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# Cayley Transforms and Symmetry Conditions for Homogeneous Siegel Domains

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**Abstract.**

In this article we first present a family of Cayley transforms of a homogeneous Siegel domain. We then give characterizations of symmetric Siegel domains among homogeneous Siegel domains in terms of norm equalities involving the Cayley transforms. Applications of these characterizations to analysis on Siegel domains (the Berezin transforms and the Poisson kernel) are also exhibited.

## Introduction

Homogeneous Siegel domains are very interesting objects for researchers working in geometry or in analysis (or in both areas). This class of domains contains Hermitian symmetric spaces, and as one sees from the study for symmetric spaces as developed in [12], [28] or else, the presence of symmetry makes the algebraic and geometric structure rich and the analysis fertile. Occasionally some of well-known facts for Hermitian symmetric spaces drastically fail to hold upon loss of symmetry. Several of these phenomena are provided in the paper [3] as striking contrasts with symmetric Siegel domains. The results announced in this article lie in the same direction, and give analytic-geometric grounds to some properties of the Laplace-Beltrami operator which the present author came across during the study of Berezin transforms. Specifically, we exhibit

(1) a norm equality which leads us to the equivalence between the commutativity of the Berezin transform with the Laplace-Beltrami operator and the symmetry of the domain,

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(2) a norm equality accounting for the equivalence between the vanishing of the Poisson kernel under the Laplace-Beltrami operator and the symmetry of the domain.

Each of the two norm equalities involves a Cayley transform which is a visible generalization of the linear fractional one in $\mathbb{C}$ mapping the right half plane onto the unit disk. Our two Cayley transforms as well as Penney's [24] differ slightly from each other in general, but coincide up to positive scalar multiples if the domain is symmetric. The Cayley transforms are presented in Section 2 in a unified manner. In the notation used there, the Cayley transform needed for the norm equality in (1) above is $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}$ (see Section 3 for $\mathbf{d}$ and $\mathbf{b}$), and the one for (2) is $\mathcal{C}_{\mathbf{d}+\mathbf{b}}$. Penney's in [24] is expressed as $\mathcal{C}_{\mathbf{d}}$. An explicit formula for the inverse Cayley transform is also given in this article. The references are [19] and [22].

The norm equalities are presented in Section 3. We outline the proof of them briefly in this article. The details are quite technical, and we refer the reader to the papers [20] and [23].

Applications to the Berezin transforms and to the Poisson kernel are exhibited in Sections 4 and 5, respectively. The proofs of these results are found in [21] and [23].

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## §1. Preliminaries

Let $V$ be a finite-dimensional real vector space and $\Omega$ an open convex cone in $V$ containing no entire line. We put $W := V_{\mathbb{C}}$, the complexification of $V$. The conjugation in $W$ with respect to the real form $V$ is written as $w \mapsto w^*$. Let $U$ be another complex vector space of finite dimension. Let $Q\colon U \times U \to W$ be an $\Omega$-positive Hermitian sesquilinear ($\mathbb{C}$-linear in the first variable and antilinear in the second) map. We have

$$\begin{cases} Q(u',u) = Q(u,u')^* & (u,\ u' \in U), \\ Q(u,u) \in \overline{\Omega} \setminus \{0\} & \text{for all } u \in U \setminus \{0\}. \end{cases}$$

The *Siegel domain* $D$ corresponding to these data is defined to be

$$D := \{(u,w) \in U \times W;\ w + w^* - Q(u,u) \in \Omega\}.$$

We always assume that $D$ is homogeneous, that is, the *Lie* group $\mathrm{Hol}(D)$ of holomorphic automorphisms of $D$ acts transitively on $D$.

By [25], we can find a split solvable Lie group $G$ which acts simply transitively on $D$. In $G$ we also have a subgroup $H$ acting linearly and simply transitively on the cone $\Omega$. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$, the Lie algebras of $G$ and $H$ respectively. Since $G$ is diffeomorphic to the complex manifold $D$, we have an integrable almost complex structure $J$ on $\mathfrak{g}$. Moreover there is a linear form $\omega$ on $\mathfrak{g}$ such that $\langle x \,|\, y \rangle_\omega := \langle [Jx, y], \omega \rangle$ defines a $J$-invariant positive definite inner product on $\mathfrak{g}$. Such linear forms $\omega$ are said to be *admissible.* Structure theory of $\mathfrak{g}$ in [25] or [26] tells us that the orthogonal complement $\mathfrak{a}$ of the derived algebra $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ is an abelian subalgebra such that $\mathfrak{a}$ acts semisimply on $\mathfrak{g}$ by adjoint representation. This gives us a root space decomposition $\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$, where $\Delta$ is a finite subset of $\mathfrak{a}^*$ explained shortly and

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n};\ [h, x] = \langle h, \alpha \rangle x \quad \text{for all } h \in \mathfrak{a}\}.$$

The dimension $r := \dim \mathfrak{a}$ is called the *rank* of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ always possesses a direct product of $r$ copies of $(ax+b)$-algebra, that is, a basis $H_1, \dots, H_r$ of $\mathfrak{a}$ such that if we put $E_k := -JH_k$, then $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \dots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $H_1, \dots, H_r$. Then the elements of $\Delta$, which we call the *roots* of $\mathfrak{g}$, are of the following form (not all possibilities need occur):

$$\begin{array}{llll} \frac{1}{2}(\alpha_m + \alpha_k) & (1 \leqq k < m \leqq r), & \frac{1}{2}(\alpha_m - \alpha_k) & (1 \leqq k < m \leqq r), \\ \frac{1}{2}\alpha_k & (1 \leqq k \leqq r), & \alpha_k & (1 \leqq k \leqq r). \end{array}$$

Moreover we have $\mathfrak{n}_{\alpha_k} = \mathbb{R} E_k$. With $\mathfrak{g}(1/2) := \sum_{i=1}^r \mathfrak{n}_{\alpha_i/2}$ and

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}, \quad \mathfrak{g}(1) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m + \alpha_k)/2},$$

we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{g}(0) + \mathfrak{g}(1/2) + \mathfrak{g}(1)$ of $\mathrm{ad}(H_1 + \cdots + H_r)$, which gives a gradation $[\,\mathfrak{g}(i), \mathfrak{g}(j)\,] \subset \mathfrak{g}(i+j)$, where we understand $\mathfrak{g}(i) = 0$ for $i > 1$. Then, we can take $\mathfrak{g}(1)$ as $V$, $(\mathfrak{g}(1/2), -J)$ as $U$ (the subspace $\mathfrak{g}(1/2)$ being $J$-invariant), and $G(0) := \exp \mathfrak{g}(0)$ as the group $H$. Set $E := E_1 + \cdots + E_r$. The cone $\Omega$ can be taken as the $H$-orbit $H(E)$, and we have a diffeomorphism from $H$ onto $\Omega$ by the orbit map $h \mapsto hE$. The sesquilinear map $Q$ is then written as

$$Q(u, u') = \frac{1}{2}\left([Ju, u'] - i[u, u']\right) \quad (u,\ u' \in U). \tag{1.1}$$

Finally we take $\mathsf{e} := (0, E)$ as a base point of $D$, so that we have a diffeomorphism $g \mapsto g \cdot \mathsf{e}$ from $G$ onto $D$.

## §2. Family of Cayley transforms

Let us put $A := \exp \mathfrak{a}$ and set for $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$

$$a(t) := \exp(t_1 H_1 + \cdots + t_r H_r). \tag{2.1}$$

For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ let $\chi_{\mathbf{s}}$ be the one-dimensional representation of $A$ defined by $\chi_{\mathbf{s}}(a(t)) = \exp(\sum_k s_k t_k)$. Let $N := \exp \mathfrak{n}$ be the subgroup corresponding to $\mathfrak{n}$. Then $G = N \rtimes A$, and $\chi_{\mathbf{s}}$ extends to a positive character of $G$ by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let $\Delta_{\mathbf{s}}$ be the function on $\Omega$ obtained by the transfer of $\chi_{\mathbf{s}}|_H$, that is, $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$. Evidently we have

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x) \qquad (h \in H,\ x \in \Omega).$$

We know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for instance [14, Corollary 2.5]).

For $\mathbf{c} \in \mathbb{R}^r$, we write $\mathbf{c} > 0$ if $c_j > 0$ for all $j = 1, \ldots, r$. Let $\mathbf{c} \in \mathbb{R}^r$ with $\mathbf{c} > 0$. Denote by $D_v$ $(v \in V)$ the directional differentiation in the direction $v \in V$: $D_v f(x) := \frac{d}{dt} f(x + tv)\big|_{t=0}$. Define a map $\mathcal{I}_{\mathbf{c}} \colon \Omega \to V^*$ through

$$\langle v, \mathcal{I}_{\mathbf{c}}(x) \rangle := -D_v \log \Delta_{-\mathbf{c}}(x) \quad (v \in V,\ x \in \Omega).$$

By [22] we know that $\mathcal{I}_{\mathbf{c}}$ is a bijection of $\Omega$ onto the dual cone $\Omega^*$, where

$$\Omega^* := \left\{ \xi \in V^*;\ \langle x, \xi \rangle > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\} \right\}.$$

Moreover, $\mathcal{I}_{\mathbf{c}}$ extends analytically to a rational map $W \to W^*$ (cf. [7, Satz 1.3.3] or [22]). To get a map inverse to $\mathcal{I}_{\mathbf{c}}$ we set for every $\mathbf{s} \in \mathbb{R}^r$

$$E_{\mathbf{s}}^* := s_1 E_1^* + \cdots + s_r E_r^*,$$

where $\langle E_i, E_j^* \rangle = \delta_{ij}$ and the $E_j^*$'s are considered as elements of $V^*$ by putting 0 on the orthogonal complement of $\mathbb{R}E_1 + \cdots + \mathbb{R}E_r$ in $V$ relative to the inner product $\langle \cdot | \cdot \rangle_\omega$. The group $H$ acts also on $\Omega^*$ simply transitively by the contragredient action $h \cdot \xi := \xi \circ h^{-1}$ $(h \in H,\ \xi \in V^*)$. We have $\mathcal{I}_{\mathbf{c}}(E) = E_{\mathbf{c}}^*$, which we choose as a base point of $\Omega^*$. Define a function $\Delta_{\mathbf{c}}^*$ on $\Omega^*$ by $\Delta_{\mathbf{c}}^*(h \cdot E_{\mathbf{c}}^*) := \chi_{\mathbf{c}}(h)$ $(h \in H)$. Let $\mathcal{I}_{\mathbf{c}}^*$ be a map $\Omega^* \to V$ obtained by

$$\langle \mathcal{I}_{\mathbf{c}}^*(\xi), f \rangle = -D_f \log \Delta_{\mathbf{c}}^*(\xi) \quad (\xi \in \Omega^*,\ f \in V^*).$$

Then, $\mathcal{I}_{\mathbf{c}}^*$ turns out to be a bijection of $\Omega^*$ onto $\Omega$, and extends analytically to a rational map $W^* \to W$.

**Proposition 2.1** ([22]).

(1) *One has* $\mathcal{I}_{\mathbf{c}}^* = \mathcal{I}_{\mathbf{c}}^{-1}$, *so that* $\mathcal{I}_{\mathbf{c}}$ *is a birational map.*

(2) $\mathcal{I}_{\mathbf{c}}$ *is holomorphic on the tube domain* $\Omega + iV$, *and* $\mathcal{I}_{\mathbf{c}}^*$ *on* $\Omega^* + iV^*$.

(3) $\mathcal{I}_{\mathbf{c}}(\Omega + iV)$ *is contained in the holomorphic domain of* $\mathcal{I}_{\mathbf{c}}^*$, *and* $\mathcal{I}_{\mathbf{c}}^*(\Omega^* + iV^*)$ *in the holomorphic domain of* $\mathcal{I}_{\mathbf{c}}$.

*Remark 2.2.* In general, we cannot have $\mathcal{I}_{\mathbf{c}}(\Omega + iV) \subset \Omega^* + iV^*$. See for example [19, §5].

Now we define our Cayley transform. Regarding $E_{\mathbf{c}}^*$ canonically as a complex linear form on $W$, we first set for $w \in W$

$$C_{\mathbf{c}}(w) := E_{\mathbf{c}}^* - 2\mathcal{I}_{\mathbf{c}}(w + E). \tag{2.2}$$

This is for the tube domain $\Omega + iV$, and the image $C_{\mathbf{c}}(\Omega + iV)$ is in $W^*$. Our Cayley transform $\mathcal{C}_{\mathbf{c}}$ for the type II domain $D$ is defined to be

$$\mathcal{C}_{\mathbf{c}}(u, w) := \big(2\langle Q(u, \cdot), \mathcal{I}_{\mathbf{c}}(w + E)\rangle, C_{\mathbf{c}}(w)\big), \tag{2.3}$$

where $u \in U$ and $w \in W$. The image $\mathcal{C}_{\mathbf{c}}(D)$ lies in $U^\dagger \oplus W^*$, where $U^\dagger$ denotes the space of antilinear forms on $U$.

Let us describe the inverse map of $\mathcal{C}_{\mathbf{c}}$. We first introduce a real inner product on $V$ by $\langle\, x \,|\, y \,\rangle_{\mathbf{c}} := \langle [Jx, y], E_{\mathbf{c}}^* \rangle$. We note here that $\langle\, x \,|\, y \,\rangle_{\mathbf{c}} = D_x D_y \log \Delta_{-\mathbf{c}}(E)$. We extend it to a complex bilinear form on $W \times W$, which we denote by the same symbol. For each $f \in W^*$, an element $\iota_{\mathbf{c}}(f) \in W$ is defined by requiring that $\langle w, f \rangle = \langle\, w \,|\, \iota_{\mathbf{c}}(f) \,\rangle_{\mathbf{c}}$ for any $w \in W$. On the other hand,

$$(u_1 \,|\, u_2)_{\mathbf{c}} = 2\langle\, Q(u_1, u_2) \,|\, E \,\rangle_{\mathbf{c}}$$

defines a Hermitian inner product on $U$. Then, $\iota_{\mathbf{c}}(F)$ ($F \in U^\dagger$) is the element in $U$ such that $\langle u, F \rangle = (\iota_{\mathbf{c}}(F) \,|\, u)_{\mathbf{c}}$ for any $u \in U$. Now for each $w \in W$, we obtain a complex linear operator $\varphi_{\mathbf{c}}(w)$ ($w \in W$) on $U$ through the formula

$$(\varphi_{\mathbf{c}}(w)u_1 \,|\, u_2)_{\mathbf{c}} = 2\langle\, Q(u_1, u_2) \,|\, w \,\rangle_{\mathbf{c}}. \tag{2.4}$$

**Theorem 2.3** ([22]).

(1) *The image* $\mathcal{C}_{\mathbf{c}}(D)$ *is bounded.*

(2) $\mathcal{C}_{\mathbf{c}}$ *maps* $D$ *onto* $\mathcal{C}_{\mathbf{c}}(D)$ *birationally and biholomorphically, and one has for* $f \in W^*$ *and* $F \in U^\dagger$

$$\begin{aligned} C_{\mathbf{c}}^{-1}(f) &= 2\mathcal{I}_{\mathbf{c}}^*(E_{\mathbf{c}}^* - f) - E, \\ \mathcal{C}_{\mathbf{c}}^{-1}(F, f) &= \big(2\varphi_{\mathbf{c}}(E - \iota_{\mathbf{c}}(f))^{-1}(\iota_{\mathbf{c}}(F)), C_{\mathbf{c}}^{-1}(f)\big). \end{aligned}$$

## §3. Norm equalities

We suppose from now on that our Siegel domain $D$ is irreducible. The Bergman metric of $D$ induces a Hermitian inner product $(\cdot\,|\,\cdot)$ on the tangent space $T_{\mathsf{e}}(D) = U + W$. Recall that by (2.3) the image $\mathcal{C}_{\mathbf{c}}(D)$ is contained in the space $U^{\dagger} + W^{*}$, on which we import a Hermitian inner product $(\cdot\,|\,\cdot)$ from $U + W$ canonically. Let $\|\cdot\|$ be the corresponding norm.

We put for $j = 1, \ldots, r$

$$d_j := \operatorname{tr} \operatorname{ad}_{\mathfrak{g}(1)}(H_j), \quad b_j := \operatorname{tr} \operatorname{ad}_{\mathfrak{g}(1/2)}(H_j). \tag{3.1}$$

Setting $\mathbf{d} := (d_1, \ldots, d_r)$ and $\mathbf{b} := (b_1, \ldots, b_r)$, we consider the Cayley transform $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}$. Recall that the domain $D$ is said to be symmetric if for every $z \in D$, there exists an involutive holomorphic automorphism $\sigma_z$ of $D$ such that $z$ is an isolated fixed point of $\sigma_z$.

**Theorem 3.1** ([20]). *The norm equality*

$$\|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g \cdot \mathsf{e})\| = \|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g^{-1} \cdot \mathsf{e})\|$$

*holds for any $g \in G$ if and only if $D$ is symmetric.*

Since $\mathcal{C}_{\mathbf{c}}(\mathsf{e}) = 0$ for any $\mathbf{c} > 0$, Theorem 3.1 can be rephrased as

**Theorem 3.2.** *The norm equality*

$$\|h \cdot 0\| = \|h^{-1} \cdot 0\|$$

*holds for any $h \in \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ G \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1}$ if and only if $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(D)$ is symmetric.*

We first indicate the proof of the "if part" of Theorem 3.2 by granting that $\mathcal{D} := \mathcal{C}_{2\mathbf{d}+\mathbf{b}}(D)$ is the *Harish-Chandra realization* of a Hermitian symmetric space if $D$ is symmetric. In this case $\mathrm{Hol}(\mathcal{D})$ is a semisimple Lie group, and we denote by $\mathsf{G}$ its connected component of the identity. Let $\mathsf{K}$ be the stabilizer of $\mathsf{G}$ at the origin. Then $\mathsf{K}$ is a maximal compact subgroup of $\mathsf{G}$. Put $\mathsf{A} := \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ A \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1}$. We have a Cartan decomposition $\mathsf{G} = \mathsf{KAK}$. Every element $h \in \mathsf{G}$ is written as $h = k_1 \mathsf{a}(t) k_2$, where $k_1,\ k_2 \in \mathsf{K}$ and $\mathsf{a}(t) := \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ a(t) \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1} \in \mathsf{A}$ with $a(t)$ as in (2.1). The only thing to be noted is that $\mathsf{K}$ is a closed subgroup of the unitary group. Therefore $\|h \cdot 0\| = \|h^{-1} \cdot 0\|$ if and only if $\|\mathsf{a}(t) \cdot 0\|$ is invariant under $t \mapsto -t$. But this is clear from the fact that

$$\mathsf{a}(t) \cdot 0 = \sum_{j=1}^{r} (2d_j + b_j) \Big( \tanh \frac{t_j}{2} \Big) E_j^{*},$$

where we note that, $D$ being irreducible and symmetric, both $d_j$ and $b_j$ are independent of $j$.

The proof of the "only if part" of Theorem 3.1 requires not only deep results due to Satake and Dorfmeister about characterizations of symmetric Siegel domains but hard computations. We also need a criterion for an irreducible Siegel domain to be quasisymmetric published in [4]. By saying that $D$ is *quasisymmetric*, we mean that the cone $\Omega$ is selfdual with respect to the inner product $\langle v_1 \mid v_2 \rangle_{\mathbf{2d+b}} = D_{v_1} D_{v_2} \log \Delta_{-\mathbf{2d-b}}(E)$ $(v_1, v_2 \in V)$, see the formula (4.1) below for the Bergman kernel of $D$ and the paper [5]. On the other hand, we define a non-associative product $v_1 v_2$ in $V$ by

$$\langle v_1 v_2 \mid v_3 \rangle_{\mathbf{2d+b}} = -\frac{1}{2} D_{v_1} D_{v_2} D_{v_3} \log \Delta_{-\mathbf{2d-b}}(E).$$

Then, $D$ is quasisymmetric if and only if this is a Jordan algebra product by [5, Theorem 2.1] or by the proof of [4, Proposition 3].

**Proposition 3.3** (D'Atri and Dotti Miatello [4]). *$D$ is quasisymmetric if and only if the following two conditions are satisfied:*

(1) $\dim \mathfrak{n}_{(\alpha_m+\alpha_k)/2}$ $(m > k)$ *is independent of* $k$, $m$,
(2) $\dim \mathfrak{n}_{\alpha_j/2}$ *is independent of* $j$.

The validity of the norm equality in Theorem 3.1 for elements $g$ in $\exp \mathfrak{g}'$, where $\mathfrak{g}'$ varies over rank two or rank three subalgebras of $\mathfrak{g}$, together with Proposition 3.3, reduces $D$ to a quasisymmetric domain after a lot of computations. Then we have a Jordan algebra structure in $V$. Furthermore, due to Dorfmeister, the linear map $\varphi := \varphi_{\mathbf{2d+b}} \colon W \to \mathrm{End}_{\mathbb{C}}\, U$ defined by (2.4) for $\mathbf{c} = \mathbf{2d} + \mathbf{b}$ turns out to be a Jordan $*$-representation, see [5] and [19]. The final reduction to a symmetric domain is to show that the Jordan structure in $V$ and the Jordan representation $\varphi$ come naturally from a Hermitian Jordan triple system (see Satake [28] and Dorfmeister [5]). Actually we use the following criterion described in [2, Corollary 1, p. 332]:

**Proposition 3.4** (Dorfmeister). *Suppose that $D$ is quasisymmetric. Then, $D$ is symmetric if and only if there is a complete set of primitive idempotents $f_1, \dots, f_r$ in the Jordan algebra $V$ such that with $U_k := \varphi(f_k)U$ we have $\varphi(Q(u_1, u_2))u_1 = 0$ for all $u_1 \in U_1$ and $u_2 \in U_2$.*

To verify this criterion, we consider $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$. It is at most 2-step nilpotent in view of our gradation of $\mathfrak{g}$. Let $N_D = \exp \mathfrak{n}_D$ be the corresponding connected and simply connected nilpotent Lie group contained in $G$. Writing the elements of $N_D$ as $n(a, b)$ $(a \in \mathfrak{g}(1)$,

$b \in \mathfrak{g}(1/2)$), we see by the Campbell-Hausdorff formula that the group operation is described as

$$n(a,b)n(a',b') = n(a + a' - \operatorname{Im} Q(b,b'), b + b').$$

The group $N_D$ acts on $U + W$ through affine transformations:

$$n(a,b) \cdot (u,w) = \Big(u + b, w + ia + \frac{1}{2}Q(b,b) + Q(u,b)\Big),$$

where $(u,w) \in U \times W$. Now verification of the criterion in Proposition 3.4 is done by inspecting the validity of the norm equality for the elements $g = n(0,u_k)n(0,u_j)$, where $u_k \in \mathfrak{n}_{\alpha_k/2}$ and $u_j \in \mathfrak{n}_{\alpha_j/2}$.

Let us proceed to the second norm equality. We know that the Silov boundary $\Sigma$ of $D$ is described as

$$\Sigma = \{(u,w) \in U \times W;\ 2\operatorname{Re} w = Q(u,u)\}. \tag{3.2}$$

Clearly $\Sigma$ is the $N_D$-orbit $N_D \cdot 0$. Indeed the orbit map $n \mapsto n \cdot 0$ gives a diffeomorphism of $N_D$ onto $\Sigma$. On the other hand, we denote by $\beta$ the *Koszul form* on $\mathfrak{g}$ given by

$$\langle x, \beta \rangle = \operatorname{tr}(\operatorname{ad}(Jx) - J\operatorname{ad}(x)) \quad (x \in \mathfrak{g}). \tag{3.3}$$

It is known by [18, Théorème 1] that $\langle\, x \,|\, y \,\rangle_\beta := \langle [Jx, y], \beta \rangle$ is the real part of the Hermitian inner product of $\mathfrak{g}$ induced by the Bergman metric of $D$ up to a positive number multiple. In particular, $\beta$ is admissible.

Let $\Psi \in \mathfrak{g}$ be the element such that $\langle\, x \,|\, \Psi \,\rangle_\beta = \operatorname{tr}\operatorname{ad}(x)$ holds for any $x \in \mathfrak{g}$. We know that $\Psi \in \mathfrak{a}$. For any $\mathbf{s} \in \mathbb{R}^r$, let $\alpha_{\mathbf{s}} \in \mathfrak{a}^*$ be the element determined by $\chi_{\mathbf{s}}(\exp T) = \exp\langle T, \alpha_{\mathbf{s}} \rangle$ for any $T \in \mathfrak{a}$. We now consider the Cayley transform $\mathcal{C}_{\mathbf{d}+\mathbf{b}}$.

**Theorem 3.5** ([23]). *The norm equality*

$$\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|^2 = \langle \Psi, \alpha_{\mathbf{d}+\mathbf{b}} \rangle$$

*holds for any $\zeta \in \Sigma$ if and only if the domain $D$ is symmetric.*

Here also we first outline the proof of the "if part" of Theorem 3.5. Suppose that $D$ is symmetric. Then, since $D$ is irreducible, one knows that both $d_j$ and $b_j$ in (3.1) are independent of $j$. Thus we can consider $\mathcal{D} := \mathcal{C}_{\mathbf{d}+\mathbf{b}}(D)$ as the Harish-Chandra realization of a bounded symmetric domain. Let $\mathsf{G}$ denote, as before, the connected semisimple Lie group $\operatorname{Hol}(\mathcal{D})^\circ$. Note that $\mathcal{C}_{\mathbf{d}+\mathbf{b}}(0) = -E^*_{\mathbf{d}+\mathbf{b}}$ in view of (2.3). Let $\mathsf{K}$ be the stabilizer of $\mathsf{G}$ at the origin. $\mathsf{K}$ is a maximal compact subgroup of $\mathsf{G}$ as well as a subgroup of the unitary group. By Theorem 3 in [16, p. 179],

$\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\Sigma)$ is contained in the Silov boundary of $\mathcal{D}$, which equals the $\mathsf{K}$-orbit $\mathsf{K}\cdot(-E^*_{\mathbf{d}+\mathbf{b}})$ (cf. Corollary in [16, p. 155]). Therefore $\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|$ is independent of $\zeta \in \Sigma$, and the equality $\|E^*_{\mathbf{d}+\mathbf{b}}\|^2 = \langle \Psi, \alpha_{\mathbf{d}+\mathbf{b}} \rangle$ is readily verified because $d_j$ and $b_j$ are independent of $j$.

To outline the proof of the "only if part", we need the complexification $G_{\mathbb{C}}$ of $G$. Let $G(0)_{\mathbb{C}}$ be the subgroup of $G_{\mathbb{C}}$ corresponding to the subalgebra $\mathfrak{g}(0)_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. We rely on the following proposition for the analysis of the norm equality in Theorem 3.5.

**Proposition 3.6** ([19]). *There exists a real analytic map $\eta \colon V \to G(0)_{\mathbb{C}}$ such that $\eta(y)E = E + iy$ and $\eta(0) = e$, the identity element of $G(0)_{\mathbb{C}}$.*

Note that if $y \in V$, then we have $(0, iy) \in \Sigma$. For $j < k$, let us write $V_{kj}$ instead of $\mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ for simplicity. Since $\mathcal{I}_{\mathbf{d}+\mathbf{b}}(E+iy) = \eta(y)\cdot E^*_{\mathbf{d}+\mathbf{b}}$, we can compute $\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(0, iy)\|^2 = \|C_{\mathbf{d}+\mathbf{b}}(iy)\|^2$ for $y$ in $V_{kj}$, in $V_{lj} + V_{lk}$, or in $V_{kj} + V_{lj}$ for $j < k < l$. Then, the validity of the norm equality for these $\zeta = (0, iy)$, together with Proposition 3.3, yields that $D$ is quasisymmetric, though the computations are by no means trivial. Once we reduce $D$ to a quasisymmetric domain, we have a Jordan algebra structure in $V$ and a Jordan $*$-representation $\varphi := \varphi_{\mathbf{d}+\mathbf{b}}$ of $W$ just as in the previous discussion. The final reduction of $D$ to a symmetric domain is done by using Proposition 3.4 and by analyzing the norm equality for

$$\zeta = \Big(u_j + u_k, \frac{1}{2}Q(u_j + u_k, u_j + u_k) + i\,\mathrm{Im}\,Q(u_j, u_k)\Big) \qquad (j < k),$$

where $u_j \in \mathfrak{n}_{\alpha_j/2}$ and $u_k \in \mathfrak{n}_{\alpha_k/2}$.

## §4. Berezin transforms

Let us first consider the Laplace-Beltrami operator on $D$ determined by the Bergman metric of $D$. Since $D$ is diffeomorphic to our split solvable Lie group $G$, we have the corresponding Laplace-Beltrami operator on $G$, which is, up to a positive number multiple, the operator $\mathcal{L}_\beta$ defined by the left invariant Riemannian metric on $G$ induced by the real inner product $\langle x \,|\, y \rangle_\beta$, where $\beta$ is the Koszul form (3.3). To express $\mathcal{L}_\beta$ in terms of elements of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, we need to fix our notation. If we regard an element $X \in U(\mathfrak{g})$ as a left invariant differential operator on $G$, we write $\widetilde{X}$, whereas we add nothing to $X$ when we regard $X$ as a right invariant differential operator on $G$. Thus if $X \in \mathfrak{g}$, we have for smooth functions $f$ on $G$

$$\widetilde{X}f(x) = \frac{d}{dt} f\big(x \exp(tX)\big)\Big|_{t=0}, \quad Xf(x) = \frac{d}{dt} f\big(\exp(-tX)x\big)\Big|_{t=0}.$$

Let $\Psi \in \mathfrak{g}$ be as in Theorem 3.5. Though Proposition 4.1 below holds for any connected Lie group, we write it down here in our situation. Let $2N := \dim \mathfrak{g}$.

**Proposition 4.1** (Urakawa [29]). *One has* $\mathcal{L}_\beta = -\widetilde{\Lambda} + \widetilde{\Psi}$, *where*

$$\Lambda := X_1^2 + \cdots + X_{2N}^2 \in U(\mathfrak{g})$$

*with an orthonormal basis* $\{X_j\}_{j=1}^{2N}$ *of* $\mathfrak{g}$ *relative to* $\langle \cdot | \cdot \rangle_\beta$.

Let $\kappa$ be the Bergman kernel of $D$. We have, up to a positive number multiple,

$$\kappa(z_1, z_2) = \Delta_{-2\mathbf{d}-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2)), \tag{4.1}$$

where $z_j = (u_j, w_j) \in D$. The *Berezin kernel* $A_\lambda$ $(\lambda \in \mathbb{R})$ on $D$ is given by

$$A_\lambda(z_1, z_2) := \left( \frac{|\kappa(z_1, z_2)|^2}{\kappa(z_1, z_1)\, \kappa(z_2, z_2)} \right)^\lambda \quad (z_1,\ z_2 \in D).$$

It is $G$-invariant:

$$A_\lambda(g \cdot z_1, g \cdot z_2) = A_\lambda(z_1, z_2) \quad (g \in G).$$

We put $a_\lambda(g) := A_\lambda(g \cdot \mathsf{e}, \mathsf{e})$ $(g \in G)$. We see easily that $a_\lambda \in L^1(G)$ with respect to the left Haar measure provided that $\lambda$ is greater than some positive number $\lambda_0 < 1$ ($\lambda_0$ can be given explicitly). We have $a_\lambda(g) = a_\lambda(g^{-1})$. Consider the space $L^2(G)$ on $G$ for the left Haar measure. The *Berezin transform* $B_\lambda$ $(\lambda > \lambda_0)$, when transferred to $L^2(G)$, is given by the convolution operator

$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a_\lambda(x) \qquad (f \in L^2(G)).$$

The integral is absolutely convergent by a standard argument.

**Theorem 4.2** ([21]). *Let* $\lambda > \lambda_0$ *be fixed. Then,* $B_\lambda$ *commutes with* $\mathcal{L}_\beta$ *if and only if* $D$ *is symmetric.*

We indicate here how Theorem 4.2 is derived from Theorem 3.1.

(1) $B_\lambda$ commutes with $\mathcal{L}_\beta \iff (-\widetilde{\Lambda} + \widetilde{\Psi})a_\lambda = (-\Lambda + \Psi)a_\lambda$.

(2) Since $a_\lambda(g) = a_\lambda(g^{-1})$, we have $\widetilde{X} a_\lambda(g) = X a_\lambda(g^{-1})$ for all $X \in U(\mathfrak{g})$ and $g \in G$.

Therefore we have

$$B_\lambda \text{ commutes with } \mathcal{L}_\beta \iff (\Lambda - \Psi)a_\lambda(g) = (\Lambda - \Psi)a_\lambda(g^{-1}) \ (\forall g \in G).$$

On the other hand, after a somewhat lengthy calculation we get

$$(\Lambda - \Psi)a_\lambda(g) = \lambda\, a_\lambda(g)\big(\lambda\|\mathcal{C}_{\mathbf{2d+b}}(g\cdot \mathsf{e})\|^2 - \langle\Psi, \alpha_{\mathbf{2d+b}}\rangle\big).$$

Thus Theorem 4.2 follows from Theorem 3.1.

## §5. Poisson kernel

Let $S(z,z')$ ($z$, $z' \in D$) be the Szegö kernel of the Siegel domain $D$. It is the reproducing kernel of the Hardy space over $D$ (cf. [11], [17]). We have, up to a positive number multiple,

$$S(z_1,z_2) = \Delta_{-\mathbf{d}-\mathbf{b}}(w_1 + w_2^* - Q(u_1,u_2)), \tag{5.1}$$

where $z_j = (u_j,w_j) \in D$. Let $\Sigma$ be the Silov boundary (3.2) of $D$. The boundary $\Sigma$ is stable under the (affine) action of $G = N_D \rtimes H$. We note that the value $S(z,\zeta)$ for $z \in D$ and $\zeta \in \Sigma$ is obtained by a simple substitution in (5.1). The *Poisson kernel* $P(z,\zeta)$ ($z \in D$, $\zeta \in \Sigma$) is defined to be

$$P(z,\zeta) := \frac{|S(z,\zeta)|^2}{S(z,z)}.$$

In what follows we set $P^G_\zeta(g) := P(g\cdot \mathsf{e},\zeta)$ ($g \in G$). Let $\mathcal{L}_\beta$ be the Laplace-Beltrami operator on $G$ introduced in the previous section. The following theorem is known.

**Theorem 5.1** (Hua-Look-Korányi-Xu). *$\mathcal{L}_\beta P^G_\zeta = 0$ for any $\zeta \in \Sigma$ if and only if the domain $D$ is symmetric.*

Hua and Look [13] gave a proof of the "if part" for the classical domains by direct and case-by-case calculations, and Korányi [15] for the case of general symmetric Siegel domains. Korányi's proof is via the mean-value property and actually shows a stronger property that the Poisson kernel is annihilated by *any* invariant differential operator without constant term. The "only if part" is due to [30]. However, Xu's proof is hardly traceable at least for the present author. The formula in Theorem 5.2 below clarifies the computation of Xu, and indeed gives it a geometric meaning. The formula together with Theorem 3.5 also yields a direct proof of Theorem 5.1. We remark here that since

$$P(g\cdot z,\zeta) = \chi_{-\mathbf{d}-\mathbf{b}}(g)P(z,g^{-1}\cdot\zeta) \qquad (g \in G),$$

it holds that $\mathcal{L}_\beta P^G_\zeta = 0$ for any $\zeta \in \Sigma$ if and only if $\mathcal{L}_\beta P^G_\zeta(e) = 0$ for any $\zeta \in \Sigma$.

**Theorem 5.2** ([23]). *One has*

$$\mathcal{L}_\beta P^G_\zeta(e) = \big(-\|\mathcal{C}_{\mathbf{d+b}}(\zeta)\|^2 + \langle\Psi,\alpha_{\mathbf{d+b}}\rangle\big)P^G_\zeta(e).$$

## References

[1] E. Damek, A. Hulanicki and R. Penney, *Admissible convergence for the Poisson-Szegö integrals*, J. Geom. Anal., **5** (1995), 49–76.

[2] J. E. D'Atri and J. Dorfmeister, *Flat totally geodesic submanifolds of quasisymmetric Siegel domains*, Geom. Dedicata, **28** (1988), 321–336.

[3] J. E. D'Atri, J. Dorfmeister and Zhao Yan Da, *The isotropy representation for homogeneous Siegel domains*, Pacific J. Math., **120** (1985), 295–326.

[4] J. E. D'Atri and I. Dotti Miatello, *A characterization of bounded symmetric domains*, Trans. Amer. Math. Soc., **276** (1983), 531–540.

[5] J. Dorfmeister, *Quasisymmetric Siegel domains and the automorphisms of homogeneous Siegel domains*, Amer. J. Math., **102** (1980), 537–563.

[6] J. Dorfmeister, *Homogeneous Siegel domains*, Nagoya Math. J., **86** (1982), 39–83.

[7] J. Dorfmeister and M. Koecher, *Relative Invarianten und nicht-assoziative Algebren*, Math. Ann., **228** (1977), 147–186.

[8] J. Dorfmeister and M. Koecher, *Reguläre Kegel*, Jber. Deutsch. Math.-Verein., **81** (1979), 109–151.

[9] M. Engliš, *Berezin transform and the Laplace-Beltrami operator*, St. Petersburg Math. J., **7** (1996), 633–647.

[10] J. Faraut and A. Korányi, Analysis on symmetric cones, Clarendon Press, Oxford, 1994.

[11] S. G. Gindikin, *Analysis in homogeneous domains*, Russian Math. Surveys, **19-4** (1964), 1–89.

[12] S. Helgason, Geometric analysis on symmetric spaces, Amer. Math. Soc. Providence, Rhode Island, 1994.

[13] L. K. Hua and K. H. Look, *Theory of harmonic functions in classical domains*, Scientia Sinica, **8** (1959), 1031–1094.

[14] H. Ishi, *Representations of the affine transformation groups acting simply transitively on Siegel domains*, J. Funct. Anal., **167** (1999), 425–462.

[15] A. Korányi, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Ann. of Math., **82** (1965), 332–350.

[16] A. Korányi, *Holomorphic and harmonic functions on bounded symmetric domains*, In "Geometry of Homogeneous Bounded Domains (C.I.M.E., 3 Ciclo, Urbino, 1967)", 1968, 127–197, Edizioni Cremonese, Rome.

[17] A. Korányi and E. M. Stein, *$H^2$ spaces of generalized half-planes*, Studia Math., **44** (1972), 379–388.

[18] J. L. Koszul, *Sur la forme hermitienne canonique des espaces homogènes complexes*, Canad. J. Math., **7** (1955), 562–576.

[19] T. Nomura, *On Penney's Cayley transform of a homogeneous Siegel domain*, J. Lie Theory, **11** (2001), 185–206.

[20] T. Nomura, *A characterization of symmetric Siegel domains through a Cayley transform*, Transform. Groups, **6** (2001), 227–260.

[21] T. Nomura, *Berezin transforms and Laplace-Beltrami operators on homogeneous Siegel domains*, Diff. Geom. Appl., **15** (2001), 90–106.

[22] T. Nomura, *Family of Cayley transforms of a homogeneous Siegel domain parametrized by admissible linear forms*, to appear in Diff. Geom. Appl.

[23] T. Nomura, *Geometric norm equality related to the harmonicity of the Poisson kernel for homogeneous Siegel domains*, to appear in J. Funct. Anal.

[24] R. Penney, *The Harish-Chandra realization for non-symmetric domains in $\mathbb{C}^n$*, in "Topics in geometry in memory of Joseph D'Atri", Ed. by S. Gindikin, Birkhäuser, Boston, 1996, 295–313.

[25] I. I. Pyatetskii-Shapiro, Automorphic functions and the geometry of classical domains, Gordon and Breach, New York, 1969.

[26] H. Rossi, *Lectures on representations of groups of holomorphic transformations of Siegel domains*, Lecture Notes, Brandeis Univ., 1972.

[27] H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal., **13** (1973), 324–389.

[28] I. Satake, Algebraic structures of symmetric domains, Iwanami Shoten and Princeton Univ. Press, Tokyo-Princeton, 1980.

[29] H. Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan, **31** (1979), 209–226.

[30] Xu Yichao, *On the Bergman kernel function of homogeneous bounded domains*, Scientia Sinica, Special Issue (II) (1979), 80–90.

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