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# An Approach to the Cartan Geometry I: Conformal Riemann Manifolds

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## Introduction

As is well known F. Klein extracted the essence of the classical geometry by saying that the geometry is the study of properties invariant under the transformations of Lie groups on homogenous spaces. This includes for instance the euclidean geometry and the conformal euclidean geometry. However, this geometry is too rigid to treat geometric objects we meet in reality. B. Riemann was thus led to introduce his geometry generalizing the euclidean geometry.

It is a natural question to ask how to generalize the Riemann's work to the case of an arbitrary classical geometry which is a homogenous space $X = G/H$, where $G$ is a Lie group and $H$ is its closed subgroup. We call any such generalization a structure modeled after the classical geometry $G/H$.

E. Cartan [1] gave an answer by introducing "a generalized space". Namely, instead of the space $X$ together with the action of $G$ on $X$, he considers the projection $\rho_G \colon G \to X$. There is on $G$ the invariant 1-form, say $\omega_G$, valued in the Lie algebra $\mathbf{g}$ of $G$. He associate to the classical geometry $G/H$ the pair $(G, \rho_G, \omega_G)$, which is in todays language a Cartan connection $\omega_G$ on a principal $H$-bundle $G$ over $X$. We recover the homogenous space structure of $X$ because the graphs of the transformations of $G$ are the integral submanifolds of the differential system $\pi_1^*\omega_G - \pi_2^*\omega_G$ on $G \times G$, where $\pi_1$ (resp. $\pi_2$) is the projection to the first (resp. second) component of $G \times G$. By the structure equation of the Lie algebra we have

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0. \tag{1}$$

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Cartan's generalized space structures are deformations of the above structure. Namely, in todays language, a generalized space structure on a manifold $M$ is a pair $(E, \rho, \omega)$ of a principal $H$-bundle $E$ over $M$ with the projection $\rho$ and a Cartan connection $\omega$ on $E$. We call it a Cartan structure modeled after the homogenous space $G/H$. $E$ will be called the frame bundle. It has the curvature form

$$K = d\omega + \frac{1}{2}[\omega, \omega]. \tag{2}$$

While developing the modern terminology for Cartan's work, C. Ehresmann [3] made an interesting comment on the problem by saying that a structure modeled after $G/H$ is a space where a homogenous space $G/H$ is attached to each point. We interpret this as saying that, on such a space, neighborhoods of each point are identified infinitesimally (up to certain order) with a neighborhood of a reference point in $G/H$.

Since Cartan's work many answers to our question are introduced, including the use of Cartan's theory of equivalence and infinite Lie groups. In this note we mainly view the development evolved around these two view points of Cartan and Ehresmann.

We note that the parameter space of structures modeled after $G/H$ on a manifold is obviously infinite dimensional. Therefore we can think of two approaches to the problem. One is to develop a way to write down all such structures and the other is to find a good way to pick one nice such structure.

A variation of the first order infinitesimal version of Ehresmann's view was started by S. S. Chern [2] under the name $G$-structure. This $G$ refers to a linear Lie group, not to our $G$, but more related to our $H$. Actually a slightly limited case of the $G$-structure was already considered by H. Weyl as a generalization of the general relativity. The theory of $G$-structures seems to mainly concerned with the first approach. For surveys see for example S. Kobayashi [5] and T. Ochiai [12].

There are also a lot of works with respect to the second approach. The Levi-Chivita's Riemann geometry may be viewed, in retrospect, as the first satisfactory fusion of Cartan's and Eheresman's viewpoints in the second approach. This is the Cartan connection, with vanishing torsion, on the orthonormal frame bundle. The case for the conformal geometry was worked out by H. Weyl [18], who extracted the conformally invariant components of the curvature tensor called Weyl tensor. CR geometry created by E. Cartan [2], N. Tanaka [16], and S. S. Chern-J. Moser [5] can be considered as the case where the model structure is the unit complex ball with the holomorphic automorphism group.

Generalizing his pioneering work on CR structures, N. Tanaka [14]–[16] introduced structures closely related to Cartan's. His work was further developed by T. Morimoto [10], [11] and K. Yamaguchi [17]. The works of A. Cap and J. Slovak for the higher codimensional CR structures are in this volume. There is also a work of R. Miyaoka [9] on the Lie's sphere geometry.

In the cases of the conformal geometry and CR geometry on a manifold $M$, we may follow the analogy with the Riemann geometry and construct the bundle, say $E_1$, using the first order Ehresmann approach. However, it is a principal $H/H_1$-bundle for a normal subgroup $H_1$ of $H$. We have to enlarge $E_1$ to a principal $H$-bundle, say $E$.

Our attempt to develop a general method to include the cases of Riemann, conformal, and CR as special cases was first outlined in [6] and completed in [7]. It is further developed by Y. Liu [8]. We constructed the above $E$ by applying the Cartan's method of prolongation to $E_1$. However, the traditional approach is to use 2-jets as was done, for example, in Kobayashi [5] and Ogiue [13]. Namely, $E_1$ may be naturally regarded as embedded in $J^1$, the space of the 1-jets of maps of $G/H$ to $M$ at a reference point $e \in G/H$. We also have the space of 2-jets $J^2$, and the projection $\rho\colon J^2 \to J^1$. We construct a section $E_1 \to J^2$. Then $E$ is defined as the subspace of $J^2$ consisting of the orbits of $H$-action passing points of the image of $E^1$ in $J^2$.

In this paper, we use the Ehresmann approach of the second order and construct $E$ as a quotient space $\rho^{-1}(E_1) \to E$ with a commutative diagram:

$$
\begin{array}{ccc}
J^1 & \leftarrow & J^2 \\
\uparrow & & \uparrow \\
E_1 & \leftarrow & \rho^{-1}E_1 \\
\| & & \downarrow \\
E_1 & \leftarrow & E
\end{array}
\tag{3}
$$

When we construct a principal $H$-bundle, say $E_2$, so that $E_1 \leftarrow E_2 \subset \rho^{-1}E_1 \subset J^2$. then the vertical downarrow in (3) will induce an isomorphism $E_2 \to E$. Therefore our frame bundle is isomorphic to the traditional one.

Once the frame bundle is constructed, we work locally and find a Cartan connection by imposing conditions on the curvature form. In Kobayashi [5] this was done using the canonical forms of $J^2$. We can adopt this method in our frame work. However, we used here a direct method using the definition of the Cartan connections.

The curvature is valued in the Lie algebra $\mathbf{g}$ of $G$, which has the grading: $\mathbf{g} = \mathbf{g}_{(-1)} + \mathbf{g}_{(0)} + \mathbf{g}_{(1)}$. We designate a suitable subspace

$\mathbf{g_n} \subset \mathbf{g}_{(0)} + \mathbf{g}_{(1)}$. A Cartan connections on $E$ is called normal when the cuvature takes value in $\mathbf{g_n}$.

In our case the set of normal connections is a family of isomorphic Cartan connections depending on one arbitrary function. It turns out that the Weyl tensor is independent of the connections in the family. Therefore we obtain a unique Weyl curvature form. However, to construct a Cartan connection globally we need to choose locally one connection from the above family in such a way they match up. We do this is this paper.

In §1 we review the case of the homogenous conformal Riemann geometry. We write down several formulas which will be used later. In §2 we construct the frame bundle and the normal Cartan connections along the line mentioned above in the case of conformal geometry. We also show that the $\mathbf{g}_{(1)}$-part of the normal Cartan connections are obtained using the conformal covariant derivative of Weyl tensor. In the end we construct a global normal conformal Cartan connection.

The literature for the conformal connection is too numerous and very difficult to give a complete reference. As a result we listed only a few which we quoted in this paper. We beg perdon for the omission.

The author is greately benefited by the discussions with Professor Keizo Yamaguchi.

## §1. The Homogeous Conformal Space

We fix a nondegenerate $m \times m$ matrix

$$(\underline{h}_{ij}), \quad i, j = 1, \ldots, m. \tag{1}$$

We consider the conformal euclidean geometry based on the metric on $\mathbf{R}^m$ given by

$$\langle dx, dx \rangle = \underline{h}_{ij}\, dx^i\, dx^j. \tag{2}$$

A) Let $\mathbf{R}^{m+2}$ be the euclidean space with the standard chart:

$$(\xi^0, \ldots, \xi^{m+1}) = (\xi^0, \xi', \xi^{m+1}), \quad \xi' = (\xi^1, \ldots, \xi^m), \tag{3}$$

from which we remove the origin obtaining the punctured euclidean space $\dot{\mathbf{R}}^{m+2}$. Dividing by the non-zero scalar mutiplication operation, we obtain the projective space

$$\rho\colon \dot{\mathbf{R}}^{m+2} \to \dot{\mathbf{R}}^{m+2}/\mathbf{C}^* = \mathbf{RP}^{m+1}. \tag{4}$$

Denote by $[\xi] = [\xi^0, \ldots, \xi^{m+1}]$ the homogenous coordinate of $\mathbf{RP}^{m+1}$.

Consider the hypersurface $\Phi^m$ in $\mathbf{RP}^{m+1}$ given by

$$\Phi^m\colon \phi(\xi) = \langle \xi', \xi' \rangle - 2\xi^0\xi^{m+1} = 0, \tag{5.1}$$

where

$$\langle \xi', \xi' \rangle = \underline{h}_{ij}\xi^i\xi^j. \tag{5.2}$$

We embed $\mathbf{R}^m$ in $\Phi^m$ by

$$\mathbf{R}^m \ni x \to \left[1, x, \frac{1}{2}\langle x, x\rangle\right] \in \Phi^m. \tag{6}$$

B) $\mathbf{R}^m$ itself is not the homogeous space. Its closure $\Phi^m$ is the homogenous conformal space, given as follows:

Denote by $\tilde{G}$ the subgroup of $GL(\mathbf{R}, m+2)$ consisting of all matrix $g$ satisfying:

$$\det g = 1, \quad \phi(g(\xi)) = \phi(\xi). \tag{7}$$

Let $G$ be the subgroup of the projective transformation group induced by $\tilde{G}$. In view of (5.1) we find that $G$ preseeves $\Phi^m$ and acts as a transformation group of $\Phi^m$.

We find that $\tilde{G}$ decomposes to the product of the translation group and the isotropy group. Namely,

$$\tilde{G} = L \cdot H, \tag{8}$$

$$L = \left\{ l(y) = \begin{pmatrix} 1 & 0 & 0 \\ y & I & 0 \\ w & y^* & I \end{pmatrix} : y = (y^1, \dots, y^m)^{\mathrm{tr}},\ w = \frac{1}{2}\langle y, y\rangle \right\} \tag{9}$$

where $(y^*)_j = \underline{h}_{jk}y^k$, and $H$ consists of matrixes of the form

$$h = h(a, t, \beta) = \begin{pmatrix} a & \gamma & b \\ 0 & t & \beta \\ 0 & 0 & a^{-1} \end{pmatrix}, \quad \text{where} \tag{10}$$

$$\begin{gathered} \det t = 1, \quad tt^* = I, \quad \beta = (\beta^1, \dots, \beta^m)^{\mathrm{tr}}, \\ \gamma_l = a(\beta^* t)_l, \quad \frac{b}{a} = \frac{1}{2}\langle \beta, \beta\rangle, \end{gathered} \tag{11}$$

where $(t^*)^i_j = \underline{h}^{ik}\underline{h}_{jl}t^l_k$ and $I$ is the identity $m \times m$-matrix. It is convenient to consider a smaller group where $a > 0$. The Lie algebra $\mathbf{g}$ of $G$ has the grading:

$$\mathbf{g} = \mathbf{g}_{(-1)} + \mathbf{g}_{(0)} + \mathbf{g}_{(1)}, \quad \text{where,} \tag{12}$$

$$\mathbf{g}_{(-1)} = \left\{\{\dot{y}\}_{(-1)} = \left(\frac{d(l(s\dot{y}))}{ds}\right)_{s=0} : \dot{y} \in \mathbf{R}^m\right\}, \tag{13.1}$$

$$\mathbf{g}_{(0)} = \mathbf{R}\pi + \{o(m)\}, \quad \text{where for } \dot{t} \in o(m) \tag{13.2}$$

$$\{\dot{t}\} = \left(\frac{dh(1, I + s\dot{t}, 0)}{ds}\right)_{s=0}, \quad \pi = \left(\frac{dh(e^s, I, 0)}{ds}\right)_{s=0}, \tag{13.3}$$

$$\mathbf{g}_{(1)} = \left\{\{\dot{\beta}\}_{(1)} = \left(\frac{dh(1, I, s\dot{\beta})}{ds}\right)_{s=0} : \dot{\beta} \in \mathbf{R}^m\right\}. \tag{13.4}$$

$$\mathbf{h} = \mathbf{g}_{(0)} + \mathbf{g}_{(1)} \text{ is the Lie algebra of } H. \tag{14}$$

We find by calculation

$$\begin{aligned} \mathrm{Ad}(h^{-1})\{\dot{y}\}_{(-1)} &= \{at^*\dot{y}\}_{(-1)} - a\langle\dot{y}, \beta\rangle\pi \\ &\quad + \{at^*\dot{y} \otimes \beta^* t - at^*\beta \otimes \dot{y}^* t\} + \{t^*(b\dot{y} - a\langle\beta, \dot{y}\rangle\beta)\}_{(1)}, \\ \mathrm{Ad}(h^{-1})\pi &= \pi + \{t^*\beta\}_{(1)} \\ \mathrm{Ad}(h^{-1})\{\dot{t}\} &= \{t^*\dot{t}t\} + \{t^*\dot{t}\beta\}_{(1)} \\ \mathrm{Ad}(h^{-1})\{\dot{\beta}\}_{(1)} &= \{a^{-1}t^*\dot{\beta}\}_{(1)}. \end{aligned} \tag{15}$$

In terms of the decomposition (8) the action of $g \in G$ on $x \in \mathbf{R}^m \subset \Phi^m$ is given by

$$T_g x = y + \frac{1}{\lambda}\Big(tx + \frac{1}{2}\langle x, x\rangle\beta\Big), \quad \text{where} \tag{16.1}$$

$$\lambda = a\Big(1 + \langle tx, \beta\rangle + \frac{1}{4}\langle\beta, \beta\rangle\langle x, x\rangle\Big). \tag{16.2}$$

We see now that $G$ acts transitively on $\Phi^m$.

We regard $\Phi^m$ as a conformal Riemann manifold as follows: We consider a metric on

$$\tilde{\Phi}^{m+1} = \rho^{-1}(\Phi^m) \tag{17}$$

given by

$$ds^2 = \langle d\xi', d\xi'\rangle - 2d\xi^0\, d\xi^{n+1}. \tag{18}$$

For simplicity we set

$$F = \mathbf{R}^m. \tag{19}$$

We have a chart $\xi = (c, x)$ on $\rho^{-1}F$ given by

$$(\xi) = \Big(c, cx, \frac{1}{2}c\langle x, x\rangle\Big). \tag{20}$$

Then we find that the metric on $\rho^{-1}F$ induced by $ds^2$ is

$$c^2(ds)_F, \quad \text{where} \quad (ds)_F = \langle dx, dx\rangle. \tag{21}$$

In view of (7) the metric $ds^2$ on $\tilde{\Phi}^{m+1}$ is invariant under the action of the matrix $g$. We find by (19), (16), and (10) that

$$(h\xi) = \Big(c_h, c_h T_h x, \frac{1}{2} c_h \langle T_h x, T_h x\rangle\Big), \quad c_h = c\lambda. \tag{22}$$

Hence $h\xi$ has the coordinate $(c_h, T_h x)$. Then it follows by (19)–(20) that $\lambda^2\langle dx, dx\rangle = \langle T_h\, dx, T_h\, dx\rangle$. Since $T_{l(y)}$ is a translation, it follows that

$$(T_g)^*(ds)_F = \lambda^{-2}(ds)_F. \tag{23}$$

We conclude that the action of $G$ on $\Phi^m$ is conformal.

C) Denote by $J_0^p(F)$ the space of 2-jets at the reference point 0 of maps of neighborhoods of $0 \in F$ into $F$. $J_0^p(F)$ has the standard chart $(y, \dots, p^k_{j_1\dots j_q}, \dots)$, where $q \le p$. If $J \in J_0^p(\mathbf{R}^m)$ is represented by a map $f = \big(f^1(x), \dots, f^m(x)\big)$

$$y = f(0), \quad p_{j_1\dots j_q}(J) = \frac{\partial^q f}{\partial x^{j_1} \dots \partial x^{j_q}}(0). \tag{24}$$

We find by calculation

$$\begin{aligned} p^k_j(T_h) &= \frac{1}{a} t^k_j, \\ p^k_{jl}(T_h) &= \frac{1}{a}\underline{h}_{jl}\beta^k - \frac{1}{a}\underline{h}_{iq}(t^k_j t^i_l + t^k_l t^i_j)\beta^q. \end{aligned} \tag{25}$$

We note that $J_0^1(T_h)$ gives informations on $a$, $t$ in (10), and we need $J_0^2$ to get $\beta$. We note also that we reach $\beta$ more quickly by using the conformal factor $\lambda^{-2}$ of $T_h$ (cf. (23) and (16)), i.e.

$$\frac{\partial \lambda}{\partial x^j}(0) = a\underline{h}_{jk}(t^{-1})^k_l \beta^l. \tag{26}$$

E) In view of (9)–(10) the Maurer-Cartan form:

$$\Omega_G = \begin{pmatrix} (\Omega_G)^0_0 & \dots (\Omega_G)^0_k & \dots (\Omega_G)^0_{m+1} \\ (\Omega_G)^j_0 & \dots (\Omega_G)^j_k & \dots (\Omega_G)^j_{m+1} \\ (\Omega_G)^{m+1}_0 & \dots (\Omega_G)^{m+1}_k & \dots (\Omega_G)^{m+1}_{m+1} \end{pmatrix} \tag{27}$$

has the relations

$$\begin{gathered} (\Omega_G)^0_0 + (\Omega_G)^{m+1}_{m+1} = 0, \quad (\Omega_G)^{m+1}_k = \underline{h}_{kl}(\Omega_G)^l_0, \\ (\Omega_G)^0_k = \underline{h}_{kl}(\Omega_G)^l_{m+1}, \quad \underline{h}_{jl}(\Omega_G)^l_k + \underline{h}_{kl}(\Omega_G)^l_j = 0, \\ (\Omega_G)^0_{m+1} = (\Omega_G)^{m+1}_0 = 0. \end{gathered} \tag{28}$$

Since $\Omega_G = g^{-1}\,dg$, we also find by (8)–(10) that

$$
(29)\qquad
\begin{aligned}
(\Omega_G)^0_0 &= d\log a - a\langle\beta, dy\rangle,\\
(\Omega_G)^j_0 &= a(t^* dy)^j = a\underline{h}^{jk}\underline{h}_{li}t^i_k\,dy^l,\\
(\Omega_G)^j_k &= (t^* dt)^j_k + a(t^* dy)^j(\beta^* t)_k - a(t^*\beta)^j(dy^* t)_k\\
&= \underline{h}^{jl}\underline{h}_{iq}t^q_l(dt^i_k + a\underline{h}_{pr}t^p_k(\beta^r\,dy^i - \beta^i\,dy^r)),\\
(\Omega_G)^j_{m+1} &= (t^* d\beta)^j + (t^*\beta)^j\,d\log a + b(t^* dy)^j - a(t^*\beta)^j\langle\beta, dy\rangle\\
&= \underline{h}^{jl}\underline{h}_{ki}t^i_l\big(d\beta^k + \beta^k(d\log a - a\langle\beta, dy\rangle) + b\,dy^k\big).
\end{aligned}
$$

We use $\omega_H$ ro denote the Maurer-Cartan form of $H$. We also set $\omega_H = h^{-1}\,dh$. Since $\omega_H$ is obtained by setting $dy = 0$ in the above

$$
(30)\qquad
\begin{aligned}
&(\omega_H)^0_0 = d\log a,\quad (\omega_H)^j_0 = 0,\quad (\omega_H)^j_k = (t^* dt)^j_k = \underline{h}^{jl}\underline{h}_{iq}t^q_l\,dt^i_k,\\
&(\omega_H)^j_{m+1} = (t^* d\beta)^j + (t^*\beta)^j\,d\log a = \underline{h}^{jl}\underline{h}_{ki}t^i_l(d\beta^k + \beta^k\,d\log a).
\end{aligned}
$$

Note that $(\Omega_G)^0_0, \dots, (\Omega_G)^j_0, \dots, (\Omega_G)^j_k\ (j > k), \dots, (\Omega_G)^j_{m+1}$ form a base.

The structure equations:

$$
(31)\qquad d(\Omega_G)^r_s + (\Omega_G)^r_t \wedge (\Omega_G)^t_s = 0
$$

$(r,\ s,\ t = 0,\ 1, \dots, m+1)$ is rewritten, due to the reltion (28), as

$$
(32)\qquad
\begin{aligned}
&d(\Omega_G)^0_0 + \underline{h}_{jk}(\Omega_G)^j_{m+1} \wedge (\Omega_G)^k_0 = 0,\\
&d(\Omega_G)^j_0 + \{(\Omega_G)^j_k - \delta^j_k(\Omega_G)^0_0\} \wedge (\Omega_G)^k_0 = 0,\\
&d(\Omega_G)^j_k + (\Omega_G)^j_i \wedge (\Omega_G)^i_k\\
&\qquad + \underline{h}_{kl}\{(\Omega_G)^j_0 \wedge (\Omega_G)^l_{m+1} - (\Omega_G)^l_0 \wedge (\Omega_G)^j_{m+1}\} = 0,\\
&d(\Omega_G)^j_{m+1} + \{(\Omega_G)^j_i + \delta^j_i(\Omega_G)^0_0\} \wedge (\Omega_G)^i_{m+1} = 0.
\end{aligned}
$$

When we regsard $\Omega_G$ as a 1-form valued in the Lie algebra $\mathbf{g}$ of $G$, the adjoint action of $H$ transforms the components of $\Omega_G$. In fact by (15)

$$
(33)\qquad
\begin{aligned}
(\mathrm{Ad}(h^{-1})\Omega_G)^i_0 &= a(t^*)^i_j(\Omega_G)^j_0,\\
(\mathrm{Ad}(h^{-1})\Omega_G)^0_0 &= (\Omega_G)^0_0 - a\underline{h}_{jk}\beta^j(\Omega_G)^k_0,\\
(\mathrm{Ad}(h^{-1})\Omega_G)^i_j &= (t^*)^i_k t^l_j(\Omega_G)^k_l\\
&\qquad + \big(a(t^*)^i_k(\beta^* t)_j - a(t^*\beta)^i t^l_j\underline{h}_{lk}\big)(\Omega_G)^k_0,\\
(\mathrm{Ad}(h^{-1})\Omega_G)^i_{m+1} &= (t^*)^i_j\Big\{a^{-1}(\Omega_G)^j_{m+1} + \beta^l(\Omega_G)^j_l + \beta^j(\Omega_G)^0_0\\
&\qquad - a\beta^j\underline{h}_{lk}\beta^l(\Omega_G)^k_0 + \frac{a}{2}\langle\beta,\beta\rangle(\Omega_G)^j_0\Big\}.
\end{aligned}
$$

## §2. Conformal Riemann Geometry

We consider the conformal Riemann geometry on a manifold $M$ based on a Riemann metric $(ds^2)_M$. We study the local aspect of the metric near a reference point, say $P_0$. Fix a chart $x = (x^1, \dots, x^m)$ on a neighborhood of $P_0$, $x(P_0) = 0$. We write

$$(1) \qquad (ds^2)_M = g_{ij}(x) dx^i\, dx^j.$$

We assume that the matrix $(g_{ij}(x))$ is conjugate to $\underline{h}_{ij}$ given in §1 (1). As in §1 we denote by $F$ the model conformal structure. We also use $F$ to denote a neighborhood of 0 of the model structure. $(ds)_F = \langle dy, dy \rangle = \underline{h}_{ij}\, dy^i\, dy^j$, where $y$ is the standard chart of $F$. $(\underline{h}^{ij})$ is the inverse matrix of $(\underline{h}_{ij})$.

A) Let $q = q_{ij}(y) dy^i\, dy^j$ be a quadratic form. We set

$$(2) \qquad \mathbf{tr}\, q = \underline{h}^{ij} q_{ij}(y).$$

Let $f$ be a map of $F$ into $M$.

**Definition 1.** We say that $f$ is an attaching map of $M$ at $f(0)$ when there is a function $c > 0$ on $F$ such that

$$(3) \quad f^*(ds^2)_M - c(ds^2)_F = O(1), \quad \mathbf{tr}\big(f^*(ds^2)_M - c(ds^2)_F\big) = O(2),$$

where $O(l)$ denotes terms in the ideal generated by $y^{i_1} \cdots y^{i_l}$. $c$ will be called the conformal factor of the attaching map. When $f$ satisfies the first equation in (3), we say that $f$ is an attaching map of order 1.

We claim that, for any attching map $g$ of order 1 and for any linear form $c = c_l y^l$ in $y$, there is an attaching map $f$ with the conformal factor $c + c_0$ such that the 1-jets of $f$ and $g$ agree. The constant $c_0$ in the above is determined by $g$. Namely, for an unknown $f$ we set

$$(4) \qquad f^i(y) = x_0^i + p_j^i y^j + \frac{1}{2} p_{jk}^i y^j y^k + \cdots .$$

Then

$$(5) \qquad f^*(ds^2)_M = g_{st}(f(y))(p_i^s + p_{ik}^s y^k)(p_j^t + p_{jl}^t y^l) dy^i\, dy^j + O(2).$$

Hence the equation for $f$ to be an attaching map as above is given by

$$(6) \qquad \begin{gathered} g_{kl}(x_0) p_i^k p_j^l = c_0 \underline{h}_{ij}, \quad \text{and with} \quad G_k^i = \underline{h}^{ij} g_{kl}(x_0) p_j^l, \\ 2G_k^i p_{it}^k + \underline{h}^{ij} \frac{\partial g_{kl}}{\partial y^s}(x_0) p_i^k p_j^l p_t^s = m c_t. \end{gathered}$$

Since the matrix $(G^i_j)$ is non-singular, our assertion follows easily.

We say that attaching maps $f_1$, $f_2$ at $x_0$ with the conformal factors $c_1$, $c_2$ are equivalent when

$$j^1_0 f_1 = j^1_0 f_2, \quad j^1_0 c_1 = j^1_0 c_2. \tag{7}$$

The equivalent classes of attaching maps will be called the frames of $M$. Denote by $E$ the set of the frames of $M$. Let $E_1$ be the space of 1-jets at 0 of attaching maps. Clearly we have the projection $E \to E_1$:

$$E \ni \text{the class of } f \to j^1_0 f \in E_1, \quad \text{and} \tag{8}$$

$$E_1 = \mathbf{R}^+(\text{the frame bundle of the metric } (ds^2)_M). \tag{9}$$

Note by (6) that $c_0$ is determined by the 1-jets information. Hence $E$ is a manifold with a standard chart:

$$(E_1, c_1, \ldots, c_m). \tag{10}$$

B) If $f$ is an attaching map at $x_0$, $f \circ T_h$ is also an attaching map at $x_0$ because $T_h$ is a conformal map of $F$. We denote by (the class of $f$)$\circ R_h$ the above frame. Let the class of $f$ has the standard chart $(x_0, p^i_j, c_1, \ldots c_m)$. Since the conformal factor of $f \circ T_h$ is $\lambda^{-2} c \circ T_h$, we see by (23)–(26) §1, the class of $f \circ T_h$ has the standard chart:

$$\begin{gathered} \left(x_0, p^i_k \frac{1}{a} t^k_j, c'_1, \ldots, c'_m\right), \quad \text{where} \\ c'_j = a^{-3} c_i t^i_j - 2a^{-2} c_0 \underline{h}_{kl} t^k_j \beta^l, \quad m c_0 = \underline{h}^{ij} g_{kl}(x_0) p^k_i p^l_j. \end{gathered} \tag{11}$$

We thus have the operation of $H$ on $E$. In particular, $E$ is a principal $H$-bundle, where the $R_h$ action of $H$ in the standard chart is given by the above formula.

C) We next discuss local trivializations of $E$. Let $\mathbf{f}(x)$ be a local section of $E$. Then the induced local trivialization of $E$ is given by

$$F \times H \ni (x, h) \to \mathbf{f}(x) \circ R_h. \tag{12}$$

Denote by

$$\left(x, p^i_j(x), c_1(x), \ldots, c_m(x)\right) \tag{13}$$

the standard chart of $\mathbf{f}(x)$. Then we see by (11) that in the standard chart the above local trivialization has the expression:

$$\begin{gathered} \left(x, p^i_k(x) \frac{1}{a} t^k_j, c_1, \ldots, c_m\right), \quad \text{where} \\ c_j = a^{-3} c_i(x) t^i_j - 2a^{-2} c_0(x) \underline{h}_{kl} t^k_j \beta^l, \\ m c_0(x) = \underline{h}^{ij} g_{kl}(x) p^k_i(x) p^l_j(x). \end{gathered} \tag{14}$$

Let us change the local section $\mathbf{f}(x)$ to $\mathbf{f}^\sharp(x) = \mathbf{f}(x) \circ R_{h(x)}$, inducing a new local chart $(x, h^\sharp)$. We see by (12) that

$$h = h(x)h^\sharp, \tag{15}$$

$$a = a(x)a^\sharp, \quad t = t(x)t^\sharp, \quad \beta = t(x)\beta^\sharp + \frac{1}{a^\sharp}\beta(x). \tag{16}$$

D) It is known how to construct a unique Cartan connection locally on $E$. Nevertheless, we want to go over the construction, because we wish to take up the problem of constructing such Cartan connection globally.

We fix a local trivialization of $E$ induced by a local section $\mathbf{f}(x)$ of $E$. We work on the domain of the above section and call it $M$. We use the induced chart $(x, h)$ of $E$.

We first follow the analogy with the Riemann manifold and construct 1-form $\Omega_F$ on $E$ valued in $\mathbf{R}^m = F$. These are the first order coframes of the conformal structure. Namely, we note that $E_1$ is the space of 1-jets of the first order attaching maps. Hence they are linear maps $T_0F \to TM$. Their dual may be regarded as $F$-valued 1-forms $\Omega_F$ on $E_1$. Composing with the projection $E \to E_1$, we thus have a well defined 1-form: $\Omega_F$ on $E$.

In terms of our chart (cf. (12) §1)

$$\Omega_F = \mathrm{Ad}(h^{-1})w_F, \quad w_F = (\ldots, w_F^j, \ldots), \quad w_F^j = p^{-1}(x)^j_k\, dx^k. \tag{17.1}$$

When we set $\Omega_F = (\Omega_F^1, \ldots, \Omega_F^m)$,

$$\Omega_F^j = a(t^{-1})^j_k p^{-1}(x)^k_l\, dx^l. \tag{17.2}$$

Note by (6) that

$$\begin{aligned} &(\ldots, w_F^j, \ldots) \text{ is a section of the 1-st order coframe bundle} \\ &\text{of the metric } \frac{1}{mc_0(x)}(ds^2)_M. \end{aligned} \tag{17.3}$$

E) A Cartan connection on $M$ has the expression

$$\Omega = \mathrm{Ad}(h^{-1})w + h^{-1}\, dh, \quad w \text{ is a } \mathbf{g}\text{-valued 1-form on } W. \tag{18}$$

Note that we have the projection $\rho_F\colon \mathbf{g} \to \mathbf{g}/\mathbf{h} = F$. By a Cartan connection of the conformal structure we mean a Cartan connection $\Omega$ such that

$$\rho\Omega = \Omega_F. \tag{19}$$

Hence the Cartan connections of $F$ are of the form

$$\Omega = \mathrm{Ad}(h^{-1})(w_F + w_H) + h^{-1}\,dh, \quad \text{where } w_H \text{ is an 1-form valued in } \mathbf{h}. \tag{20}$$

To determine $\Omega$ we have to determine $w_H$. We do this by using the curvature of $\Omega$.

F) The curvature form $K$ of $\Omega$ is given by

$$K = d\Omega + \frac{1}{2}[\Omega, \Omega] = \mathrm{Ad}(h^{-1})k, \quad \text{where } k = dw + \frac{1}{2}[w, w]. \tag{21}$$

We set (cf. (13) §1)

$$w_H = w_\pi \pi + \{w_0\} + \{w_{\mathbf{h}}\}_{(1)}, \quad \text{where} \tag{22.1}$$

$$\begin{gathered} w_\pi \text{ is } \mathbf{R}\text{-valued}, \quad w_0 = ((w_0)^i_j) \text{ is } o(m)\text{-valued}, \\ w_{\mathbf{h}} = (w^1_{\mathbf{h}}, \ldots, w^m_{\mathbf{h}}) \text{ is } \mathbf{R}^m\text{-valued}. \end{gathered} \tag{22.2}$$

In the above, $o(m)$ is with respect to the quadratic form (1) §1. In view of (32) §1 we then find that

$$k = \{k_F\}_{(-1)} + k_\pi \pi + \{k_0\} + \{k_{\mathbf{h}}\}_{(1)}, \quad \text{where} \tag{23.1}$$

$$\begin{aligned} k^j_F &= dw^j_F + ((w_0)^j_k - w_\pi \delta^j_k) \wedge w^k_F, \\ k_\pi &= dw_\pi + \underline{h}_{jk} w^j_{\mathbf{h}} \wedge w^k_F, \\ (k_0)^j_k &= d(w_0)^j_k + (w_0)^j_l \wedge (w_0)^l_k + \underline{h}_{kl}(w^j_F \wedge w^l_{\mathbf{h}} + w^j_{\mathbf{h}} \wedge w^l_F), \\ k^j_{\mathbf{h}} &= dw^j_{\mathbf{h}} + ((w_0)^j_k + w_\pi \delta^j_k) \wedge w^k_{\mathbf{h}}, \end{aligned} \tag{23.2}$$

We note that, since $w_0$ is $o(m)$-valued, $k_0$ defined by the above formula is also $o(m)$-valued.

G) We first examine the case when

$$K^j_F = 0, \quad K_\pi = 0, \tag{24.1}$$

which is (by (15) §1 and (21)) equivalent to the conditions:

$$k^j_F = 0, \quad k_\pi = 0. \tag{24.2}$$

We set

$$w_\pi = w_{\pi l} w^l_F, \quad (w_0)^j_k = (w_0)^j_{kl} w^l_F, \quad w^j_{\mathbf{h}} = (w_{\mathbf{h}})^j_l w^l_F. \tag{25}$$

We first write down the condition on $w_0$, $w_\pi$ which is equivalent to the condition: $k_F = 0$. In view of the formula for $w^j_F$ in (17.1) we find by calculation that

$$
(26)\qquad
\begin{aligned}
&dw^j_F + q^j_k \wedge w^k_F = 0, \quad \text{where}\\
&q^j_k = q^j_{kl} w^l_F, \quad q^j_{kl} = (p^{-1})^j_{j_1} \frac{\partial p^{j_1}_k}{\partial x^i} p^i_l.
\end{aligned}
$$

Therefore $k^j_F = 0$ if and only if we can find $A^j_{kl}(y)$ such that

$$
(27)\qquad (w_0)^j_k - w_\pi \delta^j_k = q^j_k + A^j_{kl} w^l_F, \quad A^j_{kl} = A^j_{lk}.
$$

Since $\underline{h}_{jk}(w_0)^j_i + \underline{h}_{ji}(w_0)^j_k = 0$, we can eliminate $w_0$ in the above. We thus find that the condition (26) implies that

$$
(28)\qquad
\begin{aligned}
&\underline{h}_{jk} A^j_{kl} + \underline{h}_{jl} A^j_{ki} = r_{kli}, \quad \text{where}\\
&r_{kli} = -(\underline{h}_{jk} q^j_{li} + \underline{h}_{jl} q^j_{ki} + 2\underline{h}_{kl} w_{\pi i}).
\end{aligned}
$$

As in the case of Riemann geometry, this equation has the unique solution. Namely,

$$
(29)\qquad
\begin{aligned}
A^j_{kl} = &-\frac{1}{2}(q^j_{kl} + q^j_{lk}) + \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{kk_1}(q^{k_1}_{lj_1} - q^{k_1}_{j_1 l})\\
&+ \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{ll_1}(q^{l_1}_{kj_1} - q^{l_1}_{j_1 k}) - \delta^j_k w_{\pi l} - \delta^j_l w_{\pi k} + \underline{h}^{ji}\underline{h}_{kl} w_{\pi i}.
\end{aligned}
$$

Therefore, it follows by (26) that

$$
(30.1)\qquad
\begin{aligned}
(w_0)^j_{kl} = &\frac{1}{2}(q^j_{kl} - q^j_{lk}) + \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{kk_1}(q^{k_1}_{lj_1} - q^{k_1}_{j_1 l})\\
&+ \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{ll_1}(q^{l_1}_{kj_1} - q^{l_1}_{j_1 k}) - \delta^j_l w_{\pi k} + \underline{h}^{ji}\underline{h}_{kl} w_{\pi i}.
\end{aligned}
$$

We check by calculation that the above $w_0$ is $o(m)$-valued. We thus find that for an arbitrary choice of $w_\pi$ there is an unique $w_0$ for which $k_F = 0$. Recalling the construction of the Levi-Civita connections, in view of (17.3) we may rewrite (30.1) as

$$
(30.2)\qquad (w_0)^j_k = (w^\sharp_0)^j_k + H^{ji}_{lk} w_{\pi i} w^l_F, \quad \text{where} H^{ji}_{lk} = \underline{h}^{ji}\underline{h}_{kl} - \delta^j_l \delta^i_k.
$$

where $(w^\sharp_0)^j_k$ is the $o(m)$-part of the Levi-Civita Cartan connection of the metric $(1/c(x))(ds^2)_M$.

We see by (23.2) that $k_\pi = 0$ if and only if

$$
(31)\qquad (dw_\pi)_{jl} = \frac{1}{2}\big(\underline{h}_{jk}(w^k_{\mathbf{h}})_l - \underline{h}_{lk}(w^k_{\mathbf{h}})_j\big), \quad dw_\pi = (dw_\pi)_{jl} w^j_F \wedge w^l_F.
$$

H) It remains to determine $\underline{h}_{jk}(w^k_{\mathbf{h}})_l + \underline{h}_{lk}(w^k_{\mathbf{h}})_j$. The formula for $k_0$ in (23.2) suggests that we may be able to obtain the above term using $k_0$ and $w_0$. In fact, when we set

$$(32) \qquad d(w_0)^j_k + (w_0)^j_l \wedge (w_0)^l_k = W^j_{kli} w^l_F \wedge w^i_F, \quad W^j_{kli} + W^j_{kil} = 0,$$

we find by calculation that

$$(33) \quad ((k_0)^j_k)_{il} = W^j_{kil} + \frac{1}{2}(\delta^j_i \underline{h}_{kk_1} w^{k_1}_{\mathbf{h}l} - \delta^j_l \underline{h}_{kk_1} w^{k_1}_{\mathbf{h}i} + \underline{h}_{kl} w^j_{\mathbf{h}i} - \underline{h}_{ki} w^j_{\mathbf{h}l}).$$

Therefore

$$(34) \qquad ((k_0)^j_k)_{ij} = W^j_{kij} + \frac{1}{2}(2-m)\underline{h}_{kl} w^l_{\mathbf{h}i} - \underline{h}_{ki} w^j_{\mathbf{h}j}.$$

In order to eliminate $w^j_{\mathbf{h}j}$ in the above, we multiply $\underline{h}^{ki}$ and add in $k$, $i$. We find

$$(35) \qquad \underline{h}^{ki}((k_0)^j_k)_{ij} = \underline{h}^{ki} W^j_{kij} + (1-m) w^j_{\mathbf{h}j}.$$

It then follows by calculation that

$$(36) \qquad \begin{aligned} A^{k_1 i_1}_{ki}\{((k_0)^j_{k_1})_{i_1 j} - W^j_{k_1 i_1 j}\} &= \frac{1}{2}(1-m)\underline{h}_{kl} w^l_{\mathbf{h}i}, \\ A^{jl}_{ki} &= \delta^j_k \delta^l_i + \frac{1}{1-m}\underline{h}_{ki}\mathbf{h}^{jl}. \end{aligned}$$

Therefore we see that $\underline{h}_{kl} w^l_{\underline{h}i} + \underline{h}_{il} w^l_{\underline{h}k}$ is determined by $k_0$ and $W$. The condition for $k_0$ becomes simpler when we note as in the case of Riemann geometry that

$$(37) \qquad ((k_0)^j_k)_{lj} = ((k_0)^j_l)_{kj}.$$

The above follows by taking the exterior derivative of $0 = k^j_F$ in (23.2) and using the formulas in (23.2). In the end the terms containing $w_{\mathbf{h}}$ cancel out.

We impose the condition:

$$(38) \qquad ((k_0)^i_j)_{ki} = 0.$$

Since the above condition is equivalent to the condition:

$$(39) \qquad ((K_0)^i_j)_{ki} = 0, \quad \text{where}(K_0)^i_j = ((K_0)^i_j)_{ki} \Omega^k_F \wedge \Omega^i_F,$$

this is a well defined curvature condition. We find by (36)

$$(40) \qquad \underline{h}_{kl} w^l_{\mathbf{h}i} + \underline{h}_{il} w^l_{\mathbf{h}k} = \frac{2}{m-1} A^{k_1 i_1}_{ki}\{W^j_{k_1 i_1 j} + W^j_{i_1 k_1 j}\}.$$

Therefore it follows by (31)

$$\underline{h}_{kl} w^l_{\mathbf{h}i} = \frac{1}{2}(dw_\pi)_{ki} + \frac{1}{m-1} A^{k_1 i_1}_{ki} \{W^j_{k_1 i_1 j} + W^j_{i_1 k_1 j}\}. \tag{41}$$

We conclude

**(42) Proposition.** *For an arbitrary 1-form $w_\pi$ in* (22.1) *there is an unique conformal Cartan connection* (20), (22) *satisfying the conditions*:

$$K_F = 0, \quad K_\pi = 0, \quad ((K_0)^i_j)_{ki} = 0.$$

$w_0$ *and* $w_{\mathbf{h}}$ *of the connection is given by* (30.2) *and* (41).

The above connections will be called normal conformal Cartan connections.

I) We next find an expression of the curvatures. For simplicity we set

$$\underline{h}_{kj} w^j_{\mathbf{h}l} = w_{[kl]} + w_{\langle kl\rangle}, \quad w_{[kl]} = w_{[lk]},\ w_{\langle kl\rangle} = -w_{\langle lk\rangle}. \tag{43}$$

By (31) and (40)

$$w_{\langle kl\rangle} = (dw_\pi)_{kl}, \quad w_{[kl]} = \frac{1}{m-1} A^{k_1 i_1}_{kl} \{W^j_{k_1 i_1 j} + W^j_{i_1 k_1 j}\}. \tag{44}$$

We then find by (33)

$$((k_0)^j_k)_{il} = W^j_{kil} + \frac{1}{2}(H^{js}_{lk}\delta^t_i - H^{js}_{ik}\delta^t_l)(w_{[st]} + w_{\langle st\rangle}). \tag{45}$$

To calculate $W^j_{kil}$, we see by (30.2)

$$dw^j_{0k} = dw^{\sharp j}_{0k} + H^{ji}_{lk}\, d(w_{\pi i}) \wedge w^l_F + w_{\pi i} H^{ji}_{lk}\, dw^l_F. \tag{46}$$

Set

$$\begin{gathered} d(w_{\pi k}) = (w_{\pi[kl]} + w_{\pi\langle kl\rangle}) w^l_F, \\ w_{\pi[kl]} = w_{\pi[lk]}, \quad w_{\pi\langle kl\rangle} = -w_{\pi\langle lk\rangle}. \end{gathered} \tag{47}$$

Since $dw_\pi = d(w_{\pi l}) \wedge w^l_F + w_{\pi i}\, dw^i_F$, we note by (44) that

$$w_{\langle kl\rangle} = w_{\pi\langle lk\rangle} - w_{\pi i}\frac{1}{2}\big((w^i_{0l})_k - (w^i_{0k})_l - \delta^i_l w_{\pi k} + \delta^i_k w_{\pi l}\big). \tag{48}$$

Therefore

$$\begin{aligned} d(w^j_{0k}) = d(w^{\sharp j}_{0k}) &+ H^{ji}_{qk}(w_{\pi[ip]} + w_{\langle pi\rangle}) w^p_F \wedge w^q_F \\ &+ w_{\pi i}\Big\{ H^{jl}_{qk}\frac{1}{2}\big((w^i_{0l})_p - w_{\pi p}\delta^i_l - (w^i_{0p})_l + w_{\pi l}\delta^i_p\big) \\ &\qquad + H^{ji}_{lk}\big(w_{\pi p}\delta^l_q - (w^l_{0q})_p\big)\Big\} w^p_F \wedge w^q_F. \end{aligned} \tag{49}$$

We also have by (30.2)

$$w^j_{0l} \wedge w^l_{0k} = w^{\sharp j}_{0l} \wedge w^{\sharp l}_{0k} + w_{\pi i}\{H^{li}_{qk}(w^{\sharp j}_{0l})_p + H^{ji}_{pl}(w^{\sharp l}_{0k})_q + H^{ji}_{pl}H^{lm}_{qk}w_{\pi m}\}w^p_F \wedge w^q_F. \tag{50}$$

Hence by (32) we find that

$$W^j_{kpq} = R^j_{kpq} + \frac{1}{2}\{H^{ji}_{qk}(w_{\pi[ip]} + w_{\langle pi\rangle}) - H^{ji}_{pk}(w_{\pi[iq]} + w_{\langle qi\rangle})\} + w_{\pi i}P^{ij}_{kpq}, \tag{51}$$

where $R^j_{lpq}w^p_f \wedge w^q_F$ is the curvature form of the metric $(1/c(x))(ds^2)_M$, and

$$\begin{aligned} P^{ij}_{kpq} = {} & \frac{1}{4}H^{jl}_{qk}\big((w^i_{0l})_p - w_{\pi p}\delta^i_l - (w^i_{0p})_l + w_{\pi l}\delta^i_p\big) \\ & - \frac{1}{4}H^{jl}_{pk}\big((w^i_{0l})_q - w_{\pi q}\delta^i_l - (w^i_{0q})_l + w_{\pi l}\delta^i_q\big) \\ & + \frac{1}{2}H^{ji}_{lk}\big((w^l_{0p})_q - w_{\pi q}\delta^l_p - (w^l_{0q})_p + w_{\pi p}\delta^l_q\big) \\ & + \frac{1}{2}\big(H^{li}_{qk}(w^{\sharp j}_{0l})_p - H^{li}_{pk}(w^{\sharp j}_{0l})_q + H^{ji}_{pl}(w^{\sharp l}_{0k})_q - H^{ji}_{ql}(w^{\sharp l}_{0k})_p\big) \\ & + \frac{1}{2}(H^{ji}_{pl}H^{lm}_{qk} - H^{ji}_{ql}H^{lm}_{pk})w_{\pi m}. \end{aligned} \tag{52}$$

Therefore we find by (45) that

$$((k_0)^j_k)_{pq} = R^j_{kpq} + \frac{1}{2}(H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)(w_{[st]} + w_{\pi[st]}) + w_{\pi i}P^{ij}_{kpq}. \tag{53}$$

Summing in $j = q$ in the above, we find by (38)

$$(H^{js}_{jk}\delta^t_p - H^{js}_{pk}\delta^t_j)(w_{[st]} + w_{\pi[st]}) = -2R^j_{kpj} - 2w_{\pi i}P^{ij}_{kpj}. \tag{54}$$

It turns out by (32) that for an indeterminant $X_{st}$ symmetric in $s$, $t$

$$Y_{kp} = (H^{js}_{jk}\delta^t_p - H^{js}_{pk}\delta^t_j)X_{st} = (2-m)X_{kp} - \underline{h}_{kp}\underline{h}^{st}X_{st}. \tag{55}$$

Hence

$$X_{st} = K^{kp}_{st}Y_{kp}, \quad \text{where } K^{kp}_{st} = \frac{1}{2-m}\delta^k_s\delta^p_t + \frac{1}{2}\frac{1}{(m-1)(m-2)}\underline{h}_{st}\underline{h}^{kp}. \tag{56}$$

Therefore we find that

$$((k_0)^j_k)_{pq} = R^j_{kpq} - (H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{kl}_{st}R^i_{kli} + w_{\pi i}\tilde{P}^{ij}_{kpq}, \tag{57}$$

where

$$\tilde{P}^{ij}_{kpq} = P^{ij}_{kpq} - (H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{kl}_{st}R^{ir}_{klr}. \tag{58}$$

We find by calculation that

$$w_{\pi i}\tilde{P}^{ij}_{kpq} = 0. \quad \text{Hence} \tag{59}$$

$$((k_0)^j_k)_{pq} = R^j_{kpq} - (H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{kp}_{st}R^r_{kpr}. \tag{60}$$

Note that

$$\begin{aligned}(H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{il}_{st}X_{il} &= \frac{1}{m-2}(H^{jl}_{pk}X_{lq} - H^{jl}_{qk}X_{lp}) \\ &+ \frac{1}{(m-1)(m-2)}(\underline{h}_{qk}\delta^j_p - \underline{h}_{pk}\delta^j_q)\underline{h}^{il}X_{il}.\end{aligned} \tag{61}$$

Then the formula (60) is rewritten as the classical formula for the Weyl tensor:

$$\begin{aligned}((k_0)^j_k)_{pq} &= R^j_{kpq} + \frac{1}{m-2}(\delta^j_p R_{kq} - \delta^j_q R_{kp} + \underline{h}_{kq}\underline{h}^{jl}R_{lp} \\ &- \underline{h}_{kp}\underline{h}^{jl}R_{lq}) + \frac{R}{(m-1)(m-2)}(\delta^j_p\underline{h}_{kq} - \delta^j_p\underline{h}^{kp}), \quad \text{where}\end{aligned} \tag{62}$$

$$R_{kl} = R^j_{klj}, \quad R = \underline{h}^{kl}R_{kl}. \tag{63}$$

(cf. formula (28.12) Chapter 2, Eisenhart [4], where the chart coframe $dx^j$ is used. We used the orthonormal coframe $w^j_F$.)

J) There is an a priori ground why the cancelation (59) takes place. This is a reflection of the fact that for normal conformal connections we have $k_\pi = 0$. In fact, for arbitrary $\mathbf{R}^m$-valued function $\beta(x)$ let us consider a Cartan connection $\underline{\Omega}$ given by

$$\underline{w}(x) = \mathrm{Ad}\big(h(1, I, \beta(x))^{-1}\big)w(x). \tag{64}$$

in (18). We see by (15) §1

$$\underline{w}_\pi(x) = w_\pi(x) - \underline{h}_{ij}\beta^j(x)w^i_F(x). \tag{65}$$

By (21) the new curvature is given by

$$\underline{k}(x) = \mathrm{Ad}\big(h(1, I, \beta(x))^{-1}\big)k(x). \tag{66}$$

We find by (15) §1 that this is a conformal Cartan connection and

$$\underline{k}_\pi = k_\pi = 0, \quad \underline{k}_0(x) = k_0(x), \quad \underline{k}_{\mathbf{h}} = k_{\mathbf{h}} + k_0\beta(x). \tag{67}$$

Therefore $\underline{\Omega}$ is a normal connection. When $w_\pi = 0$, we see by (53) that the formula for $k_0$ is given by (60). We see by the above, when

$w_\pi = \underline{h}_{ij}\beta^j(x)w_F^i(x)$, the formula for $k_0$ is still given by (60). This means that (59) must be true.

The above formula also prove the followings: Let $\underline{k}_{\mathbf{h}}(x)$ be the $\mathbf{g}_{(1)}$-part of the curvature form when $\underline{w}_\pi = 0$. Then

$$k_{\mathbf{h}}^j = \underline{k}_{\mathbf{h}}^j + \underline{h}^{ki} w_{\pi i}(k_0)_k^j. \tag{68}$$

**(69) Proposition.** *Any two normal conformal Cartan connections are isomorphic.*

K) We next write down the expression of $k_{\mathbf{h}}$. In view of (68) it is enough to consider the case $w_\pi = 0$. We then find by (41) and (36) that with $w_{\mathbf{h}}^j = w_{\mathbf{h}k}^j w_F^k$

$$\begin{aligned} w_{\mathbf{h}k}^j &= \frac{1}{m-1}\underline{h}^{ji} A_{ik}^{pq}(W_{pqr}^r + W_{qpr}^r) \\ &= \frac{1}{m-1}\underline{h}^{jp}(W_{pkr}^r + W_{kpr}^r) - \frac{2}{(m-1)^2}\delta_k^j \underline{h}^{pq} W_{pqr}^r. \end{aligned} \tag{70}$$

Since $w_{\pi[ip]} = 0$ by (47) and $w_{\langle ip\rangle} = 0$ by (44) when $w_\pi = 0$ we see by (51)

$$W_{kpr}^r = R_{kp}. \tag{71}$$

Therefore we find

$$w_{\mathbf{h}}^j = \frac{2}{m-1}\Big(\underline{h}^{jl} R_{lk} - \frac{1}{m-1}\delta_k^j R\Big) w_F^k. \tag{72}$$

It then follows by (23.2) and (68)

$$\begin{aligned} k_{\mathbf{h}}^j = \frac{2}{m-1}\Big\{&\underline{h}^{jl}(dR_{lk} - R_{li}(w_0)_k^i) + \underline{h}^{rl} R_{lk}(w_0)_r^j \\ &- \frac{1}{m-1}\delta_k^j dR\Big\} \wedge w_F^k + \underline{h}^{ki} w_{\pi i}(k_0)_k^j. \end{aligned} \tag{73}$$

We can also express $k_{\mathbf{h}}^j$ by $k_0$ and its derivatives, provided $m > 3$. By (21) (or by calculation) we find by (23.2) that

$$d(k_0)_k^j = (k_0)_l^j \wedge (w_0)_k^l - (w_0)_l^j \wedge (k_0)_k^l + \underline{h}_{kl}(k_{\mathbf{h}}^j \wedge w_F^l - w_F^j \wedge k_{\mathbf{h}}^l). \tag{74}$$

Noting that for $\alpha = \alpha_{jk}\gamma^j \wedge \gamma^k$ with $\alpha_{jk} = -\alpha_{kj}$ and $\beta = \beta_l \gamma^l$

$$\alpha \wedge \beta = \frac{1}{3}(\alpha_{jk}\beta_l + \alpha_{kl}\beta_j + \alpha_{lj}\beta_k)\gamma^j \wedge \gamma^k \wedge \gamma^l, \tag{75}$$

we find by (38) that

$$
\begin{aligned}
3((dk_0)^j_k)_{pqj} = {} & ((k_0)^j_l)_{pq}((w_0)^l_k)_j - ((k_0)^l_k)_{pq}((w_0)^j_l)_j \\
& - ((k_0)^l_k)_{qj}((w_0)^j_l)_p - ((k_0)^l_k)_{jp}((w_0)^j_l)_q \\
& + \underline{h}_{kl}\{(k^j_{\mathbf{h}})_{qj}\delta^l_p + (k^j_{\mathbf{h}})_{jp}\delta^l_q - (m-3)(k^l_{\mathbf{h}})_{pq}.
\end{aligned} \tag{76}
$$

Therefore

$$
(3-m)k^i_{\mathbf{h}} - 2(k^j_{\mathbf{h}})_{pj} w^p_F \wedge w^i_F = \tilde{d}k^i_0, \quad \text{where} \tag{77}
$$

$$
\begin{aligned}
(\tilde{d}k^i_0)_{pq} w^p_F \wedge w^q_F = {} & \underline{h}^{il}\{3((dk_0)^j_l)_{pqj} w^p_F \wedge w^q_F - ((w_0)^r_l)_j (k_0)^j_r \\
& + ((w_0)^j_r)_j (k_0)^r_l + 2(w_0)^j_r \wedge ((k_0)^r_l)_{pj} w^p_F\}.
\end{aligned} \tag{78}
$$

We then conclude that

$$
k^i_{\mathbf{h}} = \frac{1}{3-m}\Big\{\tilde{d}k^i_0 + \frac{1}{2-m}(\tilde{d}k^j_0)_{pj} w^p_F \wedge w^i_F\Big\}. \tag{79}
$$

For future use we rewrite the formula for $\tilde{d}k^i_0$ in (78). We set, with the proper symmetry, $k_0 = (k_0)_{ij} w^i_F \wedge w^j_F$, $dk_0 = (dk_0)_{ijl} w^i_F \wedge w^j_F \wedge w^l_F$, $df = (df)_i w^i_F$ for a function $f$. Then

$$
\begin{aligned}
3(dk_0)_{ijl} = {} & (d(k_0)_{jl})_i + (d(k_0)_{li})_j + (d(k_0)_{ij})_l + (k_0)_{ir}(w^r_{0l})_j \\
& - (k_0)_{ir}(w^r_{0j})_l + (k_0)_{jr}(w^r_{0i})_l - (k_0)_{jr}(w^r_{0l})_i \\
& + (k_0)_{lr}(w^r_{0j})_i - (k_0)_{lr}(w^r_{0i})_j \\
& + 2\{(k_0)_{ij}(w_\pi)_l + (k_0)_{jl}(w_\pi)_i + (k_0)_{li}(w_\pi)_j\}.
\end{aligned} \tag{80}
$$

Therefore by (38)

$$
\begin{aligned}
3(dk^l_{0i})_{pql} = {} & (d(k^l_{0i})_{pq})_l + (k^l_{0i})_{pr}(w^r_{0l})_q - (k^l_{0i})_{pr}(w^r_{0q})_l \\
& + (k^l_{0i})_{qr}(w^r_{0p})_l - (k^l_{0i})_{qr}(w^r_{0l})_p + 2(k^l_{0i})_{pq}(w_\pi)_l.
\end{aligned} \tag{81}
$$

We then find by (78)

$$
\begin{aligned}
\underline{h}_{il}\, \tilde{d}k^l_0 = {} & (d(k^l_{0i})_{pq})_l w^p_F \wedge w^q_F - (w^r_{0i})_l (k_0)^l_r + (w^j_{0l})_j (k_0)^l_i \\
& + 2(w^r_{0p})_l w^p_F \wedge (k^l_{0i})_{qr} w^q_F + 2(w_\pi)_l k^l_{0i}.
\end{aligned} \tag{82}
$$

J) We will show that $k_{\mathbf{h}}$ is also obtained by the conformal covariant derivatives of $K_0$. We first recall the definitions. This is valid for any principal $H$-bundle $E$ with a Cartan connection $\Omega$ (18) given in terms of a local trivialization of $E$. We are considering any homogenous space $G/H$.

Consider a curve $\mathbf{f}_t = (x_t, h_t)$ in $E$. We denote its tangent vectors $\dot{\mathbf{f}}_t$ by

$$(83) \qquad (\dot{x}_t, \dot{h}_t) \quad \text{where} \quad \dot{x}_t = dx/dt, \; h_{t+\epsilon} \equiv h_t(I + \epsilon \dot{h}_t)$$

(mod. $\epsilon^2$). $\dot{h}_t$ is $\mathbf{h}$-valued. Let $\Omega(\dot{\mathbf{f}}_t)$ be the evaluation of $\Omega$ at $\dot{\mathbf{f}}_t$. Then

$$(84) \qquad \Omega(\dot{\mathbf{f}}_t) = \mathrm{Ad}((h_t)^{-1}) w(x_t, dx(\dot{x}_t)) + \dot{h}_t.$$

$\mathbf{f}_t$ is called the parallel displacement of $\mathbf{f}_0$ over the curve $x(t)$ in $M$ when

$$(85) \qquad \rho_{\mathbf{h}} \Omega(\dot{\mathbf{f}}_t) = 0.$$

Clearly, given $x(t)$ and $\mathbf{f}_0$ there is an unique parallel displacement. Namely, $h_t$ is obtained by solving the ordinary differential equation:

$$(86) \qquad \dot{h}_t = -\mathrm{Ad}((h_t)^{-1}) w_H(x_t, dx(\dot{x}_t)).$$

Let $\mathbf{f}_t$ be a parallel displacement and $h_1$ be a fixed element in $H$. Then $R_{h_1}\mathbf{f}$ is also a parallel displacement. Hence it is enough to consider the case: $\mathbf{f}_0 = (x_0, I)$.

Let $\mathcal{X}_t$ be a vector field along the curve $\mathbf{f}_t$. When we express $\mathcal{X}_t = (X_t, \dot{\phi}_t)$ with $\dot{\phi}_t \in \mathbf{h}$ as in (83), $\Omega(\mathcal{X}_t) = \mathrm{Ad}((h_t)^{-1}) w(x_t, dx(X_t)) + \dot{\phi}_t$. We say that $\mathcal{X}_t$ is the parallel displacement of $\mathcal{X}_0$ along $\mathbf{f}_t$ when for all $t$

$$(87) \qquad \Omega(\mathcal{X}_t) = \Omega(\mathcal{X}_0).$$

This means that $X_t$ is determined by the equation:

$$(88.1) \qquad \begin{aligned} &\rho_{\mathbf{g}/\mathbf{h}} \mathrm{Ad}((h_t)^{-1}) w_F(x_t, dx(X_t)) \\ &= \rho_{\mathbf{g}/\mathbf{h}} \mathrm{Ad}((h_0)^{-1}) w_F(x_0, dx(X_0)), \end{aligned}$$

and $\dot{\phi}_t$ is determined by the equation:
(88.2)

$$\begin{aligned} &\mathrm{Ad}((h_t)^{-1}) w_H(x_t, dx(X_t)) + \rho_{\mathbf{h}} \mathrm{Ad}((h_t)^{-1}) w_F(x_t, dx(X_t)) + \dot{\phi}_t \\ = \; &\mathrm{Ad}((h_0)^{-1}) w_H(x_0, dx(X_0)) + \rho_{\mathbf{h}} \mathrm{Ad}((h_0)^{-1}) w_F(x_0, dx(X_0)) + \dot{\phi}_0. \end{aligned}$$

Let $\mathcal{V} = (V, \dot{\psi})$ be a vector field on $E$, where $V = V^j(x,h)\partial/\partial x^j$, $\dot{\psi} = \dot{\psi}(x,h) \in \mathbf{h}$. Pick a tangent vector $\dot{x}_0$ of $M$ at $x_0$ and $\mathbf{f}_0 \in E$ over $x_0$. By a conformal covariant derivative at $\mathbf{f}_0$ of $\mathcal{V}$ to the direction $\dot{x}_0$ is defined as follows: Take a curve $x_t$ in $M$ such that the tangent vector at $t = 0$ is $\dot{x}_0$. Let $\mathbf{f}_t$ be the parallel displacement of $\mathbf{f}_0$ and $\mathcal{V}_t$ be the parallel displacement of $\mathcal{V}_{\mathbf{f}_0}$ along $\mathbf{f}_t$. Then

$$(89) \qquad \nabla_{\dot{x}_0} \mathcal{V}_{\mathbf{f}_0} = \lim_{t \to 0} \frac{1}{t}(\mathcal{V}_{\mathbf{f}_t} - \mathcal{V}_t).$$

Let $\Theta$ be a differential $g$-form on $E$. For a vector field $\mathcal{X}$ on $E$ we define the covariant derivative $\nabla_{\mathcal{X}}\Theta$ of $T$ by $\mathcal{X}$ as follows: For any vector fields $\mathcal{V}_1, \dots, \mathcal{V}_g$

$$\begin{aligned}\nabla_{\mathcal{X}}\Theta(\mathcal{V}_1, \dots, \mathcal{V}_g) = \mathcal{X}(\Theta(\mathcal{V}_1, \dots, \mathcal{V}_g)) - \Theta(\nabla_{\mathcal{X}}\mathcal{V}_1, \dots, \mathcal{V}_g) \\ - \cdots - \Theta(\mathcal{V}_1, \dots, \nabla_{\mathcal{X}}\mathcal{V}_g).\end{aligned} \tag{90}$$

Let $e^\gamma$ be a base of $\mathbf{g}$. Set $\Omega = \Omega_\gamma e^\gamma$. Denote by $\dots, \mathcal{W}^\gamma, \dots$ the base of vector fields dual to $\dots, \Omega_\gamma, \dots$. For any parallel displacement $\mathbf{f}_t$ of a frame we see clearly $(\mathcal{W}^\gamma)_{\mathbf{f}_t}$ is the parallel displacement of $(\mathcal{W}^\gamma)_{\mathbf{f}_0}$ along $\mathbf{f}_t$. Therefore for any vector field $\mathcal{X}$

$$\nabla_{\mathcal{X}}\Theta(\mathcal{W}_1, \dots, \mathcal{W}_g) = \mathcal{X}(\Theta(\mathcal{W}_1, \dots, \mathcal{W}_g)). \tag{91}$$

We now consider the case of a normal conformal Cartan connection. We set $K_0 = \rho_{(0)} \operatorname{Ad}(h^{-1})k_0$, where $\rho_{(0)}$ is the projection to the degree 0 part of the grading (12) §1, and calculate $\nabla_{\mathcal{W}_l} K_0$. We have by (15) §1

$$w_F^j = (p^{-1})^j_k(x)dx^k, \quad w_F^l(x, dx(\mathcal{W}_j)) = a^{-1}t^l_j, \tag{92}$$

$$\mathcal{W}_l = a^{-1}p(x)^i_r t^r_l \frac{\partial}{\partial x^i} + W_{l\pi}\pi + \{W_{l0}\} + \{W_{l\mathbf{h}}\}_{(1)}, \tag{93.1}$$

where

$$W_{l\pi}(x, h) = \underline{h}_{kr} t^k_l \beta^r - w_{\pi i}(x) a^{-1} t^i_l, \tag{93.2}$$

$$\begin{aligned}&W_{l0}(x, h) = t^*\big(\beta \otimes (t_l)^* - t_l \otimes \beta^* - a^{-1}t^i_l(w_0)_i(x)\big)t, \\ &\text{with } t_l = (t^1_l, \dots, t^m_l).\end{aligned} \tag{93.3}$$

The above means that as differential operators

$$W_{l\pi}(x, I) = -w_{\pi l}(x)\Big(\frac{\partial}{\partial a}\Big)_{a=1}, \tag{93.4}$$

$$W_{l0}(x, I) = -(w^j_{0k})_l(x)\Big(\frac{\partial}{\partial t^j_k}\Big)_{t=I}, \tag{93.5}$$

We then find that

$$K_0(\mathcal{W}_p, \mathcal{W}_q) = \frac{1}{2}a^{-2}(t^j_p t^r_q - t^j_q t^r_p)t^*(k_0)_{jr}t. \tag{94}$$

By calculation we now find by (82) and (91) that

$$\underline{h}_{il}\, \tilde{d}k^l_0 = ((\nabla_{\mathcal{W}_l} K_0)^l_i)_{(x,I)}. \tag{95}$$

K) The normal conformal Cartan connections, defined locally, depend on arbitrary functions $w_\pi$. They determine a unique class up to local isomorphism. In order to define globally a normal conformal Cartan connection, we have to choose $w_\pi$ for each local trivialization of the conformal frame bundle in such a way that they match up on the intersections on the domains of trivializations.

Let $(x, \underline{h})$ be a local trivialization. Then for a $\mathbf{h}$-valued function $h(x)$ we have

$$h = h(x)\underline{h}. \tag{96}$$

For a normal conformal Cartan connection $\Omega$ we have two expressions:

$$\Omega = \mathrm{Ad}(h^{-1})w + h^{-1}\,dh = \mathrm{Ad}(\underline{h}^{-1})\underline{w} + \underline{h}^{-1}\,d\underline{h}. \tag{97}$$

Therefore we find by (15) §1 and (30) §1

$$\underline{w}^i_F = a(x)t^*(x)^i_j w^j_F(x), \tag{98.1}$$

$$\underline{w}_\pi(x) = w_\pi(x) - a(x)\underline{h}_{jk}\beta(x)^k w^j(x) + d\log a(x), \tag{98.2}$$

where as in (10) §1 we set $h(x) = h\big(a(x), t(x), \beta(x)\big)$.

To find such $w_\pi$ as above we recall that our chart $(x, h)$ is induced by a section $\mathbf{f}(x) = \big(x, p^i_j(x), c_1(x), \ldots, c_m(x)\big)$ (cf. (12)–(13)) of the frame bundle. We also have $c_0(x) = \underline{h}^{ij}g_{kl}(x)p^k_i(x)p^l_j(x)$. The chart $(x, \underline{h})$ is induced by $\underline{\mathbf{f}}(x) = R_{h(x)}\mathbf{f}(x) = \big(x, \underline{p}^i_j(x), \underline{c}_1(x), \ldots, \underline{c}_m(x)\big)$. Hence by (11)

$$\tag{99} \begin{aligned} \underline{c}_j(x) &= \frac{1}{a(x)^3}t^l_j(x)c_l(x) - \frac{2}{a(x)^2}\underline{h}_{il}t^i_j(x)\beta^l(x)c_0(x), \\ \underline{c}_0 &= \frac{1}{a(x)^2}c_0(x). \end{aligned}$$

We then find by (98.1) that

$$w_\pi = -\frac{1}{2}d\log c_0 + \frac{1}{2}\frac{c_l}{c_0}w^l_F \tag{100}$$

obeys the transformation law (98). When the above $w_\pi$ is chosen we call it the global normal conformal Cartan connection.

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