}\right).
\end{aligned}
$$

If Δ is equal to $c\vartheta^{-2}$ with c a nonzero constant, then the last expression is equal to

$$
\left(Q + \frac{v_1(\Delta)}{\Delta}\right) v_1\left(\frac{\xi}{\vartheta}\right) - Pv_2\left(\frac{\xi}{\vartheta}\right).
$$

Thus $\Omega = \xi/\vartheta$ satisfies the first equation in (Ct). In a similar way, we verify that $\Omega = \xi/\vartheta$ satisfies other two equalities of $(\mathrm{Ct})^{\Delta}$ with $\Delta = c\vartheta^{-2}$. The subspaces $s(P,Q,R,S;\vartheta)$ and $t(P,Q,R,S;\vartheta)$ are characterized by the equations $I(\vartheta,\xi) = 0$ and (4), respectively. Thus we have

$$
\begin{aligned}
&\frac{1}{c}\left(\Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega)\right) \\
&\quad = \frac{1}{\vartheta^2}\frac{v_3(\xi)\vartheta - \xi v_3(\vartheta)}{\vartheta^2} + \frac{2v_1(\vartheta)}{\vartheta^3}\frac{v_2(\xi)\vartheta - \xi v_2(\vartheta)}{\vartheta^2} \\
&\qquad - \frac{2v_2(\vartheta)}{\vartheta^3}\frac{v_1(\xi)\vartheta - \xi v_1(\vartheta)}{\vartheta^2} \\
&\quad = 2\vartheta^{-4} I(\vartheta,\xi).
\end{aligned}
$$

Therefore the image of $s(P,Q,R,S;\vartheta)$ is contained in $t(P,Q,R,S;\vartheta)$. Q.E.D.

The rest of this section is devoted to the proof of Theorem 6.1. Suppose ϕ: $(x,y,z) \mapsto (X,Y,Z)$ is a contact transformation. From Propositions 5.1 and 5.2, we see that $\{1, X, Z, W = 2Y - XZ\}$ is a linear basis of the solution space $\mathcal{T}(P,Q,R,S;\Delta)$ of the system $(\mathrm{Ct})^{\Delta}$, and $\{1, X, Z\}$ a basis of $t(P,Q,R,S;\Delta)$ of the system $(\text{Ct-s})^{\Delta}$, where $(P,Q,R,S) = S(\phi)$ and $\Delta = \Delta(\phi)$; the contact Schwarzian derivative and the contact Jacobian of ϕ.

From Proposition 5.3, we see that, if $\Delta(\phi) = c\vartheta^{-2}$ with c a nonzero constant, then ϑ satisfies the equations (Sp). Therefore by Proposition 6.2, we conclude that there exists a linear basis $\{\vartheta, \xi, \zeta, \eta\}$ (where ϑ satisfies $\Delta = c\vartheta^{-2}$) of the solution space $\mathcal{S}(P,Q,R,S)$ such that

$$
X = \frac{\xi}{\vartheta}, \quad Z = \frac{\zeta}{\vartheta}, \quad W = \frac{\eta}{\vartheta}.
$$

It remains to prove that the basis $\{\vartheta, \xi, \zeta, \eta\}$ is symplectic in the sence of Proposition 4.4. We remind that $I(\alpha, \beta)$ is constant, if α and β are solutions of the PDE system (Sp).

By using the functions ϑ, ξ, ζ and η, we calculate the contact Jacobian $\Delta(\phi)$ as follows:

$$
\begin{aligned}
\Delta(\phi) &= v_1(X)v_2(Z) - v_1(Z)v_2(X) \\
&= \frac{1}{\vartheta^4}\Big\{\big(v_1(\xi)\vartheta - \xi v_1(\vartheta)\big)\big(v_2(\zeta)\vartheta - \zeta v_2(\vartheta)\big) \\
&\qquad\qquad - \big(v_1(\zeta)\vartheta - \zeta v_1(\vartheta)\big)\big(v_2(\xi)\vartheta - \xi v_2(\vartheta)\big)\Big\} \\
&= \frac{1}{\vartheta^3}\Big\{\big(v_1(\xi)v_2(\zeta) - v_1(\zeta)v_2(\xi)\big)\vartheta \\
&\qquad\qquad + \xi\big(- v_1(\vartheta)v_2(\zeta) + v_1(\zeta)v_2(\vartheta)\big) \\
&\qquad\qquad - \zeta\big(- v_1(\xi)v_2(\vartheta) + v_1(\vartheta)v_2(\xi)\big)\Big\} \\
&= \frac{1}{\vartheta^3}\Big\{\big(v_1(\xi)v_2(\zeta) - v_1(\zeta)v_2(\xi)\big)\vartheta \\
&\qquad\qquad - \frac{1}{2}\zeta\big(v_3(\xi)\vartheta - \xi v_3(\vartheta)\big) + \frac{1}{2}\xi\big(v_3(\zeta)\vartheta - \zeta v_3(\vartheta)\big)\Big\} \\
&= I(\xi, \zeta)\vartheta^{-2}.
\end{aligned} \tag{9}
$$

Since $\Delta(\phi) = c\vartheta^{-2}$, we conclude

$$I(\xi, \zeta) = c.$$

The function W satisfies (5). If $W = \eta/\vartheta$ and $\Delta = c\vartheta^{-2}$, the equality (5) is replaced by

$$
\begin{aligned}
0 &= \Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega) - 2\Delta^2 \\
&= c\Big(\vartheta^{-2}v_3\Big(\frac{\eta}{\vartheta}\Big) - v_1(\vartheta^{-2})v_2\Big(\frac{\eta}{\vartheta}\Big) + v_2(\vartheta^{-2})v_1\Big(\frac{\eta}{\vartheta}\Big) - 2c\vartheta^{-4}\Big) \\
&= 2c\vartheta^{-4}(I(\vartheta, \eta) - c),
\end{aligned}
$$

and thus

$$I(\vartheta, \eta) = c.$$

Since $X = \xi/\vartheta$ and $Z = \zeta/\vartheta$ are in the solution space $t(P, Q, R, S; \vartheta)$, ξ and ζ are in the solution space $s(P, Q, R, S; \vartheta)$. Therefore ξ and ζ are skew orthogonal to ϑ;

$$I(\vartheta, \xi) = I(\vartheta, \zeta) = 0$$

(cf. Proposition 4.5).

In order to prove the skew orthogonalities $I(\eta,\xi) = I(\eta,\zeta) = 0$, we prove the following

Lemma 6.3. *For solutions ϑ, ξ, ζ and η of* (Sp), *put $X = \xi/\vartheta$, $Z = \zeta/\vartheta$ and $W = \eta/\vartheta$. Then the equalities $v_i(W) = v_i(X)Z - Xv_i(Z)$ $(i = 1, 2)$ hold, if and only if $I(\eta,\xi) = I(\eta,\zeta) = 0$.*

Proof. For each point $p = (x, y, z)$ in the domain, the evaluation map $|_p$:

$$\mathcal{S}(P,Q,R,S) \ni \vartheta \mapsto \Big(\vartheta(p), v_1(\vartheta)(p), v_2(\vartheta)(p), \frac{1}{2}v_3(\vartheta)(p)\Big) \in \mathbb{K}^4$$

is a symplectic linear map with respect to the skew products I on $\mathcal{S}(P,Q,R,S)$ and J on $\mathbb{K}^4$. If ϑ, ξ, ζ and η is a symplectic basis, then the matrix

$$\begin{pmatrix} \vartheta \\ \xi \\ \zeta \\ \eta \end{pmatrix}\Bigg|_p = \begin{pmatrix} \vartheta(p) & v_1(\vartheta)(p) & v_2(\vartheta)(p) & v_3(\vartheta)(p)/2 \\ \xi(p) & v_1(\xi)(p) & v_2(\xi)(p) & v_3(\xi)(p)/2 \\ \zeta(p) & v_1(\zeta)(p) & v_2(\zeta)(p) & v_3(\zeta)(p)/2 \\ \eta(p) & v_1(\eta)(p) & v_2(\eta)(p) & v_3(\eta)(p)/2 \end{pmatrix}$$

is in the symplectic group $Sp_2(\mathbb{K})$. Since the complex structure J satisfies $J^{-1} = J^t = -J$, the transpose of this matrix is also in $Sp_2(\mathbb{K})$. Thus we have

$$0 = v_i(\eta)(p)\vartheta(p) - \eta(p)v_i(\vartheta)(p) + \xi(p)v_i(\zeta)(p) - v_i(\xi)(p)\zeta(p)$$

for $i = 1, 2$. These are equivalent to

$$\begin{aligned} 0 &= v_i\Big(\frac{\eta}{\vartheta}\Big)(p) - v_i\Big(\frac{\xi}{\vartheta}\Big)(p)\frac{\zeta(p)}{\vartheta(p)} + \frac{\xi(p)}{\vartheta(p)}v_i\Big(\frac{\zeta}{\vartheta}\Big)(p) \\ &= v_i(W)(p) - v_i(X)(p)Z(p) + X(p)v_i(Z)(p). \end{aligned}$$

This completes the proof. Q.E.D.

Since $\phi = (X, Y, Z)$ is contact, $W = 2Y - XZ$ satisfies the equality of the above lemma. Therefore we conclude the equalities

$$I(\eta,\xi) = I(\eta,\zeta) = 0,$$

and complete the proof of the first half of Theorem 6.1.

Now we prove the latter half of the theorem. Suppose $\{\vartheta,\xi,\zeta,\eta\}$ is a symplectic basis of the solution space $\mathcal{S}(P,Q,R,S)$, and put $X = \xi/\vartheta$, $Z = \zeta/\vartheta$ and $W = \eta/\vartheta$. From Lemma 6.3, it follows that the function $Y = (W + XZ)/2$ satisfies the equalities $v_i(Y) = Zv_i(X)$ $(i = 1, 2)$. Put

$\Delta = v_1(X)v_2(Z) - v_2(X)v_1(Z)$. Then the same caluculation as (9) shows that Δ is equal to $I(\xi,\zeta)\vartheta^{-2}$, where $I(\xi,\zeta)$ is a nonzero constant, and thus Δ does not vanish. Therefore the map $\phi = (X, Y, Z)$ is a contact transformation (cf. Corollary 3.4).

Since X and Z are solutions of $(\text{Ct-s})^{\Delta}$ with $\Delta = c\vartheta^{-2}$ (cf. Proposition 6.2), it follows, from Proposition 5.1, that the given function P, Q, R and S are the contact Schwarzian derivatives of ϕ; $(P, Q, R, S) = S(\phi)$, and also that the contact Jacobian $\Delta(\phi)$ is equal to $c\vartheta^{-2}$.

This completes the proof of Theorem 6.1.

§7. Integrability conditions

The higher derivatives of the solutions of the PDE systems are determined by the lower terms. For the systems (Sp) and (Ct), the initial values $\vartheta(p)$, $v_1(\vartheta)(p)$, $v_2(\vartheta)(p)$ and $v_3(\vartheta)(p)$ at a fixed point $p = (x, y, z)$ in the domain determine all higher derivatives

$$v_{i_1} v_{i_2} \cdots v_{i_n}(\vartheta)(p).$$

Here the coefficient functions are required to satisfy certain compatibility conditions, which we call the *integrability condition* of the PDE system. The solution spaces of (Sp) and (Ct) is of dimension at most 4. The maximality holds, if and only if the integrability condition is fulfilled. On the other hand, the solution spaces of the PDE systems (Sp-s) and (Ct-s) are parameterized by the initial values $\vartheta(p)$, $v_1(\vartheta)(p)$ and $v_2(\vartheta)(p)$. Therefore the maximal dimension is equal to 3. The PDE systems that satisfy the maximality for the dimension of the solution spaces are said to be *integrable.*

As we investigated in the previous two sections, if the system (Sp) is integrable, then for each nonzero solution ϑ of (Sp), the other systems $(\text{Sp-s})^{\vartheta}$, $(\text{Ct-s})^{\vartheta^{-2}}$ and $(\text{Ct})^{\vartheta^{-2}}$ are integrable (see Proposition 6.2). If one of the other systems is integrable, then, by Theorem 6.1, there exists a contact transformation whose contact Schwarzian derivatives are equal to P, Q, R and S, and thus they necessarily satisfy the condition (IC) (see Remark 4.2). Theorem 4.1 says that the condition (IC) is the integrability condition of this system. Therefore, in order to clarify the integrability conditions of those PDE systems, it remains to prove Theorem 4.1.

Denote by $\mathcal{F}$ the ring of quadruples (f_0, f_1, f_2, f_3) of functions f_i of (x, y, z), and define differential operators V_1 and V_2 on $\mathcal{F}$ by

$$V_i(f_0, f_1, f_2, f_3) = (g_{0i}, g_{1i}, g_{2i}, g_{3i}),$$

where g_{ji} are defined by

$$
\begin{aligned}
g_{01} &= v_1(f_0) + M_{11}f_1 + M_4 f_2 + \frac{2}{3}(*)f_3 \\
g_{11} &= (v_1 + Q)(f_1) + f_0 + Rf_2 + 2M_4 f_3 \\
g_{21} &= (v_1 - Q)(f_2) - Pf_1 - 2M_{11}f_3 \\
g_{31} &= v_1(f_3) - \frac{1}{2}f_2 \\
g_{02} &= v_2(f_0) + M_4 f_1 + M_{22}f_2 + \frac{2}{3}(**)f_3 \\
g_{12} &= (v_2 + R)(f_1) + Sf_2 + 2M_{22}f_3 \\
g_{22} &= (v_2 - R)(f_2) + f_0 - Qf_1 - 2M_4 f_3 \\
g_{32} &= v_2(f_3) + \frac{1}{2}f_1.
\end{aligned}
$$

$(*)$ and $(**)$ read as

$$
\begin{aligned}
(*) &= -v_1(M_4) + v_2(M_{11}) + 2QM_4 - RM_{11} - PM_{22} \\
(**) &= -v_1(M_{22}) + v_2(M_4) - SM_{11} + 2RM_4 - QM_{22}.
\end{aligned}
$$

The idea how these operators V_1 and V_2 came out is the following: suppose the equation (Sp) has a 4-dimensional solution space, and fix a non-zero solution ϑ. If we consider the module over the ring of functions on (x, y, z) with ϑ, $v_1(\vartheta)$, $v_2(\vartheta)$, $v_3(\vartheta)$ as a formal basis;

$$\{f_0\vartheta + f_1 v_1(\vartheta) + f_2 v_2(\vartheta) + f_3 v_3(\vartheta) \mid f_i\colon \text{ functions on } (x, y, z)\},$$

then on this module the operators v_1 and v_2 operate naturally. Reminding that ϑ is a solution of (Sp), and using g_{ji}, we actually calculate the operations as

$$
\begin{aligned}
&v_i\big(f_0\vartheta + f_1 v_1(\vartheta) + f_2 v_2(\vartheta) + f_3 v_3(\vartheta)\big) \\
&\quad = g_{0i}\vartheta + g_{1i}v_1(\vartheta) + g_{2i}v_2(\vartheta) + g_{3i}v_3(\vartheta).
\end{aligned}
$$

Here we used the equalities

$$
\begin{aligned}
v_2 v_1(\vartheta) &= \frac{1}{2}\big(v_4(\vartheta) + v_3(\vartheta)\big) \\
v_1 v_2(\vartheta) &= \frac{1}{2}\big(v_4(\vartheta) - v_3(\vartheta)\big) \\
v_1 v_2 v_1(\vartheta) &= \frac{1}{3}\big(v_2 {v_1}^2(\vartheta) + v_1 v_4(\vartheta)\big) \\
v_2 v_1 v_2(\vartheta) &= \frac{1}{3}\big(v_2 {v_1}^2(\vartheta) + v_2 v_4(\vartheta)\big)
\end{aligned}
$$

$$v_3v_1(\vartheta) = v_1v_3(\vartheta) = \frac{1}{3}\big(2v_2{v_1}^2(\vartheta) - v_1v_4(\vartheta)\big)$$
$$v_3v_2(\vartheta) = v_2v_3(\vartheta) = \frac{1}{3}\big(-2v_1{v_2}^2(\vartheta) + v_2v_4(\vartheta)\big).$$

Proposition 7.1. *Define the operator V_3 by $V_3 = [V_2, V_1]$. The following three conditions on the four functions P, Q, R and S are equivalent:*

(a) *the PDE system* (Sp) *is integrable, that is, the solution space is of dimension equal to* 4,

(b) $[V_3, V_i] = 0, \quad (i = 1,\ 2)$, *and*

(c) *P, Q, R and S satisfy the relations* (IC).

If ϑ_i $(i = 0,\ 1,\ 2,\ 3)$ are 4 independent solutions of (Sp), then the vectors

$$\big(\vartheta_i(p), v_1(\vartheta_i)(p), v_2(\vartheta_i)(p), v_3(\vartheta_i)(p)\big), \quad (i = 0,\ 1,\ 2,\ 3)$$

are linearly independent at each point $p = (x, y, z)$. Thus the dimension of the solution space of (Sp) is at most 4.

The higher derivatives $v_{i_1}v_{i_2}\cdots v_{i_n}(\vartheta)(p)$ can be expressed as unique linear combinations of $\{\vartheta(p), v_1(\vartheta)(p), v_2(\vartheta)(p), v_3(\vartheta)(p)\}$. Those coefficients are uniquely determined, provided the operators V_1 and V_2 satisfy the Heisenberg relation $[V_3, V_i] = 0$.

The equivalence of the conditions (a) and (b) is deduced from these facts and the definition of V_i. The equivalence of (b) and (c) is deduced by the following calculation: for elements

$$e_0 = (1,0,0,0) \qquad e_1 = (0,1,0,0)$$
$$e_2 = (0,0,1,0) \qquad e_3 = (0,0,0,1)$$

of $\mathcal{F}$, we define functions a_{ij}^k by

$$[V_3, V_i](e_j) = \sum_{k=0}^{3} a_{ij}^k e_k, \qquad (i = 1,\ 2, \quad j = 0,\ 1,\ 2,\ 3).$$

All these functions are polynomials of P, Q, R, S and their derivatives.

Let c_i $(i = 0,\ 1,\ 2,\ 3)$ be functions of (x, y, z) defined by

$$c_0 = -6\big(-v_3(P) + 2(v_1 - 2Q)(M_{11}) + 4PM_4\big)$$
$$c_1 = -4\big(-3v_3(Q) + 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22}\big)$$
$$c_2 = 6 - 2\big(-3v_3(R) + 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11}\big)$$
$$c_3 = -6\big(-v_3(S) + 2(v_2 + 2R)(M_{22}) - 4SM_4\big),$$

so that the condition (IC) amounts to the condition $c_0 = c_1 = c_2 = c_3 = 0$. Our purpose is to explain a_{ij}^k as polynomials of the derivatives of c_0, c_1, c_2 and c_3.

The equalities $a_{i0}^k = 0$ ($\forall i$, $\forall k$) are obvious, since we have $[v_3, v_i] = 0$. The other functions are as follows:

$$
\begin{aligned}
a_{11}^0 &= \frac{1}{12}\big(-v_1(c_1) + v_2(c_0) - 2Rc_0 + 4Qc_1 - 2Pc_2\big),\\
a_{11}^1 &= c_1, \qquad a_{11}^2 = c_0,\\
a_{12}^0 &= \frac{1}{12}\big(-v_1(c_2) + v_2(c_1) - Sc_0 + Rc_1 + Qc_2 - Pc_3\big),\\
a_{12}^1 &= c_2, \qquad a_{12}^2 = c_1,\\
a_{13}^0 &= 2\big(-v_1(a_{12}^0) + v_2(a_{11}^0) - Pa_{22}^0 + 2Qa_{12}^0 - Ra_{11}^0\big)\\
&\qquad -\frac{1}{3}\big(M_{22}c_0 - 2M_4c_1 + M_{11}c_2\big),\\
a_{13}^1 &= 2a_{12}^0, \qquad a_{13}^2 = -2a_{11}^0,\\
a_{21}^0 &= a_{12}^0, \qquad a_{21}^1 = c_2, \qquad a_{21}^2 = c_1,\\
a_{22}^0 &= \frac{1}{12}\big(-v_1(c_3) + v_2(c_2) - 2Sc_1 + 4Rc_2 - 2Qc_3\big),\\
a_{22}^1 &= c_3, \qquad a_{22}^2 = c_2,\\
a_{23}^0 &= 2\big(-v_1(a_{22}^0) + v_2(a_{12}^0) - Qa_{22}^0 + 2Ra_{12}^0 - Sa_{11}^0\big)\\
&\qquad -\frac{1}{3}\big(M_{22}c_1 - 2M_4c_2 + M_{11}c_3\big),\\
a_{23}^1 &= 2a_{22}^0, \qquad a_{23}^2 = 2a_{12}^0,\\
a_{ij}^3 &= 0 \quad (\forall i,\ j = 0, \dots, 3),
\end{aligned}
$$

all of which are polynomials of the derivatives of c_0, c_1, c_2, c_3. The necessary and sufficient condition for (b) is the vanishings of all $a_{ij}^k = 0$. Therefore we established the equivalence of (b) and (c).

§8. Connection formula of contact Schwarzian derivatives

In this section, we use the notation $(x, y, z) = (x^1, x^3, x^2)$ for the coordinate functions. The superscript is unusual, but this simplifies the expressions that follow. And then the contact form on $\mathbb{K}^3$ is $dx^3 - x^2\, dx^1$, and three vector fields v_1, v_2 and v_3 are

$$v_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}, \quad v_2 = \frac{\partial}{\partial x^2} \quad \text{and} \quad v_3 = \frac{\partial}{\partial x^3}.$$

Proposition 8.1. *The contact Schwarzian derivatives of a composition of contact transformations* $(x^1, x^3, x^2) \mapsto (y^1, y^3, y^2) \mapsto (z^1, z^3, z^2)$

satisfy the formula

$$(10)\quad S_{\{ijk\}}(z,x) = S_{\{ijk\}}(y,x) + \sum_{p,q,r=1}^{2} S_{\{pqr\}}(z,y)\frac{v_i(y^p)v_j(y^q)v_k(y^r)}{\Delta(y,x)}.$$

In the rest of this section, we prove the above proposition. We use the following notations:

$$v_1 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}, \quad v_2 = \frac{\partial}{\partial x^2},$$
$$u_1 = \frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^3}, \quad u_2 = \frac{\partial}{\partial y^2}.$$

For a contact transformation $(x^1, x^3, x^2) \mapsto (y^1, y^3, y^2)$ and a function $f(y^1, y^3, y^2)$, it holds that $v_i(f) = \sum_{p=1}^{2} u_p(f)v_i(y^p)$ (see Lemma 3.5), and differentiating this by v_j, it holds that

$$v_jv_i(f) = \sum_{p=1}^{2} u_p(f)v_jv_i(y^p) + \sum_{p,q=1}^{2} u_qu_p(f)v_j(y^q)v_i(y^p).$$

Using this, we calculate $s_{[ij,k]}(z,x)$ as follows:

$$\begin{aligned} s_{[ij,k]}(z,x) &= v_iv_j(z^1)v_k(z^2) - v_iv_j(z^2)v_k(z^1) \\ &= \sum_{p,q} \left(u_p(z^1)u_q(z^2) - u_p(z^2)u_q(z^1)\right)v_iv_j(y^p)v_k(y^q) \\ &\quad + \sum_{p,q,r} s_{[pq,r]}(z,y)v_i(y^p)v_j(y^q)v_k(y^r). \end{aligned}$$

Concerning the first summation of the last expression, the sum for $(p,q) = (1,1)$ and $(2,2)$ is equal to 0, and for $(p,q) = (1,2)$ and $(2,1)$, it is equal to

$$\Delta(z,y)s_{[ij,k]}(y,x).$$

By summing up them for (i,j,k) cyclically permuted, and dividing it by 3, we thus get

$$(11)\qquad \Delta(z,x)S_{\{ij,k\}}(y,x),$$

which is the first term of the expression (10).

For the second term

$$\sum_{p,q,r} s_{[pq,r]}(z,y)v_i(y^p)v_j(y^q)v_k(y^r),$$

we calculate the sum of them for (i,j,k) cyclically permuted as follows:

(i) for $(p,q,r)=(1,1,1)$, the summand is equal to

$$3\Delta(z,y)S_{\{111\}}(z,y)v_i(y^1)v_j(y^1)v_k(y^1),$$

where we used the formula $\Delta(z,x)=\Delta(z,y)\Delta(y,x)$ (Lemma 3.6).

(ii) the sum over $(p,q,r)=(1,1,2)$, $(1,2,1)$ and $(2,1,1)$ is equal to

$$3\Delta(z,y)S_{\{112\}}(z,y)\times \\ \big(v_i(y^2)v_j(y^1)v_k(y^1)+v_i(y^1)v_j(y^2)v_k(y^1)+v_i(y^1)v_j(y^1)v_k(y^2)\big),$$

(iii) the sum over $(p,q,r)=(1,2,2)$, $(2,1,2)$ and $(2,2,1)$ is calculated in the same way as above, and

(iv) for $(p,q,r)=(2,2,2)$, it is equal to

$$3\Delta(z,y)S_{\{222\}}(z,y)v_i(y^2)v_j(y^2)v_k(y^2).$$

Adding up them with (11), we finally get the formula (10).

§9. Contact transformations with vanishing contact Schwarzian derivatives

We regard $\mathbb{K}^4$ a linear symplectic space with the symplectic form ω defined by

$$\begin{aligned}\omega((a_0,a_1,a_2,a_3),(b_0,b_1,b_2,b_3)) &= (a_0,a_1,a_2,a_3)J(b_0,b_1,b_2,b_3)^t \\ &= a_0b_3-a_3b_0+a_1b_2-a_2b_1,\end{aligned}$$

where the complex structure J is defined in (8).

Let ξ_i $(i=0,\dots,3)$ be functions defined by

$$\begin{aligned}\xi_0(x,y,z)&=1, & \xi_1(x,y,z)&=x,\\ \xi_2(x,y,z)&=z, & \xi_3(x,y,z)&=2y-xz.\end{aligned}$$

Then we easily verify that these functions form a symplectic basis of the solution space $\mathcal{S}(0,0,0,0)$ of the PDE system (Sp) with $P=Q=R=S=0$. Using these functions ξ_i, we define a map $\varphi\colon \mathbb{K}^3\to\mathbb{K}^4$ by

$$\varphi\colon (x,y,z)\mapsto(\xi_0,\xi_1,\xi_2,\xi_3). \tag{12}$$

The contact structure of $\mathbb{K}^3$ and the symplectic structure of $\mathbb{K}^4$ are related to each other via the map φ as follows: for each point $p\in\mathbb{K}^3$, the pull back of the skew orthogonal plane $\varphi(p)^{\perp}$ coincides with the contact distribution at p. This follows from the equalities

$$\omega\big(\varphi(p),v_i(\varphi)(p)\big)=0,\qquad (i=1,\ 2),$$

and implies the following

Lemma 9.1. *The projectification of the $Sp_2(\mathbb{K})$-action on $\mathbb{K}^4$ induces a contact transformation of $\mathbb{K}^3$ (or on a domain in $\mathbb{K}^3$) through the map φ.*

The image of φ is an affine 3-space that does not pass the origin of $\mathbb{K}^4$. Therefore, if a Lagrangian plane intersects with the image of φ, the inverse image of the Lagrangian plane by φ is a Legendrian curve in $\mathbb{K}^3$. Moreover it follows that

Lemma 9.2. *The inverse image of a Lagrangian plane by φ is a graph*

$$x \mapsto (x, ax^2 + 2bx + c, 2ax + 2b)$$

of a quadratic function $y = ax^2 + 2bx + c$, provided it is nonempty.

Proof. It suffices to consider Lagrangian planes that are the images of linear maps

$$(u, v) \mapsto \big(u, v, 2(bu + av), 2(cu + bv)\big).$$

The inverse image of those planes by φ are described by

$$x \mapsto \big(x,\ y = ax^2 + 2bx + c,\ z = 2(ax + b)\big).$$

and actually they satisfy $dy/dx = z$. Q.E.D.

The symplectic structure on $\mathbb{K}^4$ and the standard contact structure on $P^3(\mathbb{K})$ are related as follows: let a map $\psi\colon \mathbb{K}^3 \to \mathbb{K}^4$ be central (here "central" mean that, at each point p, the line through the origin of $\mathbb{K}^4$ and $\psi(p)$ is transversal to the tangent space of the image of ψ at $\psi(p)$). If ψ mediates between the contact structure on the source and the symplectic structure on the target as the map φ mediates, then the projectification of ψ is a contact map $\mathbb{K}^3 \to P^3(\mathbb{K})$.

In order to prove the following lemma, it suffices to use the same argument as in the proof of Lemma 6.3.

Lemma 9.3. *Let $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ be a symplectic basis of the solution space $\mathcal{S}(0,0,0,0)$. Then the map*

$$(x, y, z) \mapsto [\eta_0; \eta_1; \eta_2; \eta_3] \in P^3(\mathbb{K})$$

is contact.

We denote by $\mathrm{Cont}(P^3(\mathbb{K}))$ the group of contact isomorphisms of the projective space $P^3(\mathbb{K})$.

$$\Phi\colon CSp_2(\mathbb{K}) \to \mathrm{Cont}(P^3(\mathbb{K})).$$

The kernel of the homomorphism consists of the scalar matrices;

$$\ker(\Phi) = \mathbb{K} \cdot I_4.$$

Thus the image of Φ is isomorphic to

$$\Phi(CSp_2(\mathbb{K})) = \begin{cases} Sp_2(\mathbb{K})/\{\pm I_4\} \times \{I_4, I'\} & (\mathbb{K} = \mathbb{R}) \\ Sp_2(\mathbb{K})/\{\pm I_4\} & (\mathbb{K} = \mathbb{C}), \end{cases}$$

where I' denotes the diagonal matrix $\mathrm{diag}(1, 1, -1, -1)$.

We denote by $[\varphi]\colon \mathbb{K}^3 \to P^3(\mathbb{K})$ the contact map obtained by projectifying the map φ in (12).

Proposition 9.4. *If a map $\phi\colon D \to D'$ is a contact transformation with Schwarzian derivative $S(\phi) = 0$, then there exists a contact transformation $[g] \in \mathrm{Im}(\Phi)$ that commutes the following diagram;*

$$\begin{array}{ccc} D & \xrightarrow{\phi} & D' \\ {\scriptstyle [\varphi]}\downarrow & & \downarrow{\scriptstyle [\varphi]} \\ P^3(\mathbb{K}) & \xrightarrow{[g]} & P^3(\mathbb{K}) \end{array}$$

Proof. Let ϕ be a contact transformation with Schwarzian derivative $S(\phi) = 0$. It follows, from Theorem 6.1, that there exists a symplectic basis $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ of the solution space $\mathcal{S}(0, 0, 0, 0)$ such that ϕ is given by the equation (7) for these functions η_i. Then we find an element $g = (g_i^j) \in CSp_2(\mathbb{K})$ such that

$$\eta_i = \sum g_i^j \xi_j,$$

where ξ_j are the functions given as above. The projectification $\Phi(g) = [g]$ satisfies the required property. Q.E.D.

Thus we obtain the

Theorem 9.5. *The set of all contact transformations whose contact Schwarzian derivatives vanish forms a Lie group isomorphic to $Sp_2(\mathbb{K})/\{\pm I_4\}$ if $\mathbb{K} = \mathbb{C}$, and to $Sp_2(\mathbb{K})/\{\pm I_4\} \times \{I_4, I'\}$ if $\mathbb{K} = \mathbb{R}$.*

As is stated in Introduction, if we transform the ordinary differential equation $y''' = 0$ by a contact transformation $\phi = (X, Y, Z)$, we get the equation

$$y''' = P + 3Q(y'') + 3R(y'')^2 + S(y'')^3,$$

where the functions P, Q, R, S on (x, y, z) form the Schwarzian derivative of ϕ; $S(\phi) = (P, Q, R, S)$, and the variable z stands for $y' = dy/dx$.

Therefore, if the Schwarzian derivative (P, Q, R, S) of $\phi\colon D \to D'$ vanishes, the map ϕ transforms the set of graphs on D of all quadratic functions $y = ax^2 + 2bx + c$ into that on D'. The inverse implication is also true, because the solution space determines the differential equation. Thus we obtain

Corollary 9.6. *If $\phi\colon D \to D'$ is a diffeomorphism between two domains D and D' in $\mathbb{K}^3$ that gives a one-to-one correspondence on the sets of graphs of quadratic functions in D and D', then ϕ is a restriction of the projectification of a linear symplectic isomorphism on $\mathbb{K}^4$ through the embedding $\varphi(x, y, z) = (1, x, z, 2y - xz)$, where we regarded the map gI' with $g \in Sp_2(\mathbb{R})$ to be symplectic in case $\mathbb{K} = \mathbb{R}$.*

Schwarzian derivative of contact transformations on contact manifolds that are projectively flat, like $P^3(\mathbb{K})$, will be discussed in a forthcoming paper [O-S2].

References

[Gun] R. Gunning, On uniformization of complex manifolds: the role of connections, Math. Notes No. 22, Princeton. Princeton University Press, 1978.

[O-S1] T. Ozawa and H. Sato, *Linearizations of ordinary differential equations by area preserving maps*, Nagoya Math. J., **156** (1999), 109–122.

[O-S2] T. Ozawa and H. Sato, *Schwarzian derivative of global transformations and cocycles in continuous group cohomology*, in preparation.

[Sat] H. Sato, *Schwarzian derivatives of contact diffeomorphisms*, *Schwarzian derivatives of contact diffeomorphisms*, Lobachevskii J. Math., **4** (1999), 89–98.

[S-Y] H. Sato and A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan, **50** (1998), 993–1013.

[Yos] M. Yoshida, Fuchsian differential equations, Aspects of Mathematics, Vieweg, Baunschweig, 1987.

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