Complete Integrability
of the Coupled KdV-mKdV System

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Abstract.

The coupled KdV-mKdV system arises as the classical part of one of superextensions of the KdV equation. For this system, we prove its complete integrability, i.e., existence of a recursion operator and of infinite series of symmetries.

Introduction

There are several supersymmetric extensions of the classical Korteweg-de Vries equation (KdV) [5], [8] and [9]. One of them is of the form (the so-called $N = 2$, $A = 1$ extension [3])

$$u_t = -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1$$
$$+ 6\psi\varphi_1 w - 6\varphi\psi_1 w - 6\varphi\psi w_1,$$

$$\varphi_t = -\varphi_3 + 3\varphi u_1 + 3\varphi_1 u - 3\psi_2 w - 3\psi_1 w_1 + 3\varphi_1 w^2 + 6\varphi ww_1,$$

$$\psi_t = -\psi_3 + 3\psi u_1 + 3\psi_1 u + 3\varphi_2 w + 3\varphi_1 w_1 + 3\psi_1 w^2 + 6\psi ww_1,$$

$$w_t = -w_3 + 3w^2 w_1 + 3uw_1 + 3u_1 w,$$

where u and w are classical (even) independent variables while φ and ψ are odd ones (here and below the numerical subscript at an unknown variable denotes it derivative over x of the corresponding order). Being completely integrable itself, this system gives rise to an interesting system of even equations

$$(1) \qquad u_t = -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1,$$
$$w_t = -w_3 + 3w^2 w_1 + 3uw_1 + 3u_1 w,$$

which can be considered as a sort coupling between the KdV (with respect to u) and the modified KdV (with respect to w) equations. In

fact, setting $w = 0$, we obtain

$$u_t = -u_3 + 6uu_1,$$

while for $u = 0$ we have

$$w_t = -w_3 + 3w^2 w_1.$$

In what follows, we prove complete integrability, cf. [2], of system (1) by establishing existence of infinite series of symmetries and/or conservation laws. Toward this end we construct a recursion operator using the techniques of deformation theory introduced in [4] and extensively described and exemplified in [5].

In the first section of the paper the theoretical background is introduced. The second section deals with particular computations and description of basic results.

§1. Geometrical and algebraic background

Here we briefly describe the geometrical theory of partial differential equations [1], [6] and algebraic foundations of computational approach to recursion operators [4], [5].

Let $\pi\colon E \to M$ be a locally trivial vector bundle and $\pi_k\colon J^k(\pi) \to M$, $k = 0, 1, \dots, \infty$, be the bundles of its k-jets. A (nonlinear) partial differential equation (PDE) of order k is a submanifold $\mathcal{E} \subset J^k(\pi)$, $k < \infty$. Its lth prolongation is a subset $\mathcal{E}^l \subset J^{k+l}(\pi)$. There exist natural mappings $\pi_{k+l+1,k+l}\colon \mathcal{E}^{l+1} \to \mathcal{E}^l$, and $\mathcal{E}$ is said to be formally integrable, if all $\mathcal{E}^l$ are smooth manifolds while $\pi_{k+l+1,k+l}$ are smooth fiber bundles. Below, only formally integrable equations are considered.

The inverse limit $\mathcal{E}^\infty \subset J^\infty(\pi)$ of the system $\{\mathcal{E}^l, \pi_{k+l,k+l-1}\}$ is called the infinite prolongation of $\mathcal{E}$ and we consider the bundle $\pi_{\mathcal{E}}\colon \mathcal{E}^\infty \to M$. This bundle enjoys the following characteristic property:

Proposition 1 (see [6]).   *Let $\pi_i\colon E_i \to M$, $i = 1, 2$, be two locally trivial vector bundles and $\Delta\colon \Gamma(\pi_1) \to \Gamma(\pi_2)$ be a linear differential operator acting from sections of π_1 to sections of π_2. Then there exists a unique differential operator $\mathcal{C}\Delta\colon \Gamma(\pi_{\mathcal{E}}^*(\pi_1)) \to \Gamma(\pi_{\mathcal{E}}^*(\pi_2))$ such that*

$$(2) \qquad j_\infty(s)^* \circ \mathcal{C}\Delta = \Delta \circ j_\infty(s)^*$$

for any formal solution s of the equation $\mathcal{E}$. The correspondence $\Delta \mapsto \mathcal{C}\Delta$ is $C^\infty(\mathcal{E}^\infty)$-linear and complies with the composition of differential operators:

$$(3) \qquad \mathcal{C}(\Delta_2 \circ \Delta_1) = \mathcal{C}\Delta_2 \circ \mathcal{C}\Delta_1,$$

where $\Delta_1 \colon \Gamma(\pi_1) \to \Gamma(\pi_2)$, $\Delta_2 \colon \Gamma(\pi_2) \to \Gamma(\pi_3)$ *are linear differential operators,* $\pi_3 \colon E_3 \to M$ *being a third vector bundle.*

As a corollary of Proposition 1, we get

Proposition 2. *The bundle* $\pi_{\mathcal{E}}$ *possesses a natural flat connection.*

Proof. It suffices to take the trivial bundle $\mathbf{1}_M \colon M \times \mathbb{R} \to M$ for the bundles π_1 and π_2 and an arbitrary vector field X for the operator Δ. Flatness is a consequence of (3). $\qquad\square$

Let us denote by $\mathrm{D}(N)$ the $C^\infty(N)$-module of vector fields on a manifold N.

Definition 1. The connection $\mathcal{C} \colon \mathrm{D}(M) \to \mathrm{D}(\mathcal{E}^\infty)$ is called the *Cartan connection* on $\mathcal{E}^\infty$.

Denote by $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty) \subset \mathrm{D}(\mathcal{E}^\infty)$ the horizontal distribution on $\mathcal{E}^\infty$ with respect to the Cartan connection (the *Cartan distribution*) and by $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$ the normalizer of $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ in $\mathrm{D}(\mathcal{E}^\infty)$. Then, since $\mathcal{C}$ is flat, $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is integrable in a formal Frobenius sense and thus $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is an ideal in $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$.

Definition 2. The quotient Lie algebra $\operatorname{sym}\mathcal{E} = \mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)/\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is called the *algebra of higher symmetries* of the equation $\mathcal{E}$.

The Cartan connection in $\pi_{\mathcal{E}}$ determines the splitting

$$\text{(4)} \qquad \mathrm{D}(\mathcal{E}^\infty) = \mathrm{D}^v(\mathcal{E}^\infty) \oplus \mathcal{C}\mathrm{D}(\mathcal{E}^\infty),$$

where $\mathrm{D}^v(\mathcal{E}^\infty)$ denotes the module of $\pi_{\mathcal{E}}$-vertical vector fields, and for any coset $S \in \operatorname{sym}\mathcal{E}$ there exists a unique vertical representative. In its turn, such a representative is uniquely determined by a section $\varphi \in \Gamma(\pi_{\mathcal{E}}^*(\pi))$ (generating section) satisfying the defining equation

$$\text{(5)} \qquad \ell_{\mathcal{E}}\varphi = 0$$

and vice versa. Here $\ell_{\mathcal{E}}$ is the universal linearization operator for $\mathcal{E}$ restricted to $\mathcal{E}^\infty$. Due to this fact, we shall identify the solutions of (5) with higher symmetries of $\mathcal{E}$.

The connection form $U_{\mathcal{E}}$ (the *structural element* of $\mathcal{E}$) of the Cartan connection $\mathcal{C}$ is an element of the module $\mathrm{D}^v(\Lambda^1(\mathcal{E}^\infty))$ of $\Lambda^1(\mathcal{E}^\infty)$-valued vertical derivations. Thus, we can introduce an operator $\partial_{\mathcal{E}} \colon \mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty)) \to \mathrm{D}^v(\Lambda^{i+1}(\mathcal{E}^\infty))$ defined by

$$\partial_{\mathcal{E}}\Omega = [\![U_{\mathcal{E}}, \Omega]\!], \quad \Omega \in \mathrm{D}^v\left(\Lambda^i(\mathcal{E}^\infty)\right),$$

where $[\![\cdot, \cdot]\!]$ is the Frölicher-Nijenhuis bracket. Since the Cartan connection is flat, one has $[\![U_\mathcal{E}, U_\mathcal{E}]\!] = 0$, from where it follows that $\partial_\mathcal{E} \circ \partial_\mathcal{E} = 0$. Thus we obtain a complex $(\mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty)), \partial_\mathcal{E})$, whose cohomology is called the $\mathcal{C}$-*cohomology* of $\mathcal{E}$ and is denoted by $H_\mathcal{C}^\bullet(\mathcal{E})$. It is easy to see that $H_\mathcal{C}^0(\mathcal{E}) = \mathrm{sym}\,\mathcal{E}$, while $H_\mathcal{C}^1(\mathcal{E})$, by standard reasons, is identified with classes of nontrivial infinitesimal deformations of $U_\mathcal{E}$ (or, which is the same, of the equation structure).

If Ω and Θ are elements of $\mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty))$ and $\mathrm{D}^v(\Lambda^j(\mathcal{E}^\infty))$ respectively, their *contraction* $\Omega \lrcorner\, \Theta$ is defined as an element of $(\mathrm{D}^v(\Lambda^{i+j-1}(\mathcal{E}^\infty)))$ This operation is inherited by the $\mathcal{C}$-cohomology groups. In particular, if $\varphi \in \mathrm{sym}\,\mathcal{E}$ and $\mathcal{R} \in H_\mathcal{C}^1(\mathcal{E})$, then $\varphi \lrcorner\, \mathcal{R} = \mathcal{R}\varphi$ is a symmetry again. In other words, the module $H_\mathcal{C}^1(\mathcal{E})$ acts on the Lie algebra of higher symmetries.

Let $\mathcal{C}\Lambda^1(\mathcal{E}^\infty) \subset \Lambda^1(\mathcal{E}^\infty)$ be the submodule of one-forms on $\mathcal{E}^\infty$ vanishing on the Cartan distribution. Then one has the direct sum decomposition

$$\Lambda^1(\mathcal{E}^\infty) = \mathcal{C}\Lambda^1(\mathcal{E}^\infty) \oplus \Lambda_h^1(\mathcal{E}^\infty)$$

dual to (4), where $\Lambda_h^1(\mathcal{E}^\infty)$ is the submodule of *horizontal* forms. This splitting is also inherited by $H_\mathcal{C}^\bullet(\mathcal{E}^\infty)$ and an element $\mathcal{R} \in H_\mathcal{C}^1(\mathcal{E}^\infty)$ acts nontrivially on $\mathrm{sym}\,\mathcal{E}$ if only it corresponds to a derivation from $\mathrm{D}^v(\mathcal{C}\Lambda^1(\mathcal{E}^\infty))$. Moreover, it can be shown that $\mathrm{im}\,\partial_\mathcal{E} \cap \mathrm{D}^v(\mathcal{C}\Lambda^1(\mathcal{E}^\infty)) = 0$ and consequently nontrivial actions can be found by solving the equation

$$(6) \qquad\qquad\qquad \partial_\mathcal{E}(\mathcal{R}) = 0.$$

Solutions of (6) are called *recursion operators* for symmetries. Any recursion operator $\mathcal{R}$ is uniquely determined by an element $\omega_\mathcal{R} \in \mathcal{C}\Lambda^1(\mathcal{E}^\infty) \otimes \Gamma(\pi_\mathcal{E}(\pi))$ satisfying the defining equation

$$(7) \qquad\qquad\qquad \ell_\mathcal{E}^{[1]}(\omega_\mathcal{R}) = 0,$$

where $\ell_\mathcal{E}^{[1]}$ is the extension of the operator $\ell_\mathcal{E}$ to the module $\mathcal{C}\Lambda^1(\mathcal{E}^\infty) \otimes \Gamma(\pi_\mathcal{E}(\pi))$. If $\varphi \in \Gamma(\pi_\mathcal{E}^*(\pi))$ is a symmetry and $S_\varphi \in \mathrm{D}^v(\mathcal{E}^\infty)$ is the corresponding vertical vector field, then the action of $\mathcal{R}$ on φ is given by

$$(8) \qquad\qquad\qquad \mathcal{R}\varphi = S_\varphi \lrcorner\, \omega_\mathcal{R}.$$

Local coordinates

Let $\mathcal{U} \subset M$ be a coordinate neighborhood in M such that the bundle π trivializes over $\mathcal{U}$, $x_1, \ldots, x_n$ be local coordinates in $\mathcal{U}$ and $u^1, \ldots, u^m$ be coordinates along the fiber in a given trivialization. Then the

adapted coordinates in $\pi_\infty^{-1}(\mathcal{U}) \subset J^\infty(\pi)$ are given by the functions u_σ^j, $j = 1, \ldots, m$, $\sigma = i_1 \ldots i_k$, $1 \le i_\alpha \le n$, uniquely defined by

$$j_\infty(f)^*(u_\sigma^j) = \frac{\partial^{|\sigma|} f^j}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

for any local section $f = (f^1, \ldots, f^m) \in \Gamma(\pi|_\mathcal{U})$. Then the Cartan connection in $J^\infty(\pi)$ is given by

$$(9) \qquad \mathcal{C}\left(\frac{\partial}{\partial x_i}\right) = D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u_{\sigma i}^j \frac{\partial}{\partial u_\sigma^j},$$

where D_i are the so-called *total derivatives*. The structural element in this case is given by the formula

$$(10) \qquad U_\pi = \sum_{j,\sigma} \omega_\sigma^j \otimes \frac{\partial}{\partial u_\sigma^j},$$

where $\omega_\sigma^j = du_\sigma^j - \sum_i u_{\sigma i}^j \, dx_i$ are the *Cartan forms* constituting a basis in the module $\mathcal{C}\Lambda^1(J^\infty(\pi))$. The forms ω_σ^j may be rewritten as $\omega_\sigma^j = d_\mathcal{C} u_\sigma^j$, where $d_\mathcal{C}$ is the *Cartan differential* acting on $f \in C^\infty(J^\infty(\pi))$ by

$$d_\mathcal{C} f = \sum \frac{\partial f}{\partial u_\sigma^j} \omega_\sigma^j.$$

This differential restricts to any submanifold $\mathcal{E}^\infty \subset J^\infty(\pi)$ and for infinite prolongations (10) transforms to

$$(11) \qquad U_\mathcal{E} = \sum_I d_\mathcal{C} u_I \otimes \frac{\partial}{\partial u_I},$$

where $\{u_I\}$ spans the set of *internal coordinates* in $\mathcal{E}^\infty$.

If an equation $\mathcal{E} \subset J^k(\pi)$ is given by the system of equalities

$$\begin{cases} F^1(x_1, \ldots, x_n, \ldots, u_\sigma^j, \ldots) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F^r(x_1, \ldots, x_n, \ldots, u_\sigma^j, \ldots) = 0, \end{cases}$$

then its infinite prolongation is defined by

$$D_\sigma F^\alpha = 0, \quad \alpha = 1, \ldots, r, \ 0 \le |\sigma| < \infty,$$

where $D_\sigma = D_{i_1} \circ \cdots \circ D_{i_k}$ for $\sigma = i_1 \ldots i_k$. Then the universal linearization operator corresponding to this system is of the form

$$(12) \qquad \ell_F = \left\| \sum_\sigma \frac{\partial F^\alpha}{\partial u_\sigma^j} D_\sigma \right\|.$$

The operator $\ell_{\mathcal{E}}$ is obtained from (12) by rewriting it in internal coordinates.

Example (Evolutionary $1+1$ equations). Consider a system $\mathcal{E}$ of evolutionary equations

$$
(13) \qquad
\begin{cases}
u_t^1 = F^1(x, t, u^1, \ldots, u_k^m), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u_t^m = F^r(x, t, u^1, \ldots, u_k^m),
\end{cases}
$$

where $x = x_1$, $t = x_2$ and $u_l^j = \partial^l u^j / \partial x^l$. Then the functions x, t, $\ldots$, u_l^j, $\ldots$ can be taken for internal coordinates on $\mathcal{E}^\infty$. The total derivatives are written down as

$$
D_x = \frac{\partial}{\partial x} + \sum_{l,j} u_{l+1}^j \frac{\partial}{\partial u_l^j}, \quad
D_t = \frac{\partial}{\partial t} + \sum_{l,j} D_x^l(F^j) \frac{\partial}{\partial u_l^j}
$$

in these coordinates. The Cartan forms on $\mathcal{E}^\infty$ are

$$
\omega_l^j = du_l^j - u_{l+1}^j \, dx - D_x^l(F^j) \, dt,
$$

while the structural element for $\mathcal{E}$ is given by

$$
(14) \qquad U_{\mathcal{E}} = \sum_{l,j} \omega_l^j \otimes \frac{\partial}{\partial u_l^j}.
$$

The restriction $\ell_{\mathcal{E}}$ of the universal linearization operator to $\mathcal{E}^\infty$ is of the form

$$
\ell_{\mathcal{E}} = \left\| \sum_{l,j} \frac{\partial F^\alpha}{\partial u_l^j} D_x^l - \mathbf{E} D_t \right\|,
$$

where $\mathbf{E}$ is the identity matrix. So, a vector function $\varphi = (\varphi^1, \ldots, \varphi^m)$, $\varphi^j = \varphi^j(x, t, \ldots, u_l^j, \ldots)$, is a symmetry of $\mathcal{E}$ if and only if

$$
(15) \qquad \sum_{l,j} \frac{\partial F^\alpha}{\partial u_l^j} D_x^l \varphi^j = D_t \varphi^\alpha
$$

for all $\alpha = 1, \ldots, m$. The corresponding vector field is given by the formula

$$
\vartheta_\varphi = \sum_{l,j} D_x^l(\varphi^j) \frac{\partial}{\partial u_l^j}.
$$

In a similar way, recursion operators are determined by vector-valued forms $\omega_{\mathcal{R}} = (\omega_{\mathcal{R}}^1, \ldots, \omega_{\mathcal{R}}^m)$, where $\omega_{\mathcal{R}}^j = \sum_{l,\alpha} \psi_\alpha^{jl} \omega_l^\alpha$, $\psi_\alpha^{jl} \in C^\infty(\mathcal{E}^\infty)$, satisfying the equations

$$(16) \qquad \sum_{l,j} \frac{\partial F^\alpha}{\partial u_l^j} D_x^l \omega_{\mathcal{R}}^j = D_t \omega_{\mathcal{R}}^\alpha.$$

To compute the left and right sides of (16), it suffices to note that

$$D_x \omega_l^j = \omega_{l+1}^j, \quad D_t \omega_l^j = D_x^l d_{\mathcal{C}} F^j = D_x^l \sum_{\alpha,be} \frac{\partial F^j}{\partial u_\beta^\alpha} \omega_\beta^\alpha.$$

Remark 1. Let $\omega = (\omega^1, \ldots, \omega^m)$, $\omega^j = \sum_{l,\alpha} \psi_\alpha^{jl} \omega_l^\alpha$, be a vector-valued Cartan form on $\mathcal{E}^\infty$. Then for any vector-valued function $\varphi = (\varphi^1, \ldots, \varphi^m)$ on $\mathcal{E}^\infty$ the action $\mathcal{R}_\omega \colon \varphi \mapsto \mathcal{R}_\omega \varphi$ is defined by $\mathcal{R}_\omega \varphi = \partial_\varphi \lrcorner \omega$. In local coordinates, this action is expressed by the formula

$$(17) \qquad (\mathcal{R}_\omega \varphi)^j = \sum_{l,\alpha} \psi_\alpha^{jl} D_x^l(\varphi^\alpha).$$

Operators of this type (i.e., expressed in terms of total derivatives) are called *C-differential* (or *total differential*) operators.

Remark 2. As it was mentioned above, operators of the form (17), provided ω satisfies (16), take symmetries of the equation at hand to symmetries of the same equation. In other words, one has $\mathcal{R}_\omega \colon \ker \ell_{\mathcal{E}} \to \ker \ell_{\mathcal{E}}$. Under not very restrictive conditions on the equation $\mathcal{E}$, this is equivalent to the operator equality

$$\ell_{\mathcal{E}} \circ \mathcal{R}_\omega = \mathcal{A} \circ \ell_{\mathcal{E}},$$

where $\mathcal{A}$ is a C-differential operator. Taking formally adjoint, one obtains

$$\mathcal{R}_\omega^* \circ \ell_{\mathcal{E}}^* = \ell_{\mathcal{E}}^* \circ \mathcal{A}^*,$$

which means, that $\mathcal{A}^*$ is a recursion operator for generating functions of conservation laws [10].

Nonlocal setting

In practice, when solving (7), one usually finds no nontrivial solutions, though the equation $\mathcal{E}$ may possess recursion operators.

Example (The Burgers equation). Consider the equation

$$u_t = u_{xx} + u u_x.$$

It is known to possess a recursion operator of the form

$$(18) \qquad \mathcal{R} = D_x + \frac{1}{2}u_0 + \frac{1}{2}u_1 D_x^{-1}.$$

Nevertheless, the only solution of the equation

$$(19) \qquad \ell_{\mathcal{E}}^{[1]}\omega \equiv (D_x^2 + u_0 D_x + u_1 - D_t)\omega = 0$$

for $\omega = \psi_0\omega_0 + \cdots + \psi_k\omega_k$ is $\alpha\omega_0$, $\alpha \in \mathbb{R}$, which provides the trivial action $\mathcal{R}_\omega \colon \varphi \mapsto \alpha\varphi$.

   To resolve this apparent contradiction, let us extend the algebra $C^\infty(\mathcal{E}^\infty)$ with an additional element u_{-1} and set

$$D_x u_{-1} = u_0,$$

$$D_t u_{-1} = u_1 + \frac{1}{2}u_0^2,$$

$$(20) \qquad d_{\mathcal{C}}u_{-1} \equiv \omega_{-1} = du_{-1} - u_0\, dx - \left(u_1 + \frac{1}{2}u_0^2\right)dt.$$

Then, solving (19) for $\omega = \psi_{-1}\omega_{-1} + \psi_0\omega_0 + \cdots + \psi_k\omega_k$, we obtain a two-parametric solution

$$\omega = \alpha\omega_0 + \beta\Omega, \quad \Omega = \omega_1 + \frac{1}{2}u_0\omega_0 + \frac{1}{2}u_1\omega_{-1}.$$

Then the action $\varphi \mapsto \mathcal{R}_\Omega\varphi = \partial_\varphi \,\lrcorner\, \Omega$ coincides exactly with (18).

   This example reflects a general scheme of computations which arises in a lot of applications [5] and is used in Section 2. Namely, in search of recursion operators for symmetries we extend the algebra $C^\infty(\mathcal{E}^\infty)$ with a new set of variables $w^1, \ldots, w^r, \ldots$ (so-called *nonlocal variables*) and respectively extend the total derivatives to vector fields

$$(21) \qquad \bar{D}_i = D_i + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}, \quad i = 1, \ldots, n,$$

in such a way that

$$(22) \qquad [\bar{D}_i, \bar{D}_j] \equiv [D_i, X_j] + [X_i, D_j] + [X_i, X_j] = 0,$$

where $X_l = \sum_\alpha X_l^\alpha \partial/\partial w^\alpha$.

   Having a solution $X_1, \ldots, X_n$ of (21), we obtain an integrable distribution on the space $\mathcal{E}^\infty \times \mathbb{R}^N$, where N is the number of nonlocal variables (the case $N = \infty$ is included). The projection $\tau \colon \overline{\mathcal{E}^\infty} = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$ is called a *covering* over $\mathcal{E}$ and N is called

its *dimension* (for an invariant geometrical definition see [7]). Similar to the local case, we define the Lie algebra $\mathrm{sym}_\tau \, \mathcal{E} = \mathrm{D}_{\mathcal{C}}(\overline{\mathcal{E}^\infty})/\mathcal{C}\mathrm{D}(\overline{\mathcal{E}^\infty})$ of *nonlocal* τ-symmetries.

We introduce *Cartan forms*

$$\theta^j = dw^j - \sum_{i=1}^n X_i^j \, dx_i, \quad j = 1, \ldots, N,$$

corresponding to nonlocal variables on $\overline{\mathcal{E}^\infty}$. The module of all Cartan forms on $\overline{\mathcal{E}^\infty}$ is denoted by $\mathcal{C}\Lambda^1(\overline{\mathcal{E}^\infty})$. We also extend the universal linearization operator $\ell_{\mathcal{E}}$ to $\overline{\mathcal{E}^\infty}$ just by changing the total derivatives D_i to $\bar{D}_i$. Let us now consider two equations, associated to this extension:

$$\bar{\ell}_{\mathcal{E}}\varphi = 0 \quad \text{and} \quad \bar{\ell}_{\mathcal{E}}^{[1]}\Omega = 0,$$

where $\varphi \in \Gamma((\pi_{\mathcal{E}} \circ \tau)^*\pi)$ and $\Omega \in \Gamma((\pi_{\mathcal{E}} \circ \tau)^*\pi) \otimes \mathcal{C}\Lambda^1(\overline{\mathcal{E}^\infty})$. Solutions of the first equation are called τ-*shadows* of nonlocal symmetries, while solutions of the second one are said to be τ-*shadows* of recursion operators in the covering τ. The following result establishes relations of shadows to symmetries and recursion operators:

Theorem 1 (see [5] and [7]). *Let* $\tau\colon \overline{\mathcal{E}^\infty} \to \mathcal{E}^\infty$ *be a covering. Then:*

1. *If* φ *is a* τ-*shadow, then there exists a covering*

$$\bar{\tau}\colon \overline{\overline{\mathcal{E}^\infty}} \to \overline{\mathcal{E}^\infty} \xrightarrow{\tau} \mathcal{E}^\infty$$

   such that φ *reconstructs up to a nonlocal* $\bar{\tau}$-*symmetry.*
2. *If* φ *is a nonlocal* τ-*symmetry and* $\mathcal{R}$ *is a recursion operator shadow, then* $\mathcal{R}\varphi$ *is a symmetry shadow.*

Remark 3. Among all one-dimensional coverings over a given equation there exists a special class consisting of those ones, for which the fields $X_1, \ldots, X_n$ in (22) are independent of nonlocal variables (so-called *abelian coverings*). To any such a covering, one can put into correspondence a differential form on $\mathcal{E}^\infty$:

$$\omega_\tau = \sum_{l=1}^n X_l \, dx_l.$$

This form is closed with respect to the so-called *horizontal differential* $d_h = \mathcal{C}d$ (cf. Proposition 1). Vice versa, to any such a form there corresponds a covering of the above mentioned type. Moreover, two closed forms determine the same class in the cohomology group $H^1(d_h)$

if and only if the corresponding coverings are equivalent. In particular, if $n = 2$, the group $H^1(d_h)$ coincides with the group of conservation laws of the equation $\mathcal{E}$ [10]. Thus, to construct a covering under consideration is the same as to find a conservation law. This fact is used in the computations below.

§2. Basic computations and results

In this section we shall discuss the complete integrability of the KdV-mKdV system given in (1), i.e.,

$$(23) \qquad u_t = -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1,$$
$$w_t = -w_3 + 3w^2w_1 + 3uw_1 + 3u_1w.$$

In order to demonstrate the complete integrability of this system, we shall construct the recursion operator for symmetries of this coupled system, leading to infinite hierarchies of symmetries and, most probably, of conservation laws. Due to the very special form of the final results, it seems that integrability of this system, which looks quite ordinary, has not been discussed before or elsewhere. In order to do this, we shall discuss conservation laws in Subsection 2.1 leading to the necessary nonlocal variables.

In Subsection 2.2 we shall discuss local and nonlocal symmetries of the system, while in Subsection 2.3 we construct the recursion operator or deformation of the equation structure (14).

2.1. Conservation laws and nonlocal variables

Here we shall construct conservation laws for (23) in order to arrive at an abelian covering of the coupled KdV-mKdV system as was shown for Burgers equation (20). So we construct $X = X(x, t, u, \ldots, w \ldots)$, $T = T(x, t, u, \ldots, w \ldots)$ such that

$$(24) \qquad\qquad D_x(T) = D_t(X)$$

and in a similar way we construct nonlocal conservation laws by the requirement

$$(25) \qquad\qquad \bar{D}_x(\bar{T}) = \bar{D}_t(\bar{X}),$$

where $\bar{D}_*$ is defined by (9); moreover $\bar{X}$, $\bar{T}$ are dependent on local variables x, t, $u, \ldots, w, \ldots$ as well as the already determined nonlocal variables, denoted here by p_* or $p_{*,*}$, which are associated to the

conservation laws (X, T) by the formal definition

$$D_x(p_*) = (p_*)_x = X,$$
$$D_t(p_*) = (p_*)_t = T.$$

Proceeding in this way, we obtained the following set of nonlocal variables

(26) $p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1},\ p_{3,2},\ p_{4,1},\ p_5,$

where their defining equations are given by

$$(p_1)_x = u,$$
$$(p_1)_t = 3u^2 + 3uw^2 - u_2 - 3ww_2,$$
$$(p_{0,1})_x = w,$$
$$(p_{0,1})_t = 3uw + w^3 - w_2,$$
$$(p_{0,2})_x = p_1,$$
$$(p_{0,2})_t = -6p_3 - u_1,$$
$$(p_{1,1})_x = \cos(2p_{0,1})p_1 w + \sin(2p_{0,1})w^2,$$
$$(p_{1,1})_t = \cos(2p_{0,1})(3p_1 uw + p_1 w^3 - p_1 w_2 + uw_1 - u_1 w - w^2 w_1)$$
$$+ \sin(2p_{0,1})(4uw^2 + w^4 - 2ww_2 + w_1^2),$$
$$(p_{1,2})_x = \cos(2p_{0,1})w^2 - \sin(2p_{0,1})p_1 w,$$
$$(p_{1,2})_t = \cos(2p_{0,1})(4uw^2 + w^4 - 2ww_2 + w_1^2)$$
$$+ \sin(2p_{0,1})(-3p_1 uw - p_1 w^3 + p_1 w_2 - uw_1 + u_1 w + w^2 w_1),$$
$$(p_{2,1})_x = \frac{1}{2}\left(4\cos(2p_{0,1})p_{1,1}w^2 - 4\sin(2p_{0,1})p_1 p_{1,1}w + w(p_1^2 - 2u + w^2)\right),$$
$$(p_{2,1})_t = \frac{1}{2}\left(4\cos(2p_{0,1})p_{1,1}(4uw^2 + w^4 - 2ww_2 + w_1^2)\right.$$
$$+ 4\sin(2p_{0,1})$$
$$\times\, p_{1,1}(-3p_1 uw - p_1 w^3 + p_1 w_2 - uw_1 + u_1 w + w^2 w_1)$$
$$+ 3p_1^2 uw + p_1^2 w^3 - p_1^2 w_2 + 2p_1 uw_1 - 2p_1 u_1 w - 2p_1 w^2 w_1$$
$$\left. -\, 8u^2 w - uw^3 + 2uw_2 - 2u_1 w_1 + 2u_2 w + w^5 + 3w^2 w_2\right),$$
$$(p_3)_x = \frac{1}{2}(-u^2 - uw^2 + ww_2),$$
$$(p_3)_t = \frac{1}{2}(-4u^3 - 9u^2 w^2 + 2uu_2 - 3uw^4 + 11uww_2 - uw_1^2 - u_1^2$$
$$+ u_1 ww_1 + 4u_2 w^2 + 6w^3 w_2 + 3w^2 w_1^2 - ww_4 + w_1 w_3 - w_2^2),$$

$$(p_{3,1})_x = \frac{1}{12}\big(\cos(2p_{0,1})$$
$$\times w(p_1^3 - 6p_1 u + 39 p_1 w^2 - 24 p_{1,1} p_{1,2} w + 12 p_3 + 6u_1)$$
$$+ 2\sin(2p_{0,1})w(12 p_1 p_{1,1} p_{1,2} + 18 p_1 w_1 + 2w^3 + 3w_2)$$
$$+ 6 p_{1,2} w(-p_1^2 + 2u - w^2)\big),$$

$$(p_{3,2})_x = \frac{1}{12}\big(2\cos(2p_{0,1})w(12 p_1 p_{1,1} p_{1,2} - 18 p_1 w_1 - 2w^3 - 3w_2)$$
$$+ \sin(2p_{0,1})$$
$$\times w(p_1^3 - 6p_1 u + 39 p_1 w^2 + 24 p_{1,1} p_{1,2} w + 12 p_3 + 6u_1)$$
$$+ 6 p_{1,1} w(-p_1^2 + 2u - w^2)\big),$$

$$(p_{4,1})_x = \frac{1}{48}\big(8\cos(2p_{0,1})w(p_1^3 p_{1,2} + 12 p_1 p_{1,1}^2 p_{1,2} - 6 p_1 p_{1,2} u$$
$$+ 3 p_1 p_{1,2} w^2 - 12 p_{1,1} p_{1,2}^2 w + 18 p_{1,1} uw - 4 p_{1,1} w^3 - 6 p_{1,1} w_2$$
$$+ 12 p_{1,2} p_3 + 6 p_{1,2} u_1)$$
$$+ 8\sin(2p_{0,1})w(p_1^3 p_{1,1} + 12 p_1 p_{1,1} p_{1,2}^2 - 6 p_1 p_{1,1} u + 3 p_1 p_{1,1} w^2$$
$$+ 12 p_{1,1}^2 p_{1,2} w + 12 p_{1,1} p_3 + 6 p_{1,1} u_1 - 18 p_{1,2} uw + 4 p_{1,2} w^3$$
$$+ 6 p_{1,2} w_2)$$
$$+ w(-p_1^4 - 24 p_1^2 p_{1,1}^2 - 24 p_1^2 p_{1,2}^2 + 12 p_1^2 u - 6 p_1^2 w^2 - 48 p_1 p_3$$
$$- 24 p_1 u_1 + 48 p_{1,1}^2 u - 24 p_{1,1}^2 w^2 + 48 p_{1,2}^2 u - 24 p_{1,2}^2 w^2$$
$$- 60 u^2 + 44 uw^2 + 24 u_2 - 13 w^4 + 6 ww_2)\big),$$

$$(p_5)_x = \frac{1}{6}(12 u^3 + 24 u^2 w^2 - 6 uu_2 + 6 uw^4 - 30 uww_2 - 3 u_2 w^2$$
$$- 8 w^3 w_2 + 6 ww_4).$$

In the previous equations, we skipped explicit formulas for $(p_{3,1})_t$, $(p_{3,2})_t$, $(p_{4,1})_t$, and $(p_5)_t$, because they are too massive, though quite important for the setting to be well defined and in order to avoid ambiguities. The reader is referred to the Appendix for them.

It is quite a striking result that functions $\cos(2p_{0,1})$, $\sin(2p_{0,1})$ appear in the presentation of the conservation laws and their associated nonlocal variables.

We should note that p_1, $p_{0,1}$, p_3, p_5 arise from **local conservation laws** and we shall call p_1, $p_{0,1}$, p_3, p_5 *nonlocalities of first order.*

In a similar way we see that $p_{0,2}$, $p_{1,1}$, $p_{1,2}$ arise from **nonlocal conservation laws**, where their x- and t-derivatives are dependent on the first order nonlocalities. For this reason $p_{0,2}$, $p_{1,1}$, $p_{1,2}$ are called *nonlocalities of second order.*

Proceeding in this way $p_{2,1}$, $p_{3,1}$, $p_{3,2}$, $p_{4,1}$ constitute *nonlocalities of third order*.

2.2. Local and nonlocal symmetries

In this section we shall present results for the construction of local and nonlocal symmetries of system (23). In order to construct these symmetries, we consider the system of partial differential equations obtained by the infinite prolongation of (23) together with the covering by the nonlocal variables

$$p_{0,1}, \; p_{0,2}, \; p_1, \; p_{1,1}, \; p_{1,2}, \; p_{2,1}, \; p_3, \; p_{3,1}, p_{3,2}, \; p_{4,1}, \; p_5.$$

So, in the augmented setting governed by (23), their total derivatives and the equations given in Subsection 2.1 we construct symmetries $Y = (Y^u, Y^w)$ which have to satisfy the symmetry condition

$$\bar{\ell}_{\mathcal{E}} Y = 0.$$

From this condition we obtained the following symmetries

$$Y_{0,1}, \; Y_{1,1}, \; Y_{1,2}, \; Y_{1,3}, \; Y_{2,1}, \; Y_{3,1}, \; Y_{3,2}, \; Y_{3,3},$$

where generating functions $Y^u_{*,*}$, $Y^w_{*,*}$ are given as

$Y^u_{0,1} = 3t(6uu_1 + 6uww_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2) + xu_1 + 2u,$

$Y^w_{0,1} = 3t(3uw_1 + 3u_1w + 3w^2w_1 - w_3) + xw_1 + w,$

$Y^u_{1,1} = u_1,$

$Y^w_{1,1} = w_1,$

$Y^u_{1,2} = \cos(2p_{0,1})(2uw - w_2) + \sin(2p_{0,1})(u_1 + 2ww_1),$

$Y^w_{1,2} = -\cos(2p_{0,1})u - \sin(2p_{0,1})w_1,$

$Y^u_{1,3} = \cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(-2uw + w_2),$

$Y^w_{1,3} = -\cos(2p_{0,1})w_1 + \sin(2p_{0,1})u,$

$Y^u_{2,1} = \dfrac{1}{2}\big(2\cos(2p_{0,1})(p_{1,1}u_1 + 2p_{1,1}ww_1 - 2p_{1,2}uw + p_{1,2}w_2)$
$\qquad\quad + 2\sin(2p_{0,1})(-2p_{1,1}uw + p_{1,1}w_2 - p_{1,2}u_1 - 2p_{1,2}ww_1)$
$\qquad\quad + 2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3\big),$

$Y^w_{2,1} = \dfrac{1}{2}\big(2\cos(2p_{0,1})(-p_{1,1}w_1 + p_{1,2}u) + 2\sin(2p_{0,1})(p_{1,1}u + p_{1,2}w_1)$
$\qquad\quad - p_1u + u_1 + ww_1\big),$

$Y^u_{3,1} = \dfrac{1}{3}\big(6uu_1 + 6uww_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2\big),$

$$Y_{3,1}^w = \frac{1}{3}\big(3uw_1 + 3u_1w + 3w^2w_1 - w_3\big),$$

$$
\begin{aligned}
Y_{3,2}^u = \frac{1}{8}\big(& \cos(2p_{0,1})(-2p_1^2uw + p_1^2w_2 - 4p_1uw_1 - 6p_1u_1w \\
& - 4p_1w^2w_1 + 2p_1w_3 + 8p_{1,1}p_{1,2}u_1 + 16p_{1,1}p_{1,2}ww_1 - 8p_{1,2}^2uw \\
& + 4p_{1,2}^2w_2 - 4p_{2,1}u_1 - 8p_{2,1}ww_1 + 10u^2w + 6uw^3 - 8uw_2 \\
& - 14u_1w_1 - 8u_2w - 11w^2w_2 - 14ww_1^2 + 2w_4) \\
& + 2\sin(2p_{0,1})(-8p_{1,1}p_{1,2}uw + 4p_{1,1}p_{1,2}w_2 - 2p_{1,2}^2u_1 - 4p_{1,2}^2ww_1 \\
& + 4p_{2,1}uw - 2p_{2,1}w_2 + 6uu_1 + 10uww_1 + 3u_1w^2 - u_3 + 2w^3w_1 \\
& - 3ww_3 - 5w_1w_2) \\
& + 4p_{1,2}(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)\big),
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,2}^w = \frac{1}{8}\big(& \cos(2p_{0,1})(p_1^2u - 2p_1u_1 - 2p_1ww_1 - 8p_{1,1}p_{1,2}w_1 + 4p_{1,2}^2u \\
& + 4p_{2,1}w_1 - 4u^2 - 3uw^2 + 2u_2 + 4ww_2 + 2w_1^2) \\
& + 2\sin(2p_{0,1}) \\
& \quad \times (4p_{1,1}p_{1,2}u + 2p_{1,2}^2w_1 - 2p_{2,1}u - 3uw_1 - 3u_1w - 3w^2w_1 + w_3) \\
& + 4p_{1,2}(-p_1u + u_1 + ww_1)\big),
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,3}^u = \frac{1}{8}\big(& 2\cos(2p_{0,1})(2p_{1,1}^2u_1 + 4p_{1,1}^2ww_1 - 4p_{2,1}uw + 2p_{2,1}w_2 - 6uu_1 \\
& - 10uww_1 - 3u_1w^2 + u_3 - 2w^3w_1 + 3ww_3 + 5w_1w_2) \\
& + \sin(2p_{0,1})(-2p_1^2uw + p_1^2w_2 - 4p_1uw_1 - 6p_1u_1w - 4p_1w^2w_1 \\
& + 2p_1w_3 - 8p_{1,1}^2uw + 4p_{1,1}^2w_2 - 4p_{2,1}u_1 - 8p_{2,1}ww_1 + 10u^2w \\
& + 6uw^3 - 8uw_2 - 14u_1w_1 - 8u_2w - 11w^2w_2 - 14ww_1^2 + 2w_4) \\
& + 4p_{1,1}(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)\big),
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,3}^w = \frac{1}{8}\big(& 2\cos(2p_{0,1})(-2p_{1,1}^2w_1 + 2p_{2,1}u + 3uw_1 + 3u_1w + 3w^2w_1 - w_3) \\
& + \sin(2p_{0,1})(p_1^2u - 2p_1u_1 - 2p_1ww_1 + 4p_{1,1}^2u + 4p_{2,1}w_1 - 4u^2 \\
& - 3uw^2 + 2u_2 + 4ww_2 + 2w_1^2) \\
& + 4p_{1,1}(-p_1u + u_1 + ww_1)\big).
\end{aligned}
$$

2.3. Recursion operator

Here we present the recursion operator $\mathcal{R}$ for symmetries for this case obtained as a higher symmetry in the Cartan covering of system

of equations (1) augmented by equations governing the nonlocal variables (26). As explained in the previous section, the recursion operator is in effect the a deformation of the equation structure (11).

As demonstrated there, this deformation is a form-valued vector field (or a vectorfield-valued one-form) and has to satisfy

$$(27) \qquad\qquad \bar{\ell}_{\mathcal{E}}^{[1]} \mathcal{R} = 0.$$

In order to arrive at a nontrivial result as was explained for Burgers' equation too (cf. Example 1), we have to introduce nonlocal variables

$$p_{0,1}, \ p_{0,2}, \ p_1, \ p_{1,1}, \ p_{1,2}, \ p_{2,1}, \ p_3, \ p_{3,1}, p_{3,2}, \ p_{4,1}, \ p_5$$

and their associated Cartan contact forms

$$\omega_{p_{0,1}}, \ \omega_{p_{0,2}}, \ \omega_{p_1}, \ \omega_{p_{1,1}}, \ \omega_{p_{1,2}}, \omega_{p_{2,1}}, \ \omega_{p_3}, \ \omega_{p_{3,1}}, \ \omega_{p_{3,2}}, \ \omega_{p_{4,1}}, \omega_{p_5}.$$

The final result, which is dependent on the nonlocal Cartan forms

$$\omega_{p_{0,1}}, \ \omega_{p_1}, \ \omega_{p_{1,1}}, \ \omega_{p_{1,2}},$$

is given by

$$(28) \qquad\qquad \mathcal{R} = R^u \frac{\partial}{\partial u} + R^w \frac{\partial}{\partial w} + \cdots ,$$

where the components R^u, R^w are given by

$$
\begin{aligned}
R_u = {} & \omega_{u_2}(-1) + \omega_u(4u + w^2) \\
& + \omega_{w_2}(-2w) + \omega_{w_1}(-w_1) + \omega_w(3uw - 2w_2) \\
& + \omega_{p_{1,2}}(-\cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(2uw - w_2)) \\
& + \omega_{p_{1,1}}(\cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2ww_1)) \\
& + \omega_{p_1}(2u_1 + ww_1) \\
& + \omega_{p_{0,1}}(2p_1 uw - p_1 w_2 + 2uw_1 + 3u_1 w + 2w^2 w_1 - w_3),
\end{aligned}
$$

$$
\begin{aligned}
(29) \quad R_w = {} & \omega_{w_2}(-1) + \omega_w(2u + w^2) + \omega_u(2w) \\
& + \omega_{p_{1,2}}(\cos(2p_{0,1})w_1 - \sin(2p_{0,1})u) \\
& + \omega_{p_{1,1}}(\cos(2p_{0,1})u + \sin(2p_{0,1})w_1) \\
& + \omega_{p_1}(w_1) + \omega_{p_{0,1}}(-p_1 u + u_1 + ww_1).
\end{aligned}
$$

We shall now present this result in a more conventional form which appeals to expressions using operators of the form D_x and D_x^{-1}. In order to do this, we first split (29) into the so-called local part and nonlocal parts, consisting of terms associated to ω_{u_2}, ω_u, ω_{w_2}, ω_{w_1}, ω_w and those

associated to $\omega_{p_{1,2}}$, $\omega_{p_{1,1}}$, ω_{p_1}, $\omega_{p_{0,1}}$ respectively. The first part will account for D_x presentation, while the second one accounts for the D_x^{-1} part.

Due to the action of contraction $\vartheta_\varphi \lrcorner \mathcal{R}$, the local part is given by the following matrix operator:

$$\begin{bmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_1 D_x + 3uw - 2w_2 \\ 2w & -D_x^2 + 2u + w^2 \end{bmatrix}.$$

The nonlocal part will be split into parts associated to ω_{p_1}, $\omega_{p_{0,1}}$ and $\omega_{p_{1,2}}$, $\omega_{p_{1,1}}$, respectively. The first one is given as

$$\begin{bmatrix} (2u_1 + ww_1)D_x^{-1} & (2p_1 uw - p_1 w_2 + 2uw_1 + 3u_1 w + 2w^2 w_1 - w_3)D_x^{-1} \\ w_1 D_x^{-1} & (-p_1 u + u_1 + ww_1)D_x^{-1} \end{bmatrix}.$$

To deal with the last part, let us introduce the notation:

$$A_1 = \cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2ww_1),$$
$$A_2 = \cos(2p_{0,1})u + \sin(2p_{0,1})w_1,$$
$$B_1 = -\cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(2uw - w_2),$$
$$B_2 = \cos(2p_{0,1})w_1 - \sin(2p_{0,1})u,$$

being the coefficients at $\omega_{p_{1,1}}$ and $\omega_{p_{1,2}}$ in (29).

According to the presentations of $(p_{1,1})_x$ and $(p_{1,2})_x$, i.e.,

$$(p_{1,1})_x = \cos(2p_{0,1})p_1 w + \sin(2p_{0,1})w^2$$
$$(p_{1,2})_x = \cos(2p_{0,1})w^2 - \sin(2p_{0,1})p_1 w,$$

we introduce their partial derivatives with respect to $p_{0,1}$, p_1, and w as

$$\alpha_1 = -2p_1 w \sin(2p_{0,1}) + 2w^2 \cos(2p_{0,1}),$$
$$\alpha_2 = w \cos(2p_{0,1}),$$
$$\alpha_3 = p_1 \cos(2p_{0,1}) + 2w \sin(2p_{0,1}),$$
$$\beta_1 = -2w^2 \sin(2p_{0,1}) - 2p_1 w \cos(2p_{0,1}),$$
$$\beta_2 = -w \sin(2p_{0,1}),$$
$$\beta_3 = 2w \cos(2p_{0,1}) - p_1 \sin(2p_{0,1}).$$

From this we arrive in a straightforward way at the last nonlocal part of the recursion operator, i.e.,

$$\begin{bmatrix} A_1 D_x^{-1}\alpha_2 D_x^{-1} & A_1 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \\ A_2 D_x^{-1}\alpha_2 D_x^{-1} & A_2 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \end{bmatrix}$$
$$+ \begin{bmatrix} B_1 D_x^{-1}\beta_2 D_x^{-1} & B_1 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \\ B_2 D_x^{-1}\beta_2 D_x^{-1} & B_2 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \end{bmatrix}$$

So, in the final form we obtain the recursion operator as

$$
\mathcal{R} = \begin{bmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_1 D_x + 3uw - 2w_2 \\ 2w & -D_x^2 + 2u + w^2 \end{bmatrix}
$$
$$
+ \begin{bmatrix} (2u_1 + ww_1)D_x^{-1} & KD_x^{-1} \\ w_1 D_x^{-1} & (-p_1 u + u_1 + ww_1)D_x^{-1} \end{bmatrix}
$$
$$
+ \begin{bmatrix} A_1 D_x^{-1}\alpha_2 D_x^{-1} & A_1 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \\ A_2 D_x^{-1}\alpha_2 D_x^{-1} & A_2 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \end{bmatrix}
$$
$$
+ \begin{bmatrix} B_1 D_x^{-1}\beta_2 D_x^{-1} & B_1 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \\ B_2 D_x^{-1}\beta_2 D_x^{-1} & B_2 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \end{bmatrix},
$$

where $K = 2p_1 uw - p_1 w_2 + 2uw_1 + 3u_1 w + 2w^2 w_1 - w_3$.

§3. Conclusion

We gave an outline of the theory of deformations of the equation structure of differential equations, leading to the construction of recursion operators for symmetries of such equations. The extension of this theory to the nonlocal setting of differential equations is essential for getting nontrivial results. The theory has been applied to the construction of the recursion operator for symmetries for a coupled KdV-mKdV system, leading to a highly nonlocal result for this system. Moreover the appearance of nonpolynomial nonlocal terms in all results, e.g., conservation laws, symmetries and recursion operator is striking and reveals some unknown and intriguing underlying structure of the equations.

Appendix

Here we present explicit formulas for $(p_{3,1})_t$, $(p_{3,2})_t$, $(p_{4,1})_t$, and $(p_5)_t$:

$$
\begin{aligned}
(p_{3,1})_t = \frac{1}{12}\big(&\cos(2p_{0,1})(3p_1^3 uw + p_1^3 w^3 - p_1^3 w_2 + 3p_1^2 uw_1 - 3p_1^2 u_1 w \\
& - 3p_1^2 w^2 w_1 - 24p_1 u^2 w + 105p_1 uw^3 + 6p_1 uw_2 - 6p_1 u_1 w_1 \\
& + 6p_1 u_2 w + 39p_1 w^5 - 27p_1 w^2 w_2 - 96p_{1,1}p_{1,2}uw^2 \\
& - 24p_{1,1}p_{1,2}w^4 + 48p_{1,1}p_{1,2}ww_2 - 24p_{1,1}p_{1,2}w_1^2 + 36p_3 uw \\
& + 12p_3 w^3 - 12p_3 w_2 - 12u^2 w_1 + 48uu_1 w + 39uw^2 w_1 + 3u_1 w^3 \\
& - 6u_1 w_2 + 6u_2 w_1 - 6u_3 w - 9w^4 w_1 - 12w^2 w_3 \\
& - 18ww_1 w_2 + 6w_1^3) \\
& + 2\sin(2p_{0,1})(36p_1 p_{1,1}p_{1,2}uw + 12p_1 p_{1,1}p_{1,2}w^3
\end{aligned}
$$

$$- 12p_1p_{1,1}p_{1,2}w_2 + 54p_1uww_1 + 54p_1u_1w^2 + 54p_1w^3w_1$$

$$- 18p_1ww_3 + 12p_{1,1}p_{1,2}uw_1 - 12p_{1,1}p_{1,2}u_1w - 12p_{1,1}p_{1,2}w^2w_1$$

$$- 9u^2w^2 + 18uw^4 + 27uww_2 - 9uw_1^2 + 9u_1ww_1 + 3u_2w^2$$

$$+ 2w^6 - 11w^3w_2 + 12w^2w_1^2 - 3ww_4 + 3w_1w_3 - 3w_2^2)$$

$$+ 6p_{1,2}(-3p_1^2uw - p_1^2w^3 + p_1^2w_2 - 2p_1uw_1 + 2p_1u_1w + 2p_1w^2w_1$$

$$+ 8u^2w + uw^3 - 2uw_2 + 2u_1w_1 - 2u_2w - w^5 - 3w^2w_2)),$$

$$(p_{3,2})_t = \frac{1}{12}\big(2\cos(2p_{0,1})(36p_1p_{1,1}p_{1,2}uw + 12p_1p_{1,1}p_{1,2}w^3$$

$$- 12p_1p_{1,1}p_{1,2}w_2 - 54p_1uww_1 - 54p_1u_1w^2 - 54p_1w^3w_1$$

$$+ 18p_1ww_3 + 12p_{1,1}p_{1,2}uw_1 - 12p_{1,1}p_{1,2}u_1w - 12p_{1,1}p_{1,2}w^2w_1$$

$$+ 9u^2w^2 - 18uw^4 - 27uww_2 + 9uw_1^2 - 9u_1ww_1 - 3u_2w^2$$

$$- 2w^6 + 11w^3w_2 - 12w^2w_1^2 + 3ww_4 - 3w_1w_3 + 3w_2^2)$$

$$+ \sin(2p_{0,1})(3p_1^3uw + p_1^3w^3 - p_1^3w_2 + 3p_1^2uw_1 - 3p_1^2u_1w$$

$$- 3p_1^2w^2w_1 - 24p_1u^2w + 105p_1uw^3 + 6p_1uw_2 - 6p_1u_1w_1$$

$$+ 6p_1u_2w + 39p_1w^5 - 27p_1w^2w_2 + 96p_{1,1}p_{1,2}uw^2$$

$$+ 24p_{1,1}p_{1,2}w^4 - 48p_{1,1}p_{1,2}ww_2 + 24p_{1,1}p_{1,2}w_1^2 + 36p_3uw$$

$$+ 12p_3w^3 - 12p_3w_2 - 12u^2w_1 + 48uu_1w + 39uw^2w_1 + 3u_1w^3$$

$$- 6u_1w_2 + 6u_2w_1 - 6u_3w - 9w^4w_1 - 12w^2w_3$$

$$- 18ww_1w_2 + 6w_1^3)$$

$$+ 6p_{1,1}(-3p_1^2uw - p_1^2w^3 + p_1^2w_2 - 2p_1uw_1 + 2p_1u_1w + 2p_1w^2w_1$$

$$+ 8u^2w + uw^3 - 2uw_2 + 2u_1w_1 - 2u_2w - w^5 - 3w^2w_2)),$$

$$(p_{4,1})_t = \frac{1}{48}\big(8\cos(2p_{0,1})(3p_1^3p_{1,2}uw + p_1^3p_{1,2}w^3 - p_1^3p_{1,2}w_2 + 3p_1^2p_{1,2}uw_1$$

$$- 3p_1^2p_{1,2}u_1w - 3p_1^2p_{1,2}w^2w_1 + 36p_1p_{1,1}^2p_{1,2}uw + 12p_1p_{1,1}^2p_{1,2}w^3$$

$$- 12p_1p_{1,1}^2p_{1,2}w_2 - 24p_1p_{1,2}u^2w - 3p_1p_{1,2}uw^3 + 6p_1p_{1,2}uw_2$$

$$- 6p_1p_{1,2}u_1w_1 + 6p_1p_{1,2}u_2w + 3p_1p_{1,2}w^5 + 9p_1p_{1,2}w^2w_2$$

$$+ 12p_{1,1}^2p_{1,2}uw_1 - 12p_{1,1}^2p_{1,2}u_1w - 12p_{1,1}^2p_{1,2}w^2w_1$$

$$- 48p_{1,1}p_{1,2}^2uw^2 - 12p_{1,1}p_{1,2}^2w^4 + 24p_{1,1}p_{1,2}^2ww_2 - 12p_{1,1}p_{1,2}^2w_1^2$$

$$+ 72p_{1,1}u^2w^2 + 18p_{1,1}uw^4 - 54p_{1,1}uww_2 + 18p_{1,1}uw_1^2$$

$$- 18p_{1,1}u_1ww_1 - 24p_{1,1}u_2w^2 - 4p_{1,1}w^6 - 32p_{1,1}w^3w_2$$

$$- 24p_{1,1}w^2w_1^2 + 6p_{1,1}ww_4 - 6p_{1,1}w_1w_3 + 6p_{1,1}w_2^2$$

$$+ 36p_{1,2}p_3uw + 12p_{1,2}p_3w^3 - 12p_{1,2}p_3w_2 - 12p_{1,2}u^2w_1$$

$$+ 48p_{1,2}uu_1w + 39p_{1,2}uw^2w_1 + 3p_{1,2}u_1w^3$$

$$- 6p_{1,2}u_1w_2 + 6p_{1,2}u_2w_1 - 6p_{1,2}u_3w - 9p_{1,2}w^4w_1$$

$$- 12p_{1,2}w^2w_3 - 18p_{1,2}ww_1w_2 + 6p_{1,2}w_1^3)$$

$$+ 8\sin(2p_{0,1})(3p_1^3p_{1,1}uw + p_1^3p_{1,1}w^3 - p_1^3p_{1,1}w_2 + 3p_1^2p_{1,1}uw_1$$

$$- 3p_1^2p_{1,1}u_1w - 3p_1^2p_{1,1}w^2w_1 + 36p_1p_{1,1}p_{1,2}^2uw$$

$$+ 12p_1p_{1,1}p_{1,2}^2w^3 - 12p_1p_{1,1}p_{1,2}^2w_2 - 24p_1p_{1,1}u^2w - 3p_1p_{1,1}uw^3$$

$$+ 6p_1p_{1,1}uw_2 - 6p_1p_{1,1}u_1w_1 + 6p_1p_{1,1}u_2w + 3p_1p_{1,1}w^5$$

$$+ 9p_1p_{1,1}w^2w_2 + 48p_{1,1}^2p_{1,2}uw^2 + 12p_{1,1}^2p_{1,2}w^4 - 24p_{1,1}^2p_{1,2}ww_2$$

$$+ 12p_{1,1}^2p_{1,2}w_1^2 + 12p_{1,1}p_{1,2}^2uw_1 - 12p_{1,1}p_{1,2}^2u_1w$$

$$- 12p_{1,1}p_{1,2}^2w^2w_1 + 36p_{1,1}p_3uw + 12p_{1,1}p_3w^3 - 12p_{1,1}p_3w_2$$

$$- 12p_{1,1}u^2w_1 + 48p_{1,1}uu_1w + 39p_{1,1}uw^2w_1 + 3p_{1,1}u_1w^3$$

$$- 6p_{1,1}u_1w_2 + 6p_{1,1}u_2w_1 - 6p_{1,1}u_3w - 9p_{1,1}w^4w_1 - 12p_{1,1}w^2w_3$$

$$- 18p_{1,1}ww_1w_2 + 6p_{1,1}w_1^3 - 72p_{1,2}u^2w^2 - 18p_{1,2}uw^4$$

$$+ 54p_{1,2}uww_2 - 18p_{1,2}uw_1^2 + 18p_{1,2}u_1ww_1 + 24p_{1,2}u_2w^2$$

$$+ 4p_{1,2}w^6 + 32p_{1,2}w^3w_2 + 24p_{1,2}w^2w_1^2 - 6p_{1,2}ww_4$$

$$+ 6p_{1,2}w_1w_3 - 6p_{1,2}w_2^2) - 3p_1^4uw - p_1^4w^3 + p_1^4w_2 - 4p_1^3uw_1$$

$$+ 4p_1^3u_1w + 4p_1^3w^2w_1 - 72p_1^2p_{1,1}^2uw - 24p_1^2p_{1,1}^2w^3 + 24p_1^2p_{1,1}^2w_2$$

$$- 72p_1^2p_{1,2}^2uw - 24p_1^2p_{1,2}^2w^3 + 24p_1^2p_{1,2}^2w_2 + 48p_1^2u^2w + 6p_1^2uw^3$$

$$- 12p_1^2uw_2 + 12p_1^2u_1w_1 - 12p_1^2u_2w - 6p_1^2w^5 - 18p_1^2w^2w_2$$

$$- 48p_1p_{1,1}^2uw_1 + 48p_1p_{1,1}^2u_1w + 48p_1p_{1,1}^2w^2w_1 - 48p_1p_{1,2}^2uw_1$$

$$+ 48p_1p_{1,2}^2u_1w + 48p_1p_{1,2}^2w^2w_1 - 144p_1p_3uw - 48p_1p_3w^3$$

$$+ 48p_1p_3w_2 + 48p_1u^2w_1 - 192p_1uu_1w - 156p_1uw^2w_1$$

$$- 12p_1u_1w^3 + 24p_1u_1w_2 - 24p_1u_2w_1 + 24p_1u_3w + 36p_1w^4w_1$$

$$+ 48p_1w^2w_3 + 72p_1ww_1w_2 - 24p_1w_1^3 + 192p_{1,1}^2u^2w$$

$$+ 24p_{1,1}^2uw^3 - 48p_{1,1}^2uw_2 + 48p_{1,1}^2u_1w_1 - 48p_{1,1}^2u_2w - 24p_{1,1}^2w^5$$

$$- 72p_{1,1}^2w^2w_2 + 192p_{1,2}^2u^2w + 24p_{1,2}^2uw^3 - 48p_{1,2}^2uw_2$$

$$+ 48p_{1,2}^2u_1w_1 - 48p_{1,2}^2u_2w - 24p_{1,2}^2w^5$$

$$- 72p_{1,2}^2w^2w_2 - 48p_3uw_1 + 48p_3u_1w + 48p_3w^2w_1 - 252u^3w$$

$$- 36u^2w^3 + 60u^2w_2 - 144uu_1w_1 + 240uu_2w - 7uw^5$$

$$+ 342uw^2w_2 + 24uww_1^2 + 144u_1^2w + 228u_1w^2w_1 + 94u_2w^3$$
$$- 24u_2w_2 + 24u_3w_1 - 24u_4w - 13w^7 + 47w^4w_2 - 4w^3w_1^2$$
$$- 78w^2w_4 - 60ww_1w_3 - 120ww_2^2 + 60w_1^2w_2\big),$$

$$(p_5)_t = \frac{1}{6}\big(54u^4 + 180u^3w^2 - 72u^2u_2 + 126u^2w^4 - 282u^2ww_2$$
$$- 12u^2w_1^2 - 84uu_1ww_1 - 174uu_2w^2 + 6uu_4 + 18uw^6$$
$$- 300uw^3w_2 - 90uw^2w_1^2 + 66uww_4 + 6uw_1w_3 + 48uw_2^2$$
$$+ 42u_1^2w^2 - 6u_1u_3 + 12u_1w^3w_1 + 42u_1ww_3 - 48u_1w_1w_2$$
$$+ 6u_2^2 - 39u_2w^4 + 162u_2ww_2 - 48u_2w_1^2 + 48u_3ww_1 + 21u_4w^2$$
$$- 42w^5w_2 - 12w^4w_1^2 + 35w^3w_4 + 120w^2w_1w_3$$
$$+ 195w^2w_2^2 + 120ww_1^2w_2 - 6ww_6 - 30w_1^4 + 6w_1w_5 - 6w_2w_4\big).$$

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