Submanifolds with Degenerate Gauss Mappings in Spheres

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§1. Introduction

Let M be a connected l-dimensional C^∞ manifold. An immersion $f\colon M \to S^n$ to the sphere (resp. $f\colon M \to \mathbb{RP}^n$ to the projective space) is called *tangentially degenerate* (or, *developable*, or, *strongly parabolic*) if its Gauss mapping $\gamma\colon M \to G_{l+1}(\mathbb{R}^{n+1})$ has rank $< l$. Here $G_{l+1}(\mathbb{R}^{n+1})$ denotes the Grassmannian of $(l + 1)$-dimensional linear subspaces in $\mathbb{R}^{n+1}$. A submanifold of S^n or $\mathbb{RP}^n$ is called *tangentially degenerate* (or, *developable*, or, *strongly parabolic*) if so is the inclusion.

In the present paper we construct new examples of tangentially degenerate compact submanifolds satisfying the equality for the inequality proved by Ferus [19]. Remark that, if we have a tangentially degenerate immersed submanifold in S^n then, via the canonical double covering $\pi\colon S^n \to \mathbb{RP}^n$, we have a tangentially degenerate immersed submanifold in $\mathbb{RP}^n$.

Remark also that the notion of tangential degeneracy is invariant under the projective transformations. Recall that $\mathbb{RP}^n = G_1(\mathbb{R}^{n+1})$ and $S^n = \widetilde{G}_1(\mathbb{R}^{n+1})$ (oriented Grassmannian) have natural projective structures, respectively. In fact, M. A. Akivis clearly stated in [3] and [4] that the study of tangentially degenerate submanifolds belongs to projective geometry. Then our standpoint is as follows: We do not need the metric structures on them for the formulation of the results, while, for the proofs of the results, we use freely the metric structures.

Let M^l be compact and connected, and $f\colon M \to S^n$ a tangentially degenerate immersion. Denote by r the maximal rank of the Gauss

mapping γ. Then, by [19], we know that there exists a number $F(l)$ (Ferus number), depending only on the dimension l of M, such that, if $r < F(l)$ then $r = 0$, therefore $M = S^l$ and $f(M)$ is a great l-sphere in S^n. The Ferus number $F(l)$ is defined by

$$F(l) := \min\{k \mid A(k) + k \geq l\},$$

where $A(k)$ is the Adams number: the maximal number of linearly independent vector fields over the sphere S^{k-1}.

In other words, if $f: M^l \to S^n$ is a tangentially degenerate immersion of rank r, and $f(M)$ is not a great l-sphere, then $F(l) \leq r$ (Ferus inequality).

Then the problem we are concerning with is the following:

Problem.   Is the inequality $r < F(l)$ best possible for the implication $r = 0$? Do there exist tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$? Moreover can we classify tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$?

In contrast to the tangentially degenerate submanifolds in $\mathbb{R}^n$ such as cylinders, cones or tangent developable of space curves, all of which have singularities when considered in $\mathbb{RP}^n$, we construct many *compact* tangentially degenerate submanifolds in the sphere, some of which even satisfy the Ferus equality. In §2 we recall, in more detail, the Ferus inequality on tangentially degenerate immersions. In §3, degeneracy of the Gauss mapping of isoparametric hypersurfaces and their focal submanifolds in the sphere is discussed as a model case. These provide us with an infinitely many tangentially degenerate submanifolds in the sphere, which satisfy the Ferus equality for $(l, r) = (2^p + 1, 2^p)$ or $(2^q + 3, 2^q)$ for $p \geq 2$, $q \geq 3$. This answers the first and the second problem affirmatively. In §9, we construct further homogeneous examples with $r = F(l)$ for $(l, r) = (3, 2), (5, 4), (6, 4), (9, 8), (10, 8), (12, 8),$ $(17, 16), (18, 16), (20, 16), (25, 14), (26, 24), (28, 24)$, and possibly more. In order to describe the idea of the construction, in §4, the Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of orthonormal 2-vectors is introduced as a circle bundle over a complex quadric Q^{n-1}. Then in §5, we study complex submanifolds $\varphi: \Sigma \to Q^{n-1}$, and using this, in §6, we construct a natural map $\Phi: M = \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ from the pullback bundle over Σ to the sphere, such that the image of Φ is the union of l-parameter family of great circles in S^n. Then we show that, on the open set of maximal rank of $d\Phi$ in M, each fiber of the circle bundle $\varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ lies in the kernel of the differential of the Gauss mapping of Φ, so Φ is tangentially degenerate. Examples in this context are given in §7.

In §8, we recall *austere* submanifolds defined by Harvey and Lawson [23]. They proved that from austere submanifolds in spheres, one can construct *special Lagrangian* and volume minimizing varieties in complex Euclidean spaces. Then we show that if a complex submanifold $\varphi \colon \Sigma \to Q^{n-1}$ is *first order isotropic*, then the corresponding map $\Phi \colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ is always an immersion and is austere. In §9, we give further examples of homogeneous *austere* submanifolds M^l, which can be considered as a generalization of *Cartan's isoparametric hypersurfaces*, see §2. More precisely, M is the total space of $\mathbb{KP}^1$-bundle over 2-plane Grassmannian $G_2(\mathbb{K}^{n+1})$ where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and each fiber of the bundle lies in the kernel of the Gauss mapping of M.

In §10, we give a classification of tangentially degenerate hypersurfaces in S^4 with $r = 2$. Moreover, using Bryant's result [9], we construct an example of tangentially degenerate immersions $M^3 \to S^4$ such that the rank of Gauss mapping is not constant.

§2. Ferus inequality for submanifolds with degenerate Gauss mapping

The proof of the Ferus inequality is achieved by considering the Levi-Civita connection of the ordinary metric on S^n, the co-nullity operator, and a matrix Riccati-type equation [18] and [19]. Here we review the outline of the proof: Let $f \colon M^l \to S^n$ be a tangentially degenerate immersion. Assume that the induced metric on M from f is complete. It is the case if M is compact. Now, assume $0 < r < l$, for the maximal rank r of the Gauss mapping of f. Let $D \subset TM$ denote the *Monge-Ampère distribution*, namely the kernel of the differential of Gauss mapping, along the open subset U of M consisting of points where the Gauss mapping has the maximal rank r. Remark that $f(U)$ is a union of totally geodesic spheres of dimension $l - r$. Take x from U. Then we get a linear mapping $D_x^\perp - \{0\} \to (D_x^\perp)^{l-r+1}$ defined by $Y \mapsto (Y, C_{X_1}Y, \ldots, C_{X_{l-r}}Y)$, where $X_1, \ldots, X_{l-r}$ are basis of D_x and $C_X Y := -\mathrm{pr}(\nabla_Y X)$ denotes the co-nullity operator, $\mathrm{pr} \colon T_x M \to D_x^\perp$ being the projection. Then, by the assumption $r \neq 0$ and by an argument on Riccati equation, we conclude that $Y, C_{X_1}Y, \ldots, C_{X_{l-r}}Y$ are linearly independent. Thus we have $l - r \leq A(r)$, Adams number. Therefore $l \leq A(r) + r$.

It is well known ([6] and [19]) that the Adams number is determined by

$$A((2k+1)2^{c+4d}) = 2^c + 8d - 1, \quad \text{where } 0 \leq c \leq 3,\ 0 \leq d.$$

Therefore the Ferus number $F(l)$ is given for $l \le 24$, by

(1) $F(l) = \{\text{the highest power of 2 not larger than } l\}.$

Remark that $F(l) = l - \mu(l)$ and $A(l) = \rho(l) - 1$, using the original notations in [6] and [19].

Regard S^n as the unit hypersphere $S^n(1)$ in $\mathbb{R}^{n+1}$ with the ordinary metric. For a submanifold M in $S^n(1)$, the *index of relative nullity* $\nu(x)$ at $x \in M$, introduced by Chern and Kuiper [17] and [30], is defined as the dimension of

$$\{X \in T_x(M) \mid \sigma(X, Y) = 0 \text{ for any } Y \in T_x(M)\},$$

σ being the second fundamental form. Notice that the rank $r(x)$ of differential mapping $d\gamma_x : T_x M \to T_{\gamma(x)} G_{l+1}(\mathbb{R}^{n+1})$ at $x \in M$ is related to $\nu(x)$ by $r(x) = l - \nu(x)$, because

(2) $\ker d\gamma_x = \{X \in T_x(M) \mid \sigma(X, Y) = 0 \text{ for any } Y \in T_x(M)\}.$

Therefore the minimum ν of $\nu(x)$ over $x \in M$ is equal to $l - r$.

Theorem 2.1 ([6] , [12] and [19]). *Let M be a complete submanifold of n-dimensional unit sphere with $\dim_{\mathbb{R}} M = l$. If $r < F(l)$, then $r = 0$ and M is totally geodesic.*

For instance, in view of (1), we see that (cf. [6])
  (i) if l is a power of 2, $r < l$ implies $r = 0$,
 (ii) if $l = 3$, $r < 2$ implies $r = 0$,
(iii) if $5 \le l \le 7$, $r < 4$ implies $r = 0$,
 (iv) if $9 \le l \le 15$, $r < 8$ implies $r = 0$,
  (v) if $17 \le l \le 24$, $r < 16$ implies $r = 0$,
 (vi) if $25 \le l \le 31$, $r < 24$ implies $r = 0$.

We have examples of compact connected tangentially degenerate embedded hypersurfaces $M^3 \subset S^4$, $M^6 \subset S^7$, $M^{12} \subset S^{13}$, $M^{24} \subset S^{25}$ with $r = 2$, 4, 8, 16 respectively; *Cartan hypersurfaces* [12] and [27]. Remark that $F(3) = 2$, $F(6) = 4$, $F(12) = 8$ and $F(24) = 16$. Each of them is defined by a real cubic polynomial, and it is a closed orbit of projective actions of SO(3), SU(3), Sp(3), F_4 on $S^n = \widetilde{G}_1(\mathbb{R}^{n+1})$, $n = 4$, 7, 13, 25, respectively. Their projective dual $M^\vee = \gamma(M) \in G_1(\mathbb{R}^{(n+1)*}) = \mathbb{R}\mathbb{P}^{n*}$ are the images of Veronese embeddings of projective planes $\mathbb{K}\mathbb{P}^2$, for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ (the Cayley's octonians) respectively. The latter are real parts with respect to properly chosen real forms of *Severi varieties* in complex projective spaces, [45]. The reason of this coincidence has not been explained fully, as far as the authors know.

In [26], the first author showed that a homogeneous compact hypersurface in $\mathbb{RP}^n$ which is tangentially degenerate is projectively equivalent to a hyperplane or a Cartan hypersurface. This result is the motivation of our study. Cartan hypersurfaces are well-known as isoparametric hypersurfaces, and we give some observation concerning this. Moreover, motivated by their sphere bundle structure, we construct many examples of tangentially degenerate submanifolds (§7, §9). Simple examples are obtained as follows:

Lemma 2.2. *Let $\Sigma^k \subset \mathbb{CP}^n$ be a complex submanifold of complex dimension k. Consider the Hopf fibration $\pi\colon S^{2n+1}(\subset \mathbb{C}^{n+1}) \to \mathbb{CP}^n$, and set $M^{2k+1} := \pi^{-1}(\Sigma) \subset S^{2n+1}$. Then M is a submanifold with degenerate Gauss mapping of S^{2n+1}. If Σ is compact and not a complex projective subspace, then the rank of Gauss mapping is equal to $2k$.*

Remark that the codimension of M^{2k+1} equals to $2n+1-(2k+1) = 2(n-k) \geq 2$. In some cases this construction provides examples of compact tangentially degenerate submanifolds satisfying the equality $r = F(l)$: $(l,r) = (3,2)$, $(5,4)$, $(9,8)$, $(17,16)$, $(25,24)$ (see §7, §9).

Proof of Lemma 2.2. The Gauss mapping $\gamma\colon M \to G_{2k+2}(\mathbb{R}^{2n+2})$ is decomposed into the projection $\pi|_M\colon M \to \Sigma$, the complex Gauss mapping $\gamma_{\mathbb{C}}\colon \Sigma \to G_{k+1}(\mathbb{C}^{n+1})$ and the natural inclusion $G_{k+1}(\mathbb{C}^{n+1}) \hookrightarrow G_{2k+2}(\mathbb{R}^{2n+2})$. Therefore M is tangentially degenerate; γ is degenerate along the Hopf fibers. We refer to the following:

Theorem 2.3 ([2] and [21]). *Let Σ^k be a k-dimensional compact complex submanifold in $\mathbb{CP}^n$ and let $\gamma_{\mathbb{C}}\colon \Sigma \to G_{k+1}(\mathbb{C}^{n+1})$ be the complex Gauss mapping of Σ in $\mathbb{CP}^n$. If the rank of $\gamma_{\mathbb{C}}$ is less than $\dim_{\mathbb{C}} \Sigma$, then Σ is necessarily a complex projective subspace $\mathbb{CP}^k$ in $\mathbb{CP}^n$.*

§3. Examples related to isoparametric hypersurfaces

Hypersurfaces in the sphere are tangentially degenerate if they have zero principal curvature. In the simplest case where the principal curvatures are constant, i.e. in the case of isoparametric hypersurfaces (see [43] for general facts), the principal curvatures are given by

$$\lambda_i = \cot\left(\theta_0 + \frac{\pi(i-1)}{g}\right), \quad 0 < \theta_0 < \frac{\pi}{g}, \quad i = 1,\ldots,g$$

where $g = 1, 2, 3, 4, 6$. Then the tangentially degenerate isoparametric hypersurfaces are

(i) $g = 1$ and M is a great hypersphere

(ii) $g = 3$ and M is the Cartan hypersurfaces ([12]).

Each isoparametric hypersurface has two focal submanifolds $M_\pm$, and some of them are tangentially degenerate. It is well known that all the shape operators S_N of $M_\pm$ have constant eigenvalues given by

   (i) 0 for $g = 2$
   (ii) $\pm 1/\sqrt{3}$ for $g = 3$
   (iii) $\pm 1, 0$ for $g = 4$
   (iv) $\pm\sqrt{3}, \pm 1/\sqrt{3}, 0$ for $g = 6$

When $g = 2$, $M_\pm$ are totally geodesic subspheres hence tangentially degenerate. Other possibilities are when $g = 4$ or 6. If the kernel of the shape operators have a common non-trivial vector, they are tangentially degenerate. When $g = 6$ and M is homogeneous, both focal submanifolds are tangentially degenerate [35]. Note that they are given by singular orbits of the linear isotropy representation of the rank two symmetric spaces $G_2/SO(4)$ and $G_2 \times G_2/G_2$. Moreover, these satisfy the Ferus equality for $(l, r) = (5, 4), (10, 8)$.

Examples of isoparametric hypersurfaces with $g = 4$ are given in [39] and [20]. Take the principal curvatures $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ so that m_1 and m_2 are the multiplicities of λ_{odd} and λ_{even} where $m_1 \leq m_2$. Examples in [39] are classified in [20] as

   (i) Homogeneous ones of Clifford type:

$$(m_1, m_2) = (1, k), (2, 2k-1), (4, 4k-1), (9, 6)$$

   (ii) Homogeneous ones with $(m_1, m_2) = (2, 2)$ and $(4, 5)$
   (iii) Inhomogeneous ones of Clifford type: $(m_1, m_2) = (3, 4k), (7, 8k)$

Following [20], we assume $\dim M_- = 2m_1 + m_2$ and $\dim M_+ = m_1 + 2m_2$. Now we summarize results and remarks on tangential degeneracy of these examples.

Proposition 3.1. *Let M be a homogeneous isoparametric hypersurface with $g = 4$ and $(m_1, m_2) = (1, k-2), k \geq 3, (2, 2k-3), (4, 4k-5), k \geq 2$. When $(m_1, m_2) = (1, k-2)$, the focal submanifolds M_+ is tangentially degenerate with $(l, r) = (2k-3, 2k-4)$; while M_- is not. When $(m_1, m_2) = (2, 2k-3), (4, 4k-5)$, M_- is tangentially degenerate with $(l, r) = (2k+1, 2k), (4k+3, 4k)$, respectively; while M_+ is not. In particular, there exist infinitely many tangentially degenerate homogeneous submanifolds in the sphere, which satisfy the Ferus equality.*

On the last assertion, we can easily show that for $p \geq 1$ and $q \geq 2$, $F(2^p + 1) = 2^p$ and $F(2^q + 3) = 2^q$ hold, hence examples are given by M_- of isoparametric hypersurfaces with $(m_1, m_2) = (2, 2^p - 3), p \geq 2$, and $(4, 2^q - 5), q \geq 3$.

Proposition 3.2. *Let M be a homogeneous isoparametric hypersurface with $g = 4$. When $(m_1, m_2) = (2, 2)$, the focal submanifolds M_+ is tangentially degenerate with $(l, r) = (6, 4)$, satisfying the Ferus equality; while M_- is not. When $(m_1, m_2) = (4, 5)$, M_- is tangentially degenerate with $(l, r) = (13, 12)$; while M_+ is not.*

Corollary 3.3. *The focal submanifold M_+ of homogeneous isoparametric hypersurface M with $g = 4$ and $(m_1, m_2) = (1, k - 2)$, $(2, 2)$, and M_- with $(2, 2k - 3)$, $(4, 4k - 5)$, $(4, 5)$ is foliated by totally geodesic subspheres of dimension 1, 2 and 1, 3, 1, respectively. Along these subspheres, the tangent space is parallel.*

Remark 3.4. As for the case (iii), M_+ is not tangentially degenerate, since by Theorem 5.8 in [20], there exists a point $x \in M_+$ at which $d(x) = \dim \bigcap_n \operatorname{Ker} S_N = 0$. In the case $(m_1, m_2) = (3, 4k)$, M_- is homogeneous (6.4 in [20]). We will discuss these and the remaining homogeneous case with $(m_1, m_2) = (9, 6)$ in another occasion, as well as all other inhomogeneous examples of Clifford type. The tangential degeneracy of M_+ for $(m_1, m_2) = (1, k - 2)$ and of M_- for $(2, 2k - 3)$, $(4, 5)$ follows from Lemma 2.2, since the odd dimensional focal submanifolds given by singular orbits of the linear isotropy representation of a Hermitian symmetric space of rank two is tangentially degenerate. Then use the classification in [24]. In the Appendix, we give a systematic proof of the propositions, which is applicable to all the homogeneous cases.

§4. Stiefel manifolds and complex quadrics

Let W be a real vector space with Euclidean inner product $\langle \, , \, \rangle$. By a 2-*frame* in W we mean an ordered set of 2 orthonormal vectors in W. Let $V_2(W)$ be the space of 2-frames in W, i.e.,

$$(3) \qquad V_2(W) = \{(\mathbf{f}_1, \mathbf{f}_2) \in W \times W \mid \langle \mathbf{f}_\alpha, \mathbf{f}_\beta \rangle = \delta_{\alpha\beta} \ (\alpha, \beta = 1, 2)\}.$$

Then $V_2(W)$ is a Stiefel manifold with $\dim_{\mathbb{R}} V_2(W) = 2 \dim_{\mathbb{R}} W - 3$. The tangent space $T_{(\mathbf{f}_1, \mathbf{f}_2)} V_2(W)$ is

$$\mathbb{R}(-\mathbf{f}_2, \mathbf{f}_1) \oplus \{(\mathbf{x}_1, \mathbf{x}_2) \in W \times W \mid \mathbf{x}_1, \mathbf{x}_2 \perp \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}\}.$$

The inner product on $W \times W$ defined by

$$\langle (\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \rangle = \langle \mathbf{x}_1, \mathbf{y}_1 \rangle + \langle \mathbf{x}_2, \mathbf{y}_2 \rangle$$
$$\text{for} \quad (\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \in W \times W$$

induces a Riemannian metric $\widetilde{g}$ on $V_2(W)$.

Let $\widetilde{G}_2(W)$ be the space of oriented 2-planes in W. Then $V_2(W)$ is a principal fiber bundle over $\widetilde{G}_2(W)$ with structure group S^1 and projection map $\pi\colon V_2(W) \to \widetilde{G}_2(W)$ defined by

$$\pi((\mathbf{f}_1, \mathbf{f}_2)) = \mathrm{span}\{\mathbf{f}_1, \mathbf{f}_2\}.$$

For each $(\mathbf{f}_1, \mathbf{f}_2) \in V_2(W)$, the fiber $\pi^{-1}(\pi(\mathbf{f}_1, \mathbf{f}_2))$ is

$$\{(\cos\theta\,\mathbf{f}_1 - \sin\theta\,\mathbf{f}_2, \sin\theta\,\mathbf{f}_1 + \cos\theta\,\mathbf{f}_2) \in V_2(W) \mid \theta \in S^1\}.$$

The Riemannian metric $\widetilde{g}$ on $V_2(W)$ is invariant by the structure group. Thus we may define a Riemannian metric g on $\widetilde{G}_2(W)$ such that π is a Riemannian submersion.

The distribution given by

(4)
$$\begin{aligned} &T'_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W)) \\ &= \{(\mathbf{x}_1, \mathbf{x}_2) \in T_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W)) \mid \mathbf{x}_1, \mathbf{x}_2 \perp \mathrm{span}\{\mathbf{f}_1, \mathbf{f}_2\}\} \end{aligned}$$

defines a connection in the principal fiber bundle $V_2(W)(\widetilde{G}_2(W), S^1)$, because $T'_{(\mathbf{f}_1,\mathbf{f}_2)}$ is complementary to the subspace $\mathbb{R}(-\mathbf{f}_2, \mathbf{f}_1)$ tangent to the fiber through $(\mathbf{f}_1, \mathbf{f}_2)$, and invariant under the S^1-action. The natural projection π of $V_2(W)$ onto $\widetilde{G}_2(W)$ induces a linear isomorphism of $T'_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W))$ onto $T_p(\widetilde{G}_2(W))$, where $\pi((\mathbf{f}_1, \mathbf{f}_2)) = p$. The complex structure $\widetilde{J}$ on $T'_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W))$ defined by

(5)
$$(\mathbf{x}_1, \mathbf{x}_2) \mapsto (-\mathbf{x}_2, \mathbf{x}_1)$$

induces a canonical complex structure J on $\widetilde{G}_2(W)$ through $d\pi$.

Let $\widetilde{Q}(\mathbb{C}^{m+1})$ be a submanifold of $S^{2m+1}(\sqrt{2})$ defined by

(6)
$$\widetilde{Q}(\mathbb{C}^{m+1}) = \{\mathbf{z} \in S^{2m+1}(\sqrt{2}) \mid {}^t\mathbf{z}\mathbf{z} = 0\}.$$

There is an identification between $\widetilde{Q}(\mathbb{C}^{m+1})$ and $V_2(\mathbb{R}^{m+1})$ as:

$$\widetilde{Q}(\mathbb{C}^{m+1}) \ni \mathbf{z} \mapsto (\mathrm{Re}\,\mathbf{z}, \mathrm{Im}\,\mathbf{z}) \in V_2(\mathbb{R}^{m+1}).$$

Then $\widetilde{G}_2(\mathbb{R}^{m+1})$ is identified with the complex quadric

(7)
$$Q^{m-1} = \{\pi(\mathbf{z}) \in \mathbb{CP}^m \mid \mathbf{z} \in \widetilde{Q}(\mathbb{C}^{m+1})\},$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{Q}(\mathbb{C}^{m+1}) & \xrightarrow{\ \sim\ } & V_2(\mathbb{R}^{m+1}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Q^{m-1} & \xrightarrow{\ \sim\ } & \widetilde{G}_2(\mathbb{R}^{m+1}). \end{array}$$

§5. Complex submanifolds in complex quadrics

Consider an isometric immersion $\varphi\colon \Sigma \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ of a Riemannian manifold Σ with $\dim_{\mathbb{R}} \Sigma = l$ to the complex quadric. Let $\eta\colon U' \to V_2(\mathbb{R}^{n+1})$ be a local cross section of the circle bundle $\pi\colon V_2(\mathbb{R}^{n+1}) \to \widetilde{G}_2(\mathbb{R}^{n+1})$ on an open set $U' \subset \widetilde{G}_2(\mathbb{R}^{n+1})$ and let U be an open neighborhood of a point p in Σ such that $\varphi(U) \subset U'$. Denote

$$(8) \qquad (\eta \circ \varphi)(q) = (\mathbf{f}_1(q), \mathbf{f}_2(q)) \quad \text{for} \quad q \in U \subset \Sigma.$$

Here $\mathbf{f}_\alpha$ is an $\mathbb{R}^{n+1}$-valued function on U with $\langle \mathbf{f}_\alpha, \mathbf{f}_\beta \rangle = \delta_{\alpha\beta}$ ($\alpha,\ \beta = 1,\ 2$). Write differential maps of $\mathbf{f}_\alpha\colon U \to \mathbb{R}^{n+1}$ ($\alpha = 1,\ 2$) as

$$(9) \qquad d\mathbf{f}_1(X) = \lambda(X)\mathbf{f}_2 + \mathbf{p}(X), \quad d\mathbf{f}_2(X) = -\lambda(X)\mathbf{f}_1 + \mathbf{q}(X)$$
$$\text{for} \quad X \in T_q(\Sigma),$$

where λ is a 1-form on U, and $\mathbf{p}$, $\mathbf{q}$ are $\mathbb{R}^{n+1}$-valued 1-forms on U such that $\mathbf{p}(X)$, $\mathbf{q}(X) \perp \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}$. Hence differentials of $\eta \circ \varphi$ and φ are given by

$$\begin{aligned}
d(\eta \circ \varphi)(X) &= (d\mathbf{f}_1(X), d\mathbf{f}_2(X)) \\
&= \big(\lambda(X)\mathbf{f}_2 + \mathbf{p}(X), -\lambda(X)\mathbf{f}_1 + \mathbf{q}(X)\big), \\
d\varphi(X) &= (d\pi \circ d(\eta \circ \varphi))(X) \\
(10) \qquad\qquad &= d\pi(\mathbf{p}(X), \mathbf{q}(X)).
\end{aligned}$$

Suppose that (Σ^m, J) is a Kähler manifold with $\dim_{\mathbb{C}} \Sigma = m$ and $\varphi\colon \Sigma^m \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ ($m < n-1$) is a holomorphic isometric immersion. Then (5), (10) and $d\varphi \circ J = J \circ d\varphi$ imply

$$\begin{aligned}
d\pi(\mathbf{p}(JX), \mathbf{q}(JX)) &= d\varphi(JX) = Jd\varphi(X) = Jd\pi(\mathbf{p}(X), \mathbf{q}(X)) \\
&= d\pi(-\mathbf{q}(X), \mathbf{p}(X)),
\end{aligned}$$

and $\mathbf{q}(X) = -\mathbf{p}(JX)$. Hence

$$d\varphi(T_q(\Sigma)) = \{d\pi(\mathbf{p}(X), -\mathbf{p}(JX)) \mid X \in T_q(\Sigma)\}.$$

With respect to the metrics on Σ and Q^{n-1}, we have

$$\begin{aligned}
\langle d\varphi(X), d\varphi(Y) \rangle &= \langle \mathbf{p}(X), \mathbf{p}(Y) \rangle + \langle \mathbf{p}(JX), \mathbf{p}(JY) \rangle \\
(11) \qquad\qquad &= \langle X, Y \rangle.
\end{aligned}$$

Let ∇^G and ∇^V be the Levi-Civita connections on $\widetilde{G}_2(\mathbb{R}^{n+1})$ and $V_2(\mathbb{R}^{n+1})$, respectively. If X and Y are vector fields on Σ, then we have

$\nabla^G_{d\varphi(X)}d\varphi(Y) = d\pi(\nabla^V_{d\varphi(X)'}d\varphi(Y)')$ where $d\varphi(X)'$ and $d\varphi(Y)'$ are the basic vector fields corresponding to $d\varphi(X)$ and $d\varphi(Y)$, respectively. Let σ^φ denote the second fundamental form of φ. Then we can see that

$$\sigma^\varphi(X,Y) = \text{component of } d\pi((\nabla_X \mathbf{p})Y, -(\nabla_X \mathbf{p})JY)$$
$$\text{orthogonal to } d\varphi(T_q(\Sigma)) \text{ in } T_{\varphi(q)}(\widetilde{G}_2(\mathbb{R}^{n+1})),$$

where $\nabla \mathbf{p}$ denotes the covariant derivative of $\mathbf{p}\colon T_q(\Sigma) \to \mathbb{R}^{n+1}$. Let $\mathbf{s}\colon T_q(\Sigma) \times T_q(\Sigma) \to \mathbb{R}^{n+1}$ be a bilinear mapping defined by

(12) $\mathbf{s}(X,Y) = \text{the component of } (\nabla_X \mathbf{p})Y \text{ orthogonal to}$
$$\{\mathbf{p}(X) \mid X \in T_q(\Sigma)\} \text{ in } \mathbb{R}^{n+1}.$$

Then $\sigma^\varphi(X,Y) = \sigma^\varphi(Y,X)$ and $\sigma^\varphi(X,Y) + \sigma^\varphi(JX,JY) = 0$ imply that

(13) $\mathbf{s}(X,Y) = \mathbf{s}(Y,X), \quad \mathbf{s}(X,Y) + \mathbf{s}(JX,JY) = 0.$

§6. Submanifolds with degenerate Gauss mapping in spheres

Let $\varphi\colon \Sigma \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ be a mapping from a differentiable manifold Σ with $\dim_\mathbb{R}\Sigma = l$ to the complex quadric, and let $\pi_\varphi\colon \varphi^*V_2(\mathbb{R}^{n+1}) \to \Sigma$ be the pullback bundle of the circle bundle $\pi\colon V_2(\mathbb{R}^{n+1}) \to \widetilde{G}_2(\mathbb{R}^{n+1})$ with respect to φ:

(14)
$$\begin{array}{ccc}
\varphi^*V_2(\mathbb{R}^{n+1}) & \xrightarrow{\psi} & V_2(\mathbb{R}^{n+1}) \\
\pi_\varphi \downarrow & & \downarrow \pi \\
\Sigma & \xrightarrow{\varphi} & \widetilde{G}_2(\mathbb{R}^{n+1}).
\end{array}$$

Let $\Phi\colon \varphi^*V_2(\mathbb{R}^{n+1}) \to S^n(1)$ be the mapping defined by

(15) $\Phi = \mathrm{pr}_1 \circ \psi,$

where $\psi\colon \varphi^*V_2(\mathbb{R}^{n+1}) \to V_2(\mathbb{R}^{n+1})$ is the bundle mapping in (14) and $\mathrm{pr}_1\colon V_2(\mathbb{R}^{n+1}) \to S^n(1)$ is the projection given by

$$\mathrm{pr}_1(\mathbf{f}_1, \mathbf{f}_2) = \mathbf{f}_1.$$

Then we have

$$\Phi(\varphi^*V_2(\mathbb{R}^{n+1})) = \bigcup_{p\in\Sigma} \{\cos\theta\,\mathbf{f}_1 + \sin\theta\,\mathbf{f}_2 \mid \pi(\mathbf{f}_1,\mathbf{f}_2) = \varphi(p),\ \theta \in S^1\}.$$

Hence $\Phi(\varphi^* V_2(\mathbb{R}^{n+1}))$ is a union of (real) l-parameter family of great circles in $S^n(1)$.

By the local triviality of the circle bundle $\pi_\varphi \colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$, every point p of Σ has an open neighborhood U such that there is a diffeomorphism $\phi_U \colon \pi_\varphi^{-1}(U) \to S^1 \times U$ defined by $\phi_U(u) = (\theta(u), \pi_\varphi(u))$ where θ is a mapping of $\pi_\varphi^{-1}(U)$ into S^1 satisfying $\theta(ua) = \theta(u)a$ for all $u \in \pi_\varphi^{-1}(U)$ and $a \in S^1$. Let η' be a local cross section of $\pi_\varphi \colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ on U, and denote $(\psi \circ \eta')(q) = (\mathbf{f}_1(q), \mathbf{f}_2(q))$ for $q \in U$. Using (7), we see that oriented 2-plane spanned by $(\mathbf{f}_1(q), \mathbf{f}_2(q)) \in V_2(\mathbb{R}^{m+1})$ is identified with $\varphi(q) \in Q^{n-1}$. We define the mapping $\Phi_U = \Phi \circ \phi_U^{-1} \colon S^1 \times U \to S^n$, where Φ is defined by (15). Then Φ_U is written as

$$(16) \qquad \Phi_U(\theta, q) = \cos\theta\, \mathbf{f}_1(q) + \sin\theta\, \mathbf{f}_2(q), \quad (\theta, q) \in S^1 \times U.$$

Let X be a tangent vector of Σ at q in $U \subset \Sigma$. Then (9) and (16) yield that the differential of Φ_U is

$$(17) \quad d\Phi_U(\partial/\partial\theta, 0) = -\sin\theta\, \mathbf{f}_1 + \cos\theta\, \mathbf{f}_2,$$

$$d\Phi_U(0, X) = \cos\theta\, (\lambda(X)\mathbf{f}_2 + \mathbf{p}(X)) + \sin\theta\, (-\lambda(X)\mathbf{f}_1 + \mathbf{q}(X))$$
$$(18) \qquad\qquad = \lambda(X)\, d\Phi_U(\partial/\partial\theta, 0) + \cos\theta\, \mathbf{p}(X) + \sin\theta\, \mathbf{q}(X).$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_l$ be an orthonormal basis of the tangent space $T_q(\Sigma)$ at $q \in U \subset \Sigma$. Then the mapping Φ_U is non-singular at $(\theta, q) \in S^1 \times U$ if and only if

$$(19) \qquad \Phi_U \wedge d\Phi_U(\partial/\partial\theta, 0) \wedge d\Phi_U(0, \mathbf{e}_1) \wedge \cdots \wedge d\Phi_U(0, \mathbf{e}_l) \neq 0.$$

By (16) and (17) we have

$$(20) \qquad\qquad\qquad \Phi_U \wedge d\Phi_U(0, \partial/\partial\theta) = \mathbf{f}_1 \wedge \mathbf{f}_2.$$

With respect to $\mathbb{R}^{n+1}$-valued 1-forms $\mathbf{p}, \mathbf{q}$ on $U \subset \Sigma$ defined by (10), denote $\mathbf{p}(\mathbf{e}_j) = \mathbf{p}_j$ and $\mathbf{q}(\mathbf{e}_j) = \mathbf{q}_j$ for $j = 1, \ldots, l$, and put

$$(21) \qquad \Psi_j = \cos\theta\, \mathbf{p}_j + \sin\theta\, \mathbf{q}_j \in T_{\Phi_U(\theta, q)}(S^n) \quad \text{for} \quad j = 1, \ldots, l.$$

Then by (18) and (20), (19) is equivalent to

$$(22) \qquad\qquad\qquad \Psi_1 \wedge \cdots \wedge \Psi_l \neq 0.$$

Consequently we obtain

Proposition 6.1. *The mapping* $\Phi_U \colon S^1 \times U \to S^n(1)$ *defined by* (16) *is non-singular at* (q, θ) *if and only if at* $q \in U \subset \Sigma$, *the mapping* $\varphi \colon \Sigma \to Q^{n-1}$ *satisfies*

$$\cos^l \theta \, \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_l$$

$$+ \cos^{l-1} \theta \sin \theta \, (\mathbf{q}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_l + \cdots + \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_{l-1} \wedge \mathbf{q}_l)$$

$$\cdots$$

$$+ \cos \theta \sin^{l-1} \theta \, (\mathbf{p}_1 \wedge \mathbf{q}_2 \wedge \cdots \wedge \mathbf{q}_l + \cdots + \mathbf{q}_1 \wedge \cdots \wedge \mathbf{q}_{l-1} \wedge \mathbf{p}_l)$$

$$+ \sin^l \theta \, \mathbf{q}_1 \wedge \cdots \wedge \mathbf{q}_l \neq 0.$$

Suppose that (Σ^m, J) is a Kähler manifold with $\dim_{\mathbb{C}} \Sigma = m$ and $\varphi \colon \Sigma^m \to Q^{n-1}$ $(m < n - 1)$ is a holomorphic isometric immersion. Let $\{\mathbf{e}_{2k-1}, \mathbf{e}_{2k} = J\mathbf{e}_{2k-1} \mid k = 1, \ldots, m\}$ be an orthonormal basis of the tangent space $T_q(\Sigma^m)$ at $q \in U \subset \Sigma^m$. Then we obtain

$$\begin{aligned}
\Psi_{2k-1} \wedge \Psi_{2k} &= (\cos \theta \, \mathbf{p}_{2k-1} + \sin \theta \, \mathbf{q}_{2k-1}) \wedge (\cos \theta \, \mathbf{p}_{2k} + \sin \theta \, \mathbf{q}_{2k}) \\
&= (\cos \theta \, \mathbf{p}_{2k-1} - \sin \theta \, \mathbf{p}_{2k}) \wedge (\cos \theta \, \mathbf{p}_{2k} + \sin \theta \, \mathbf{p}_{2k-1}) \\
&= \mathbf{p}_{2k-1} \wedge \mathbf{p}_{2k}, \quad (k = 1, \ldots, m)
\end{aligned}$$

and $\Psi_1 \wedge \cdots \wedge \Psi_{2m} = \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_{2m}$. By (22) we get

Proposition 6.2. *Let* (Σ^m, J) *be a Kähler manifold of* $\dim_{\mathbb{C}} \Sigma = m$ *and let* $\varphi \colon \Sigma^m \to Q^{n-1}$ $(m < n - 1)$ *be a holomorphic immersion. Then the mapping* $\Phi \colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ *defined by* (15) *is non-singular at each point in* $\pi_\varphi^{-1}(q)$ *if and only if at* $q \in \Sigma^m$, φ *satisfies*

$$(23) \qquad\qquad \mathbf{p}_1 \wedge \cdots \wedge \mathbf{p}_{2m} \neq 0.$$

Remark 6.3. Jensen-Rigoli-Yang [28] studied holomorphic curves $(m = 1)$ in complex quadrics. A point p in the holomorphic curve $\varphi \colon \Sigma^1 \to Q^{n-1}$ is called a *real point* if $\mathbf{p}_1 \wedge \mathbf{p}_2 = 0$ at p. They showed that if any $p \in \Sigma^1$ is a real point, then $\varphi(\Sigma^1)$ is contained in a totally geodesic Q^1 in Q^{n-1}.

Suppose that a holomorphic immersion $\varphi \colon \Sigma^m \to Q^{n-1}$ satisfies (23) at each point of Σ^m. Let $V = d\phi_U^{-1}(\partial/\partial\theta, 0)$ be a tangent vector of the fiber $\pi_\varphi^{-1}(q)$ of the submersion $\pi_\varphi \colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ at $q \in \Sigma$. Then $\mathbb{R} \, d\Phi(V) + \mathrm{span}\{\Psi_j \mid j = 1, \ldots, 2m\} = \mathbb{R} \, d\Phi_U(\partial/\partial\theta, 0) + \{d\Phi_U(0, X) \mid X \in T_q(\Sigma)\}$. Denote σ^Φ the second fundamental form of the immersion $\Phi \colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$. Since each fiber $\pi_\varphi^{-1}(q)$ is a great circle of $S^n(1)$, we have

$$(24) \qquad\qquad \sigma^\Phi(V, V) = 0.$$

On the other hand, if we denote D the Euclidean connection of $\mathbb{R}^{n+1}$, then using (21) we get

$$D_{\partial/\partial\theta}\Psi_{2k-1} = -\sin\theta\,\mathbf{p}_{2k-1} - \cos\theta\,\mathbf{p}_{2k},$$
$$D_{\partial/\partial\theta}\Psi_{2k} = -\sin\theta\,\mathbf{p}_{2k} + \cos\theta\,\mathbf{p}_{2k-1},$$

$(k = 1, \ldots, m)$ and both of these terms are contained in the tangent space of $\Phi(\varphi^* V_2(\mathbb{R}^{n+1}))$. Hence we obtain

(25) $$\sigma^{\Phi}(V, d\phi_U^{-1}(0, X)) = 0 \quad \text{for} \quad X \in T_q(\Sigma).$$

Combining with (2), (24) and (25), we obtain a generalization of Lemma 2.2:

Theorem 6.4. *Let* $\varphi\colon \Sigma^m \to Q^{n-1}$ $(m < n-1)$ *be a holomorphic immersion from a Kähler manifold* Σ^m *to the complex quadric for which* (23) *holds. Then with respect to the immersion* $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ *given by* (15), *any tangent line of the fiber* $\pi_\varphi^{-1}(p)$ *at each* $u \in \pi_\varphi^{-1}(p)$, $p \in \Sigma^m$ *lies in the kernel of differential of the Gauss mapping of* φ. *Hence* Φ *is tangentially degenerate.*

Remark 6.5. Let $\pi\colon S^{2m+1}(1) \to \mathbb{CP}^m$ be the Hopf fibration, and let $f\colon M \to \mathbb{CP}^m$ be an isometric immersion from a *real* l-dimensional Riemannian manifold M. Then there is a natural immersion $F\colon \pi^{-1}(M) \to S^{2m+1}(1)$ such that the following diagram is commutative:

(26)
$$
\begin{array}{ccc}
\pi^{-1}(M) & \xrightarrow{\;F\;} & S^{2m+1}(1) \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
M & \xrightarrow{\;f\;} & \mathbb{CP}^m.
\end{array}
$$

It can be seen that any tangent line of each fiber $\pi^{-1}(p)$ $(p \in M)$ of the S^1-bundle $\pi\colon \pi^{-1}(M) \to M$ lies in

$$\{X \in T_x(\pi^{-1}M) \mid \sigma^F(X, Y) = 0 \text{ for any } Y \in T_x(M)\}, \quad (\pi(x) = p)$$

if and only if l is even and M is a complex submanifold of $\mathbb{CP}^m$.

Let $\varphi\colon \Sigma^k \to Q^{2m}$ $(k < 2m)$ be a holomorphic immersion from a Kähler manifold Σ^k with $\dim_{\mathbb{C}} \Sigma = k$ to the complex quadric Q^{2m} satisfying (23). Then the immersion $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^{2m+1}(1)$ defined by (15) is congruent to the inverse image of a complex submanifold M^k in $\mathbb{CP}^m$ if and only if $\varphi(\Sigma^k)$ is contained in a totally geodesic $\mathbb{CP}^m$ in Q^{2m} [16] and [29]. In fact the set of fibers of the Hopf fibration π is identified with totally geodesic $\mathbb{CP}^m$ in Q^{2m}.

§7. Examples

Now we give examples of homogeneous complex submanifolds of complex quadrics, which satisfy the assumption of Theorem 6.4.

Let $M(m + 1, \mathbb{C})$ be the space of all complex $(m + 1) \times (m + 1)$ matrices, and let

$$\mathrm{Sym}^{\mathbb{C}}(m + 1) = \{A \in M(m + 1, \mathbb{C}) \mid {}^tA = A\},$$
$$\mathrm{Sym}_0^{\mathbb{C}}(m + 1) = \{A \in \mathrm{Sym}^{\mathbb{C}}(m + 1) \mid \mathrm{trace}\, A = 0\}.$$

Then $\mathrm{Sym}^{\mathbb{C}}(m + 1)$ (resp. $\mathrm{Sym}_0^{\mathbb{C}}(m + 1)$) can be considered as a real $(m + 1)(m + 2)$ (resp. $m(m + 3)$)-dimensional vector space with the inner product given by

$$(27) \qquad \langle A_1, A_2 \rangle = \frac{1}{2}\,\mathrm{Re}(\mathrm{trace}(A_1 A_2^*)).$$

Similarly, let

$$\mathrm{Sym}^{\mathbb{R}}(m + 1) = \{A \in M(m + 1, \mathbb{R}) \mid {}^tA = A\},$$
$$\mathrm{Sym}_0^{\mathbb{R}}(m + 1) = \{A \in \mathrm{Sym}^{\mathbb{R}}(m + 1) \mid \mathrm{trace}\, A = 0\}.$$

Then $\mathrm{Sym}_0^{\mathbb{R}}(m + 1)$ can be considered as a real $m(m + 3)/2$-dimensional vector space with the inner product given by

$$(28) \qquad \langle B_1, B_2 \rangle = \frac{1}{2}\,\mathrm{Re}(\mathrm{trace}(B_1 B_2)).$$

Let $S_r(\mathrm{Sym}^{\mathbb{C}}(m + 1))$ (resp. $S_r(\mathrm{Sym}_0^{\mathbb{C}}(m + 1))$) be the hypersphere of radius $r > 0$ in $\mathrm{Sym}^{\mathbb{C}}(m + 1)$ (resp. $\mathrm{Sym}_0^{\mathbb{C}}(m + 1)$), and let $\mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m + 1))$ (resp. $\mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m + 1))$) be the complex projective space which is considered as the quotient space of $S_r(\mathrm{Sym}^{\mathbb{C}}(m + 1))$ (resp. $S_r(\mathrm{Sym}_0^{\mathbb{C}}(m + 1))$) obtained by identifying v with λv, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Let $\pi: S_r(\mathrm{Sym}^{\mathbb{C}}(m + 1)) \to \mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m + 1))$ (resp. $\pi: S_r(\mathrm{Sym}_0^{\mathbb{C}}(m + 1)) \to \mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m + 1))$) be the *Hopf fibration*.

Define a mapping $\widetilde{\varphi}_m : S^{2m+1}(\sqrt{2}) \to \mathrm{Sym}^{\mathbb{C}}(m + 1)$ as

$$(29) \qquad \widetilde{\varphi}_m(\mathbf{z}) = \frac{1}{\sqrt{2}}\mathbf{z}^t\mathbf{z} = \frac{1}{\sqrt{2}}\begin{pmatrix} z_0^2 & z_0 z_1 & \cdots & z_0 z_m \\ z_1 z_0 & z_1^2 & \cdots & z_1 z_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m z_0 & z_m z_1 & \cdots & z_m^2 \end{pmatrix}$$

for $\mathbf{z} = (z_j) \in S^{2m+1}(\sqrt{2})$. Then it can be verified that $\langle \widetilde{\varphi}_m(\mathbf{z}), \widetilde{\varphi}_m(\mathbf{z}) \rangle = 1$ with respect to (27) for $\mathbf{z} \in S^{2m+1}(\sqrt{2})$ and $\widetilde{\varphi}_m$ induces a mapping

φ_m of $\mathbb{CP}^m$ into $\mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1)) \cong \mathbb{CP}^{m(m+3)/2}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
S^{2m+1}(\sqrt{2}) & \xrightarrow{\;\widetilde{\varphi}_m\;} & S_1(\mathrm{Sym}^{\mathbb{C}}(m+1)) \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{CP}^m & \xrightarrow{\;\varphi_m\;} & \mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1)).
\end{array}
$$

φ_m is nothing but the *complex Veronese embedding*.

If we restrict $\widetilde{\varphi}_m$ to the submanifold $\widetilde{Q}(\mathbb{C}^{m+1})$ which is given by (6), then the image $\widetilde{\varphi}_m(\widetilde{Q}(\mathbb{C}^{m+1}))$ is contained in the submanifold

$$(30) \qquad \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) = \{A \in S_1(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid \mathrm{trace}\, A^2 = 0\}.$$

Putting $W = \mathrm{Sym}_0^{\mathbb{C}}(m+1)$ in the argument in §4, we have a commutative diagram:

$$
\begin{array}{ccc}
\widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) & \xrightarrow{\;\sim\;} & V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)) \\
\pi \downarrow & & \downarrow \pi \\
Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) & \xrightarrow{\;\sim\;} & \widetilde{G}_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)),
\end{array}
$$

where we can write

$$Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) = \{\pi(A) \in \mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid \mathrm{trace}\, A^2 = 0\}.$$

Consider a Riemannian metric on $\widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ which is induced by the inclusion into $\mathrm{Sym}_0^{\mathbb{C}}(m+1)$ of (27). Then we have the Riemannian metric on $Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ such that the fibering $\pi\colon \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ is a Riemannian submersion. Hence $\widetilde{\varphi}_m$ induces a mapping φ_m of Q^{m-1} into $Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \cong Q^{(m-1)(m+4)/2}$ such that the following diagram is commutative:

$$(31) \qquad
\begin{array}{ccc}
\widetilde{Q}(\mathbb{C}^{m+1}) & \xrightarrow{\;\widetilde{\varphi}_m\;} & \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \\
\pi \downarrow & & \downarrow \pi \\
Q^{m-1} & \xrightarrow{\;\varphi_m\;} & Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)).
\end{array}
$$

We can see that the image of $\widetilde{Q}(\mathbb{C}^{m+1})$ under $\widetilde{\varphi}_m$ is given by

$$\widetilde{\varphi}_m(\widetilde{Q}(\mathbb{C}^{m+1})) = \{A \in \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid A^2 = 0,\ AA^*A = 4A\}.$$

The special orthogonal group $\mathrm{SO}(m+1)$ acts on Q^{m-1} isometrically as

$$(32) \qquad \pi(\mathbf{z}) \mapsto \pi(T\mathbf{z}), \quad (T \in \mathrm{SO}(m+1),\ \mathbf{z} \in \widetilde{Q}(\mathbb{C}^{m+1})).$$

This action is transitive [30, p. 279] and the isotropy subgroup at the point $o = \pi({}^t(1, i, 0, \dots, 0)) \in Q^{m-1}$ is $SO(2) \times SO(m-1)$. $SO(m+1)$ acts on $Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ isometrically as

(33)
$$\pi(A) \mapsto \pi(TAT^{-1}),$$

$$(T \in SO(m+1), \ A \in \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1))).$$

By (29) and (31), we have

$$\varphi_m(\pi(T\mathbf{z})) = \pi(T\varphi_m(\mathbf{z})T^{-1}).$$

Hence *the embedding* $\varphi_m \colon Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ *given by* (29) *and* (31) *is equivariant with respect to the actions* (32) *and* (33) *of* $SO(m+1)$.

We calculate differential of $\varphi_m \colon Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ at $\pi(\mathbf{f}_1 + i\mathbf{f}_2) \in Q^{m-1}$. Here we identify (31) with

$$
\begin{array}{ccc}
V_2(\mathbb{R}^{m+1}) & \xrightarrow{\ \widetilde{\varphi}_m\ } & V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)) \\
\pi \downarrow & & \downarrow \pi \\
\widetilde{G}_2(\mathbb{R}^{m+1}) & \xrightarrow{\ \varphi_m\ } & \widetilde{G}_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)).
\end{array}
$$

Let $t \mapsto (\mathbf{f}_1(t), \mathbf{f}_2(t))$ be a curve in $V_2(\mathbb{R}^{m+1})$ such that $(\mathbf{f}_1(0), \mathbf{f}_2(0)) = (\mathbf{f}_1, \mathbf{f}_2)$ and $(\mathbf{f}_1'(0), \mathbf{f}_2'(0)) = (\mathbf{x}_1, \mathbf{x}_2) \in T'_{(\mathbf{f}_1, \mathbf{f}_2)}(V_2(\mathbb{R}^{m+1}))$ (cf. (4)). Then we have

$$\widetilde{\varphi}_m(\mathbf{f}_1(t), \mathbf{f}_2(t)) = \frac{1}{\sqrt{2}}\left(\mathbf{f}_1(t){}^t\mathbf{f}_1(t) - \mathbf{f}_2(t){}^t\mathbf{f}_2(t), \mathbf{f}_1(t){}^t\mathbf{f}_2(t) + \mathbf{f}_2(t){}^t\mathbf{f}_1(t)\right)$$

and

$$d\widetilde{\varphi}_m(\mathbf{x}_1, \mathbf{x}_2) = \frac{d}{dt}\bigg|_{t=0} \widetilde{\varphi}_m(\mathbf{f}_1(t), \mathbf{f}_2(t))$$

$$= \frac{1}{\sqrt{2}}\left(\mathbf{x}_1{}^t\mathbf{f}_1 + \mathbf{f}_1{}^t\mathbf{x}_1 - \mathbf{x}_2{}^t\mathbf{f}_2 - \mathbf{f}_2{}^t\mathbf{x}_2, \mathbf{x}_1{}^t\mathbf{f}_2 + \mathbf{f}_1{}^t\mathbf{x}_2 + \mathbf{x}_2{}^t\mathbf{f}_1 + \mathbf{f}_2{}^t\mathbf{x}_1\right)$$

$$\in T'_{\widetilde{\varphi}_m(\mathbf{f}_1, \mathbf{f}_2)}\left(V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1))\right).$$

Hence

$$d\varphi_m(d\pi(\mathbf{x}_1, \mathbf{x}_2))$$

$$= \frac{1}{\sqrt{2}}d\pi\left(\mathbf{x}_1{}^t\mathbf{f}_1 + \mathbf{f}_1{}^t\mathbf{x}_1 - \mathbf{x}_2{}^t\mathbf{f}_2 - \mathbf{f}_2{}^t\mathbf{x}_2, \mathbf{x}_1{}^t\mathbf{f}_2 + \mathbf{f}_1{}^t\mathbf{x}_2 + \mathbf{x}_2{}^t\mathbf{f}_1 + \mathbf{f}_2{}^t\mathbf{x}_1\right)$$

implies that φ_m is holomorphic, i.e., $d\varphi_m \circ J = J \circ d\varphi_m$. With respect to the notation (9), we obtain

$$\mathbf{p}(d\pi(\mathbf{x}_1, \mathbf{x}_2)) = \frac{1}{\sqrt{2}}(\mathbf{x}_1{}^t\mathbf{f}_1 + \mathbf{f}_1{}^t\mathbf{x}_1 - \mathbf{x}_2{}^t\mathbf{f}_2 - \mathbf{f}_2{}^t\mathbf{x}_2) \in \mathrm{Sym}_0^{\mathbb{R}}(m+1),$$

and

$$\langle \mathbf{p}(d\pi(\mathbf{x}_1, \mathbf{x}_2)), \mathbf{p}(d\pi(\mathbf{y}_1, \mathbf{y}_2)) \rangle = \frac{1}{2}({}^t\mathbf{x}_1\mathbf{y}_1 + {}^t\mathbf{x}_2\mathbf{y}_2)$$

(34)
$$= \frac{1}{2}\langle d\pi(\mathbf{x}_1, \mathbf{x}_2), d\pi(\mathbf{y}_1, \mathbf{y}_2) \rangle.$$

In particular, the holomorphic embedding $\varphi_m \colon Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ satisfies (23). Therefore, applying Theorem 6.4, we obtain homogeneous examples of tangentially degenerate submanifolds:

$$\Phi \colon \varphi_m^* V_2(\mathbb{R}^{m+1}) \to S^{(m^2+3m-2)/2}(1)$$

§8. Austere submanifolds in spheres

A submanifold M in a Riemannian manifold is called *austere* [23] if for each normal vector ξ, the set of eigenvalues of the shape operator A_ξ is invariant under multiplication of -1. Clearly austere submanifolds are minimal, and they are closely related to *special Lagrangian submanifolds* (see also [10]). In fact, Harvey and Lawson showed (Theorem 3.17 in [23]) that from any compact austere submanifold of S^n, one can construct an $n+1$-dimensional special Lagrangian cone of least mass in $\mathbb{R}^{2n+2}$. In this section we will show that if Σ is a complex submanifold of *first order isotropic* in Q^{n-1}, then the corresponding submanifold M in S^n with 2-parameter family of great circles is austere, as well as tangentially degenerate.

We will use notations of §5 and §6. Let $(\Sigma^m, \langle\ ,\ \rangle, J)$ be a Kähler manifold of $\dim_{\mathbb{C}} \Sigma = m$ and let $\varphi \colon \Sigma^m \to Q^{n-1} \cong G_2(\mathbb{R}^{n+1})$ $(m < n - 1)$ be a holomorphic isometric immersion, i.e., (11) holds. We say that holomorphic immersion $\varphi \colon \Sigma \to Q^{n-1}$ is *first order isotropic* (cf. (9), (18) and (34)) if

(35)
$$\langle \mathbf{p}(X), \mathbf{p}(Y) \rangle = \frac{1}{2}\langle X, Y \rangle.$$

This condition is independent of the choice of local cross section $\eta' \colon U \to \varphi^* V_2(\mathbb{R}^{n+1})$ for an open subset $U \subset \Sigma$. Moreover, using (11) and (35), we see that

Proposition 8.1. *Let $\varphi: \Sigma \to Q^{n-1}$ be a holomorphic isometric immersion of a Kähler manifold and let $\iota: Q^{n-1} \to \mathbb{CP}^n$ be the inclusion. Then φ is first order isotropic if and only if $d(\iota \circ \varphi)(X)$ is an isotropic vector for each $X \in T\Sigma$.*

For holomorphic curves (i.e., 1-dimensional complex submanifolds) in the complex quadric, the definition of *first order isotropic* is given by Jensen-Rigoli-Yang [28]. In this case, (23) holds for φ, and by Proposition 6.1, the mapping $\Phi: \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ defined by (15) is an immersion. Let $\Phi_U = \Phi \circ \phi_U^{-1}: S^1 \times U \to S^n$ be the local expression of Φ defined by (16). For each $\theta \in S^1$, we define a linear mapping $\Psi_\theta: T_q(\Sigma) \to T_{\Phi_U(\theta,q)}(S^n)$ as

$$\Psi_\theta(X) = \cos\theta\,\mathbf{p}(X) + \sin\theta\,\mathbf{q}(X) = \cos\theta\,\mathbf{p}(X) - \sin\theta\,\mathbf{p}(JX).$$

By (18), we have $\Psi_\theta(X) = d\Phi_{U(\theta,q)}(0,X) - \lambda(X)d\Phi_{U(\theta,q)}(\partial/\partial\theta, 0)$. Then (17), (18), (23) and $\mathbf{q}(X) = -\mathbf{p}(JX)$, imply

$$\begin{aligned}
d\Phi_U(T_{(\theta,q)}(S^1 \times U)) &= \mathbb{R}\,d\Phi_U(\partial/\partial\theta) \oplus \{\mathbf{p}(X) \mid X \in T_q(\Sigma)\} \\
&= \mathbb{R}\,d\Phi_U(\partial/\partial\theta) \oplus \{\Psi_\theta(X) \mid X \in T_q(\Sigma)\}.
\end{aligned}$$

With respect to the submersion $\pi_\varphi: \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$, $d\phi_U^{-1}(\partial/\partial\theta)$ is a unit vertical vector and $\{\Psi_\theta(X) \mid X \in T_q(\Sigma)\}$ is the image of horizontal subspace in $T_{\phi_U^{-1}(\theta,q)}(\varphi^* V_2(\mathbb{R}^{n+1}))$ under $d\Phi$. Using (35) we see that for each θ, $\langle\Psi_\theta(X), \Psi_\theta(Y)\rangle = (1/2)\langle X, Y\rangle$ and in particular Ψ_θ is injective. Hence we have

Proposition 8.2. *Let $\varphi: \Sigma \to Q^{n-1}$ be a first order isotropic holomorphic isometric immersion from a Kähler manifold to the complex quadric, and let $\Phi: \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ be the corresponding immersion defined by (15). Then restriction of the differential of the projection $\pi_\varphi: \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ to the horizontal subspaces is a homothety with respect to the metric on $\varphi^* V_2(\mathbb{R}^{n+1})$ induced by Φ.*

Let D denote the Euclidean covariant derivative on $\mathbb{R}^{n+1}$. Then we obtain

$$\begin{aligned}
D_{\Psi_\theta(X)}\Psi_\theta(Y) &= \cos\theta\,\{(\nabla_X\mathbf{p})Y + \mathbf{p}(\nabla_X Y) + \lambda(X)\mathbf{p}(JY)\} \\
&\quad - \sin\theta\,\{(\nabla_X\mathbf{p})JY + \mathbf{p}(\nabla_X(JY)) - \lambda(X)\mathbf{p}(Y)\}.
\end{aligned}$$

Then (25) yields

$$\sigma^\Phi\left(d\phi_U^{-1}(0,X), d\phi_U^{-1}(0,Y)\right) = \cos\theta\,\mathbf{s}(X,Y) - \sin\theta\,\mathbf{s}(X,JY)$$

where $X, Y \in T_q(\Sigma)$. Hence by virtue of (13), we obtain
(36)
$$\sigma^\Phi\left(d\phi_U^{-1}(0, X), d\phi_U^{-1}(0, Y)\right) + \sigma^\Phi\left(d\phi_U^{-1}(0, JX), d\phi_U^{-1}(0, JY)\right) = 0.$$

As a generalization of Theorem 1 in [29], we get

Theorem 8.3. *Let $\varphi\colon \Sigma \to Q^{n-1}$ be a first order isotropic holomorphic isometric immersion from a Kähler manifold to the complex quadric. Then the corresponding immersion $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ defined by (15) is austere.*

Proof. Let U be an open subset in Σ and let $\Phi_U = \Phi \circ \phi_U^{-1}\colon S^1 \times U \to S^n$ be the local expression of Φ as (16). By Proposition 8.2, at each point $u = \phi_U^{-1}(\theta, q) \in \pi_\varphi^{-1}(U)$, we can choose an orthonormal basis $\{V, \mathbf{e}_1, \ldots, \mathbf{e}_{2m}\}$ of the tangent space $T_u(\varphi^* V_2(\mathbb{R}^{n+1}))$ such that $V = d\phi_U^{-1}(\partial/\partial\theta, 0)$ is a unit vertical vector and $\mathbf{e}_j = \mathcal{H}\, d\phi_U^{-1}(0, \mathbf{v}_j)$, $(j = 1, \ldots, 2m)$ are horizontal vector with respect to the submersion $\pi_\varphi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$, where $\mathcal{H}$ denotes the horizontal component of the tangent vector and $\mathbf{v}_1, \ldots, \mathbf{v}_{2m}$ is a basis of orthogonal vectors of $T_q(\Sigma)$ with $J\mathbf{v}_{2k-1} = \mathbf{v}_{2k}$ for $k = 1, \ldots, m$. Then by Theorem 6.4 and (36), we can see that Φ is an austere immersion. $\qquad\square$

Remark 8.4. It is well-known (cf. [11] and [28]) that there is a one-to-one correspondence between *totally isotropic holomorphic curves* in Q^{2m-1} and *pseudoholomorphic surfaces (superminimal surfaces, or isotropic minimal surfaces)* in S^{2m}. Hence from full minimal 2-spheres in S^{2m}, we can construct 3-dimensional full austere submanifolds in S^{2m} [29]. As for tangential degeneracy, see §10.

§9. Examples satisfying Ferus equality in spheres

In this section, we give further examples of homogeneous submanifolds with degenerate Gauss mapping, and list up those which satisfy Ferus equality.

Let $\mathbb{K}$ be the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers or the field $\mathbb{H}$ of quaternions. In the natural way, $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. The conjugate of each element $q \in \mathbb{H}$ is defined as follows:

$$\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} \quad \text{for} \quad q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

where $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is usual basis for $\mathbb{H}$. Define a number d by

$$d = \begin{cases} 1 & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}, \\ 4 & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

Let $\mathbf{x} \in \mathbb{K}^{n+1}$ be a column vector. The usual inner product on $\mathbb{K}^{n+1} = \mathbb{R}^{(n+1)d}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathrm{Re}(\mathbf{x}^*\mathbf{y}) \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{K}^{n+1}$$

where $\mathrm{Re}(\mathbf{x}^*\mathbf{y})$ denotes the real part of $\mathbf{x}^*\mathbf{y}$.

The projective space $\mathbb{KP}^n$ over $\mathbb{K}$ is considered as the quotient space of the unit $((n+1)d-1)$-dimensional sphere $S^{(n+1)d-1} = \{\mathbf{x} \in \mathbb{K}^{n+1} \mid \mathbf{x}^*\mathbf{x} = 1\}$ obtained by identifying $\mathbf{x}$ with $\mathbf{x}\lambda$ where $\lambda \in S^{d-1} = \{t \in \mathbb{K} \mid |t| = 1\}$. Let $\pi \colon S^{(n+1)d-1} \to \mathbb{KP}^n$ be the Hopf fibration and denote $\pi(\mathbf{x}) = [\mathbf{x}] \in \mathbb{KP}^n$ for $\mathbf{x} \in S^{(n+1)d-1}$. Then the canonical metric in $\mathbb{KP}^n$ is the invariant metric such that π is a Riemannian submersion.

Let

$$V_2(\mathbb{K}^{n+1}) = \{(\mathbf{u}_1, \mathbf{u}_2) \in S^{(n+1)d-1} \times S^{(n+1)d-1} \mid \mathbf{u}_1^*\mathbf{u}_2 = 0\}$$

be the Stiefel manifold over $\mathbb{K}$. Then $T_{(\mathbf{u}_1, \mathbf{u}_2)}(V_2(\mathbb{K}^{n+1}))$, the tangent space at $(\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1})$ is

$$T_{(\mathbf{u}_1, \mathbf{u}_2)}(V_2(\mathbb{K}^{n+1})) = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid$$
$$\langle \mathbf{x}_1, \mathbf{u}_1 \rangle = \langle \mathbf{x}_2, \mathbf{u}_2 \rangle = 0, \ \mathbf{x}_1^*\mathbf{u}_2 + \mathbf{u}_1^*\mathbf{x}_2 = 0\}.$$

The subspaces $T_0(\mathbf{u}_1, \mathbf{u}_2)$, $T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$, $T_\mu(\mathbf{u}_1, \mathbf{u}_2)$ of $T_{(\mathbf{u}_1, \mathbf{u}_2)}(V_2(\mathbb{K}^{n+1}))$ are defined as

$$T_0(\mathbf{u}_1, \mathbf{u}_2) = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \mathbf{x}_\alpha^*\mathbf{u}_\beta = 0 \text{ for } \alpha, \beta = 1, 2\}$$
$$T_\lambda(\mathbf{u}_1, \mathbf{u}_2) = \{(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \lambda \in \mathbb{K}\}$$
$$T_\mu(\mathbf{u}_1, \mathbf{u}_2) = \{(\mathbf{u}_1\mu_1, \mathbf{u}_2\mu_2) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \mu_1, \mu_2 \in \mathrm{Im}\,\mathbb{K}\}.$$

Then we have $T_{(\mathbf{u}_1, \mathbf{u}_2)}(V_2(\mathbb{K}^{n+1})) = T_0(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\lambda(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\mu(\mathbf{u}_1, \mathbf{u}_2)$.

Put

$$F_2(\mathbb{K}^{n+1}) = \{([\mathbf{u}_1], [\mathbf{u}_2]) \in \mathbb{KP}^n \times \mathbb{KP}^n \mid \mathbf{u}_1^*\mathbf{u}_2 = 0\}.$$

Then $V_2(\mathbb{K}^{n+1})$ is a principal fiber bundle over $F_2(\mathbb{K}^{n+1})$ with structure group $S^{d-1} \times S^{d-1}$ and projection map $\pi \colon V_2(\mathbb{K}^{n+1}) \to F_2(\mathbb{K}^{n+1})$ defined by

$$\pi(\mathbf{u}_1, \mathbf{u}_2) = ([\mathbf{u}_1], [\mathbf{u}_2]).$$

The distribution $T_0(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ defines a connection in the principal fiber bundle $V_2(\mathbb{K}^{n+1})(F_2(\mathbb{K}^{n+1}), S^{d-1} \times S^{d-1})$, because this is complementary to the subspace $T_\mu(\mathbf{u}_1, \mathbf{u}_2)$ tangent to the fiber through

$(\mathbf{u}_1, \mathbf{u}_2)$, and invariant under the $S^{d-1} \times S^{d-1}$-action. The projection π of $V_2(\mathbb{K}^{n+1})$ onto $F_2(\mathbb{K}^{n+1})$ induces a linear isomorphism of $T_0(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ onto $T_p(F_2(\mathbb{K}^{n+1}))$, where $\pi((\mathbf{u}_1, \mathbf{u}_2)) = p$. We mention that $F_2(\mathbb{K}^{n+1})$ is considered as the total space of "tautological $\mathbb{KP}^1$-bundle" over 2-plane Grassmann manifold $\widetilde{G}_2(\mathbb{K}^{n+1})$ with projection $\pi(\mathbf{u}_1, \mathbf{u}_2) \mapsto \operatorname{span}_{\mathbb{K}}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Let $\mathfrak{M}^{\mathbb{K}}(n+1)$ be the space of all $(n+1) \times (n+1)$ matrices over $\mathbb{K}$. The inner product on $\mathfrak{M}^{\mathbb{K}}(n+1) = \mathbb{R}^{(n+1)^2 d}$ is defined as

$$\langle A, B \rangle = \frac{1}{2} \operatorname{Re} \operatorname{trace}(AB^*) \quad \text{for} \quad A, B \in \mathfrak{M}^{\mathbb{K}}(n+1).$$

Let

$$\operatorname{Herm}^{\mathbb{K}}(n+1) = \{A \in \mathfrak{M}^{\mathbb{K}}(n+1) \mid A^* = A\}$$

$$\operatorname{Herm}_0^{\mathbb{K}}(n+1) = \{A \in \operatorname{Herm}^{\mathbb{K}}(n+1) \mid \operatorname{trace} A = 0\}$$

$$S(\operatorname{Herm}_0^{\mathbb{K}}(n+1)) = \{A \in \operatorname{Herm}_0^{\mathbb{K}}(n+1) \mid \langle A, A \rangle = 1\}$$

$$U^{\mathbb{K}}(n+1) = \{P \in \mathfrak{M}^{\mathbb{K}}(n+1) \mid P^*P = E\}$$

where $A^* = {}^t\overline{A}$ and E is the identity matrix. $U^{\mathbb{K}}(n+1)$ acts on $V_2(\mathbb{K}^{n+1})$, $F_2(\mathbb{K}^{n+1})$ and $\widetilde{G}_2(\mathbb{K}^{n+1})$ transitively as

$$P \cdot (\mathbf{u}_1, \mathbf{u}_2) = (P\mathbf{u}_1, P\mathbf{u}_2) \text{ for } (\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1})$$

$$P \cdot \pi(\mathbf{u}_1, \mathbf{u}_2) = \pi(P\mathbf{u}_1, P\mathbf{u}_2) \text{ for } \pi(\mathbf{u}_1, \mathbf{u}_2) \in F_2(\mathbb{K}^{n+1})$$

$$P \cdot \operatorname{span}_{\mathbb{K}}\{\mathbf{u}_1, \mathbf{u}_2\} = \operatorname{span}_{\mathbb{K}}\{P\mathbf{u}_1, P\mathbf{u}_2\} \text{ for } \operatorname{span}_{\mathbb{K}}\{\mathbf{u}_1, \mathbf{u}_2\} \in \widetilde{G}_2(\mathbb{K}^{n+1})$$

and $P \in U^{\mathbb{K}}(n+1)$. Hence as homogeneous spaces, we have

$$V_2(\mathbb{K}^{n+1}) = U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(n-1)$$

$$F_2(\mathbb{K}^{n+1}) = U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(1) \times U^{\mathbb{K}}(1) \times U^{\mathbb{K}}(n-1)$$

$$\widetilde{G}_2(\mathbb{K}^{n+1}) = U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(2) \times U^{\mathbb{K}}(n-1).$$

Define a map $\widetilde{\varphi}_n^{\mathbb{K}} \colon V_2(\mathbb{K}^{n+1}) \to S(\operatorname{Herm}_0^{\mathbb{K}}(n+1))$ as follows:

(37) $$\widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{u}_1 \mathbf{u}_1^* - \mathbf{u}_2 \mathbf{u}_2^* \quad \text{for} \quad (\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1}).$$

It is easy to see that $\widetilde{\varphi}_n^{\mathbb{K}}$ gives a map $\varphi_n^{\mathbb{K}} \colon F_2(\mathbb{K}^{n+1}) \to S(\operatorname{Herm}_0^{\mathbb{K}}(n+1))$ such that $\widetilde{\varphi}_n^{\mathbb{K}} = \varphi_n^{\mathbb{K}} \circ \pi$. For simplicity, we denote $\varphi = \varphi_n^{\mathbb{K}}$.

$U^{\mathbb{K}}(n+1)$ acts on $\operatorname{Herm}_0^{\mathbb{K}}(n+1)$ orthogonally as

$$P(A) = PAP^* \quad \text{for} \quad P \in U^{\mathbb{K}}(n+1), \ A \in \operatorname{Herm}_0^{\mathbb{K}}(n+1).$$

It is well-known that for each $A \in \mathrm{Herm}_0^{\mathbb{K}}(n+1)$, there exists $P \in U^{\mathbb{K}}(n+1)$ that PAP^* is a diagonal matrix whose elements are real numbers. The map φ is equivariant, since

$$
\begin{aligned}
\varphi(P \cdot \pi(\mathbf{u}_1, \mathbf{u}_2)) &= \varphi(\pi(P\mathbf{u}_1, P\mathbf{u}_2)) = \widetilde{\varphi}_n^{\mathbb{K}}(P\mathbf{u}_1, P\mathbf{u}_2) \\
&= (P\mathbf{u}_1)(P\mathbf{u}_1)^* - (P\mathbf{u}_2)(P\mathbf{u}_2)^* = P\varphi(\pi(\mathbf{u}_1, \mathbf{u}_2))P^* \\
&\quad \text{for any } \pi(\mathbf{u}_1, \mathbf{u}_2) \in F_2(\mathbb{K}^{n+1}).
\end{aligned}
$$

Take an element $(\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)$ with $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1$. Then the curve $t \mapsto \pi(\mathbf{u}_1 \cos t + \mathbf{x}_1 \sin t, \mathbf{u}_2 \cos t + \mathbf{x}_2 \sin t)$ is tangent to $d\pi(\mathbf{x}_1, \mathbf{x}_2)$ at $\pi(\mathbf{u}_1, \mathbf{u}_2)$ in $F_2(\mathbb{K}^{n+1})$. We have

$$
\begin{aligned}
(38) \qquad \varphi\big(&\pi(\mathbf{u}_1 \cos t + \mathbf{x}_1 \sin t, \mathbf{u}_2 \cos t + \mathbf{x}_2 \sin t)\big) \\
&= (\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*)\cos^2 t + (\mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*)\sin^2 t \\
&\quad + (\mathbf{u}_1\mathbf{x}_1^* + \mathbf{x}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{x}_2^* - \mathbf{x}_2\mathbf{u}_2^*)\cos t \sin t.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
(39) \qquad d\varphi(d\pi(\mathbf{x}_1, \mathbf{x}_2)) &= \mathbf{u}_1\mathbf{x}_1^* + \mathbf{x}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{x}_2^* - \mathbf{x}_2\mathbf{u}_2^* \\
&\quad \text{for } (\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)
\end{aligned}
$$

and

$$
(40) \qquad \|d\varphi(d\pi(\mathbf{x}_1, \mathbf{x}_2))\| = \sqrt{2}.
$$

Take an element $(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda) \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ with $|\lambda| = 1$. Then the curve $t \mapsto \pi(\mathbf{u}_1 \cos t - \mathbf{u}_2\bar{\lambda}\sin t, \mathbf{u}_2 \cos t + \mathbf{u}_1\lambda \sin t)$ is tangent to $d\pi(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda)$ at $\pi(\mathbf{u}_1, \mathbf{u}_2)$ in $F_2(\mathbb{K}^{n+1})$. We have

$$
\begin{aligned}
(41) \qquad \varphi\big(&\pi(\mathbf{u}_1 \cos t - \mathbf{u}_2\bar{\lambda}\sin t, \mathbf{u}_2 \cos t + \mathbf{u}_1\lambda \sin t)\big) \\
&= (\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*)\cos 2t - (\mathbf{u}_1\lambda\mathbf{u}_2^* + \mathbf{u}_2\bar{\lambda}\mathbf{u}_1^*)\sin 2t.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
(42) \qquad d\varphi(d\pi(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda)) &= -2(\mathbf{u}_1\lambda\mathbf{u}_2^* + \mathbf{u}_2\bar{\lambda}\mathbf{u}_1^*) \\
&\quad \text{for } (-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda) \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2)
\end{aligned}
$$

and

$$
(43) \qquad \|d\varphi(d\pi(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda))\| = 2.
$$

By (40) and (43), the map φ is an immersion. If $\widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{u}_1, \mathbf{u}_2) = \widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{v}_1, \mathbf{v}_2)$ holds for $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in V_2(\mathbb{K}^{n+1})$, then $\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^* = \mathbf{v}_1\mathbf{v}_1^* - \mathbf{v}_2\mathbf{v}_2^*$

and $(\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*)^2 = (\mathbf{v}_1\mathbf{v}_1^* - \mathbf{v}_2\mathbf{v}_2^*)^2$ imply $\mathbf{u}_1\mathbf{u}_1^* = \mathbf{v}_1\mathbf{v}_1^*$ and $\mathbf{u}_2\mathbf{u}_2^* = \mathbf{v}_2\mathbf{v}_2^*$. By comparing the components, we see that $\mathbf{v}_1 = \mathbf{u}_1\mu_1$ and $\mathbf{v}_2 = \mathbf{u}_2\mu_2$ for some $\mu_1, \mu_2 \in \mathbb{K}$ with $|\mu_1| = |\mu_2| = 1$ and hence φ is an embedding.

Let σ be the second fundamental form of φ. Take a tangent vector $(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda) \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ ($|\lambda| = 1$) at $(\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1})$. (41) implies

$$\frac{d^2}{dt^2}\bigg|_{t=0} \varphi\big(\pi(\mathbf{u}_1 \cos t - \mathbf{u}_2\bar{\lambda} \sin t, \mathbf{u}_2 \cos t + \mathbf{u}_1\lambda \sin t)\big)$$
$$= -4(\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*),$$

which is proportional to $\varphi(\pi(\mathbf{u}_1, \mathbf{u}_2))$. Hence we get $\sigma(d\pi(X), d\pi(X)) = 0$ for any $X \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ and

(44) $\qquad \sigma(d\pi(X), d\pi(Y)) = 0 \quad$ for any $\quad X, Y \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2),$

by polarization. Put $(\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)$ with $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1$. Using (38), we obtain

$$\frac{d^2}{dt^2}\bigg|_{t=0} \varphi\big(\pi(\mathbf{u}_1 \cos t + \mathbf{x}_1 \sin t, \mathbf{u}_2 \cos t + \mathbf{x}_2 \sin t)\big)$$
$$= 2(-\mathbf{u}_1\mathbf{u}_1^* + \mathbf{u}_2\mathbf{u}_2^* + \mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*).$$

By taking the component which is orthogonal to $\varphi(\pi(\mathbf{u}_1, \mathbf{u}_2))$, and using (39) and (42), we have

$$\sigma\big(d\pi(\mathbf{x}_1, \mathbf{x}_2), d\pi(\mathbf{x}_1, \mathbf{x}_2)\big) = 2(\mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*)$$

for any $(\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)$. Hence

(45) $\qquad \sigma\big(d\pi(\mathbf{x}_1, \mathbf{x}_2), d\pi(\mathbf{y}_1, \mathbf{y}_2)\big) = \mathbf{x}_1\mathbf{y}_1^* + \mathbf{y}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{y}_2^* - \mathbf{y}_2\mathbf{x}_2^*$
$$\text{for any } (\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)$$

holds by polarization. Finally we consider the curve in $V_2(\mathbb{K}^{n+1})$ defined as:

$$t \mapsto \big(\mathbf{u}_1 \cos t + ((\mathbf{x}_1 - \mathbf{u}_2\bar{\lambda})/\sqrt{2}) \sin t, \mathbf{u}_2 \cos t + ((\mathbf{x}_2 + \mathbf{u}_1\lambda)/\sqrt{2}) \sin t\big)$$
$$((\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2), \ \|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1 \text{ and } |\lambda| = 1),$$

which is tangent to $((\mathbf{x}_1, \mathbf{x}_2) + (-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda))/\sqrt{2}$ at $(\mathbf{u}_1, \mathbf{u}_2)$. Then we have

$$\frac{d^2}{dt^2}\bigg|_{t=0} \varphi\Big(\pi\Big(\mathbf{u}_1 \cos t + \Big(\frac{\mathbf{x}_1 - \mathbf{u}_2\bar{\lambda}}{\sqrt{2}}\Big) \sin t, \mathbf{u}_2 \cos t + \Big(\frac{\mathbf{x}_2 + \mathbf{u}_1\lambda}{\sqrt{2}}\Big) \sin t\Big)\Big)$$
$$= -3(\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*) + (\mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*)$$
$$\quad - \big((\mathbf{x}_1\lambda)\mathbf{u}_2^* + \mathbf{u}_2(\mathbf{x}_1\lambda)^* + (\mathbf{x}_2\bar{\lambda})\mathbf{u}_1^* + \mathbf{u}_1(\mathbf{x}_2\bar{\lambda})^*\big).$$

By (39), we get

$$\frac{1}{2}\sigma\big(d\pi(\mathbf{x}_1,\mathbf{x}_2)+d\pi(-\mathbf{u}_2\bar\lambda,\mathbf{u}_1\lambda),d\pi(\mathbf{x}_1,\mathbf{x}_2)+d\pi(-\mathbf{u}_2\bar\lambda,\mathbf{u}_1\lambda)\big)$$
$$=\mathbf{x}_1\mathbf{x}_1^*-\mathbf{x}_2\mathbf{x}_2^*.$$

Then (44) and (45) yield

(46) $$\sigma\big(d\pi(\mathbf{x}_1,\mathbf{x}_2),d\pi(X)\big)=0$$

for any $(\mathbf{x}_1,\mathbf{x}_2)\in T_0(\mathbf{u}_1,\mathbf{u}_2)$ and $X\in T_\lambda(\mathbf{u}_1,\mathbf{u}_2)$.

Hence the index of relative nullity [30] of the embedding $\varphi\colon F_2(\mathbb{K}^{n+1})\to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ is equal to $\dim_\mathbb{R} T_\lambda(\mathbf{u}_1,\mathbf{u}_2)=d$ at any $x=\pi(\mathbf{u}_1,\mathbf{u}_2)\in F_2(\mathbb{K}^{n+1})$.

Our examples $\varphi\colon F_2(\mathbb{K}^{n+1})\to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))=S^m$ satisfying Ferus equality $r=F(l)$ (or equivalently $\nu=\mu(l)$) for $l\le 32$ are as follows:

Embedding	l	r	ν	m
$\varphi_2^\mathbb{R}\colon F_2(\mathbb{R}^3)\to S(\mathrm{Herm}_0^\mathbb{R}(3))$	3	2	1	4
$\varphi_3^\mathbb{R}\colon F_2(\mathbb{R}^4)\to S(\mathrm{Herm}_0^\mathbb{R}(4))$	5	4	1	8
$\varphi_2^\mathbb{C}\colon F_2(\mathbb{C}^3)\to S(\mathrm{Herm}_0^\mathbb{C}(3))$	6	4	2	7
$\varphi_5^\mathbb{R}\colon F_2(\mathbb{R}^6)\to S(\mathrm{Herm}_0^\mathbb{R}(6))$	9	8	1	19
$\varphi_3^\mathbb{C}\colon F_2(\mathbb{C}^4)\to S(\mathrm{Herm}_0^\mathbb{C}(4))$	10	8	2	14
$\varphi_2^\mathbb{H}\colon F_2(\mathbb{H}^3)\to S(\mathrm{Herm}_0^\mathbb{H}(3))$	12	8	4	13
$\varphi_9^\mathbb{R}\colon F_2(\mathbb{R}^{10})\to S(\mathrm{Herm}_0^\mathbb{R}(10))$	17	16	1	53
$\varphi_5^\mathbb{C}\colon F_2(\mathbb{C}^6)\to S(\mathrm{Herm}_0^\mathbb{C}(6))$	18	16	2	33
$\varphi_3^\mathbb{H}\colon F_2(\mathbb{H}^4)\to S(\mathrm{Herm}_0^\mathbb{H}(4))$	20	16	4	26
$\varphi_{13}^\mathbb{R}\colon F_2(\mathbb{R}^{14})\to S(\mathrm{Herm}_0^\mathbb{R}(14))$	25	24	1	103
$\varphi_7^\mathbb{C}\colon F_2(\mathbb{C}^8)\to S(\mathrm{Herm}_0^\mathbb{C}(8))$	26	24	2	62
$\varphi_4^\mathbb{H}\colon F_2(\mathbb{H}^5)\to S(\mathrm{Herm}_0^\mathbb{H}(5))$	28	24	4	43

Note that $\varphi_2^\mathbb{K}\colon F_2(\mathbb{K}^3)\to S(\mathrm{Herm}_0^\mathbb{K}(3))=S^{3d+1}$ is nothing but the Cartan's isoparametric (minimal) hypersurfaces.

With respect to $\varphi\colon F_2(\mathbb{K}^{n+1})\to S(\mathrm{Herm}_0^\mathbb{K}(n+1))$, (45) implies

(47) $$\sigma\big(d\pi(\mathbf{x},0),d\pi(\mathbf{y},0)\big)+\sigma\big(d\pi(0,\mathbf{x}),d\pi(0,\mathbf{y})\big)$$
$$=\sigma\big(d\pi(\mathbf{x},0),d\pi(0,\mathbf{y})\big)=0,$$

where $\mathbf{x}^*\mathbf{u}_1 = \mathbf{x}^*\mathbf{u}_2 = \mathbf{y}^*\mathbf{u}_1 = \mathbf{y}^*\mathbf{u}_2 = 0$ at any $\pi(\mathbf{u}_1, \mathbf{u}_2) \in F_2(\mathbb{K}^{n+1})$. Hence the embedding φ is austere (§8) and minimal.

Remark 9.1. When $\mathbb{K} = \mathbb{R}$, the submanifold $F_2(\mathbb{R}^{n+1})$ is given in §7. Also when $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$, the submanifold $F_2(\mathbb{K}^{n+1})$ is obtained as a pull-back bundle over 2-plane Grassmannian $G_2(\mathbb{K}^{n+1})$ of canonical d-dimensional sphere bundle over real Grassmannian $G_{d+1}^{\mathbb{R}}(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ with respect to the embedding:

Case $\mathbb{K} = \mathbb{C}$: $G_2(\mathbb{C}^{n+1}) \to G_3^{\mathbb{R}}(\mathrm{Herm}_0^{\mathbb{C}}(n+1))$,

$$\mathrm{span}_{\mathbb{C}}\{\mathbf{u}, \mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{R}}\{\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*, \mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*, \mathbf{i}(\mathbf{u}\mathbf{v}^* - \mathbf{v}\mathbf{u}^*)\}$$

Case $\mathbb{K} = \mathbb{H}$: $G_2(\mathbb{H}^{n+1}) \to G_5^{\mathbb{R}}(\mathrm{Herm}_0^{\mathbb{H}}(n+1))$,

$$\mathrm{span}_{\mathbb{H}}\{\mathbf{u}, \mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{R}}\{\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*, \mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*,$$
$$\mathbf{u}\mathbf{i}\mathbf{v}^* - \mathbf{v}\mathbf{i}\mathbf{u}^*, \mathbf{u}\mathbf{j}\mathbf{v}^* - \mathbf{v}\mathbf{j}\mathbf{u}^*, \mathbf{u}\mathbf{k}\mathbf{v}^* - \mathbf{v}\mathbf{k}\mathbf{u}^*\},$$

where $\mathbf{u}^*\mathbf{u} = \mathbf{v}^*\mathbf{v} = 1$ and $\mathbf{u}^*\mathbf{v} = 0$. For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, $\varphi \colon F_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ satisfies B. Y. Chen's equality (cf. Theorem 4.1 in [15])

$$\delta\big((n-1)d, (n-1)d\big) = (n-1)(nd+1)d$$
$$(n = (2n-1)d, \ n_1 = n_2 = (n-1)d, \ H = 0, \ \epsilon = 1).$$

§10. Hypersurfaces with degenerate Gauss mappings in the four dimensional sphere

In this section, we study the simplest case $n = 4$, $l = 3$, $r = F(3) = 2$.

Recall that the Cartan hypersurface $M^3 \subset S^4$ is a homogeneous space of $\mathrm{SO}(3)$ and written as $M = \mathrm{O}(3)/\big(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)\big)$. The Gauss mapping $\gamma \colon M^3 \to G_4(\mathbb{R}^5) \cong G_1(\mathbb{R}^{5*}) = \mathbb{RP}^{4*}$ into the dual projective space, has the constant rank 2. Moreover its image $\gamma(M)$, that is the projective dual in this case, is a linear projection $\mathbb{RP}^2 \subset \mathbb{RP}^{4*}$ of the Veronese surface $\mathbb{RP}^2 \subset \mathbb{RP}^{5*}$ in the sense of algebraic geometry [26]: The Veronese surface has the crucial property that its secant variety is of positive codimension in $\mathbb{RP}^{5*}$ (cf. [45]). Notice that it lifts to the Veronese surface $i \colon \mathbb{RP}^2 \hookrightarrow \widetilde{G}_4(\mathbb{R}^5) \cong \widetilde{G}_1(\mathbb{R}^{5*}) = S^{4*}$, in the sense of differential geometry. The liftability means just that M is orientable. Consider the double covering $\pi \colon S^2 \to \mathbb{RP}^2$ and take the fiber product $\widetilde{M}$ of π and $\gamma \colon M \to \mathbb{RP}^2$:

$$
\begin{array}{ccc}
\widetilde{M} & \longrightarrow & S^2 \\
\Pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{\ \gamma\ } \mathbb{RP}^2 & \xrightarrow{\ i\ } S^{4*}.
\end{array}
$$

We call $\widetilde{M}$ the *doubled Cartan hypersurface*. Then we have the tangentially degenerate immersion $\widetilde{M} \to S^4$, that is the composition of the double covering $\Pi \colon \widetilde{M} \to M$ and the inclusion $M \subset S^4$. $\widetilde{M}$ is connected and realized by $O(3)/(SO(1) \times O(1) \times O(1))$ as a homogeneous space. Also remark, by the spherical-projective duality, that $\widetilde{M}$ is the total space of the associated $\widetilde{G}_1(\mathbb{R}^2) = S^1$ bundle over S^2 to the normal bundle of the immersion $i \circ \pi \colon S^2 \to S^{4*}$.

Then we give the following characterization of the diffeomorphism type of compact connected tangentially degenerate hypersurfaces in S^4, using the result of Asperti [5].

Theorem 10.1. *Let M' be a compact connected 3-dimensional manifold, and $f' \colon M'^3 \to S^4$ a tangentially degenerate immersion. Assume that the rank of the Gauss mapping of f' is everywhere 2. Then, there exists a tangentially degenerate immersion $f \colon \widetilde{M} \to S^4$ from the doubled Cartan hypersurface $\widetilde{M}$ with $f(\widetilde{M}) = f'(M')$.*

Proof. We may assume M' is orientable. If not, we take its connected double covering. Then the Gauss mapping $\gamma(f') \colon M' \to \mathbb{RP}^{4*}$ of f' lifts to $\widetilde{\gamma}(f') \colon M' \to S^{4*}$. Then rank $\widetilde{\gamma}(f')$ is identically 2. Therefore $\widetilde{\gamma}(f')$ is decomposed into π and j, where $\pi \colon M' \to N'$ is an S^1-fibration and $j \colon N' \to S^{4*}$ is an immersion. Since M' is connected, N' is connected. Moreover we may assume N' is orientable without loss of generality. (If not, we may take connected double covers of M' and N'). Since $\widetilde{\gamma}(f')$ is an immersion, we see that the second fundamental form of j has no singular quadratic form [27] and [26]. Then, j has non-vanishing normal curvature, relatively to the ordinary metric. Now we recall that any oriented immersed surfaces in S^4 with non-vanishing normal curvature is parameterized by an immersion $S^2 \to S^4$ with the normal Euler number ± 4 [5]. Thus we can assume that $N' = S^2$ and the normal Euler number of j is equal to 4. If it is equal to -4, we take another lifting $\widetilde{\gamma}(f')$, changing the orientation of M'. Now consider the associated $\widetilde{G}_1(\mathbb{R}^2) = S^1$ bundle M'' over S^2 to the normal bundle of the immersion $j \colon S^2 \to S^{4*}$. Then we see that the Euler number of the S^1-bundle $M'' \to S^2$ is equal to 4. Since the diffeomorphism type of such bundles is uniquely determined, and since also the immersion $\widetilde{M} \to S^4$ induces the S^1 fibration $\widetilde{M} \to S^2$ with Euler number 4 as well, we see that there exists a diffeomorphism $\rho \colon \widetilde{M} \to M''$. On the other hand, by the projective duality, if we set $f = f'' \circ \rho \colon \widetilde{M} \to S^4$, then we see that f is a tangentially degenerate immersion and $f(\widetilde{M}) = f'(M')$.   □

Lastly we proceed to construct an example of tangentially degenerate immersions from a compact submanifold M of dimension 3 to S^4, the rank of whose Gauss mapping is not constant 2.

Recall that for a Riemannian surface Σ, a holomorphic immersion $\varphi \colon \Sigma \to Q^{n-1}$ is called *first-order isotropic* if the complex derivative $\varphi' \colon \Sigma \to \mathbb{CP}^n$ lies in Q^{n-1} again. This condition is equivalent to that the *tangent developable*, the union of tangent lines, to φ is contained in Q^{n-1} (cf. Proposition 8.1). A holomorphic immersion $\varphi \colon \Sigma \to Q^{n-1}$ *has no real point* if $\varphi'(\Sigma) \cap \mathbb{RP}^n = \emptyset$, which is the case for first-order isotropic immersion. Using the notation in §8, let $M = \varphi^* V_2(\mathbb{R}^{n+1})$ be the pull-back bundle over the Riemannian surface Σ, and let Φ be given by (15).

Theorem 10.2 ([29]). *If $\varphi \colon \Sigma \to Q^{n-1}$ has no real point, then $f \colon M^3 \to S^n$ is a tangentially degenerate immersion. If $\varphi \colon \Sigma \to Q^{n-1}$ is a first-order isotropic immersion, then $f \colon M^3 \to S^n$ is a minimal tangentially degenerate immersion, with respect to the ordinary metric on S^n.*

Now, in the case $n = 4$, there exist first-order isotropic holomorphic immersions (unramified) $\varphi \colon S^2 (= \mathbb{CP}^1) \to Q^3 \subset \mathbb{CP}^4$, [9]. Thus we have

Proposition 10.3. *There exist a minimal tangentially degenerate immersion $f \colon M^3 \to S^4$ such that M is a circle bundle over S^2, and that the oriented Gauss mapping $\widetilde{\gamma} \colon M \to \widetilde{G}_4(\mathbb{R}^5) = S^4$ splits into a fibration $M \to S^2$ and a ramified minimal immersion $X \colon S^2 \to S^4$. The rank of $\widetilde{\gamma}$ is not constant 2.*

Proof. Take $\gamma_3 \colon S^2 \to Q^3$ of [9], page 237. The corresponding complex contact curve $\lambda_3 \colon S^2 \to \mathbb{CP}^3$ has the ramification degree 2. Therefore the induced minimal immersion $X = \pi \circ \lambda_3 \colon S^2 \to S^4$ is ramified as well, and X is a parameterization of the image of $\widetilde{\gamma}$. $\square$

Remark that, in [8], it is proved that there exist minimal immersions $\Sigma \to S^4$ from any *compact* Riemann surface Σ of arbitrary genus. Then, by taking their directrix, we have first-order isotropic homomorphic mappings $\varphi \colon \Sigma \to Q^3$, however, in general, ramified.

§11. Appendix

Here we give proofs of Propositions in §3. Let D^1, D^2, D^3, D^4 be the curvature distributions with respect to the principal curvatures λ_1, λ_2, λ_3, λ_4. Using an orthonormal basis of each D^i, take an orthonormal frame $e_1, \ldots, e_{2(m_1+m_2)}$ of M. Then the shape operator B_ζ of M_- is

given by [33] and [34]

$$B_\zeta = c \begin{pmatrix} 0 & B_{12} & B_{13} \\ {}^tB_{12} & 0 & B_{23} \\ {}^tB_{13} & {}^tB_{23} & 0 \end{pmatrix}, \quad c = (\sin(\operatorname{arccot}\lambda_4))^{-1}$$

where $B_{ij} = \left(\Lambda_{k4}^l/(\lambda_i - \lambda_4)\right)$, $1 \le i, j \le 3$ is the $m_i \times m_j$ matrix with

$$\Lambda_{k4}^l = \langle \nabla_{e_k} e_4, e_l \rangle, \quad e_k \in D^i, \ e_l \in D^j$$

and $\zeta \in D^4$ is a unit normal vector for M_-.

Similarly, the shape operator C_ζ of M_+ is given by

$$C_\zeta = c' \begin{pmatrix} 0 & C_{23} & C_{24} \\ {}^tC_{23} & 0 & C_{34} \\ {}^tC_{24} & {}^tC_{34} & 0 \end{pmatrix}, \quad c' = (\sin(\operatorname{arccot}\lambda_1))^{-1}$$

where $C_{ij} = \left(\Lambda_{k1}^l/(\lambda_i - \lambda_1)\right)$, $2 \le i, j \le 4$ is the $m_i \times m_j$ matrix with

$$\Lambda_{k1}^l = \langle \nabla_{e_k} e_1, e_l \rangle, \quad e_k \in D^i, \ e_l \in D^j$$

and $\zeta \in D^1$ is a unit normal vector for M_+.

M_- or M_+ is tangentially degenerate if there exists a frame with respect to which a certain row (then column) of B_ζ (C_ζ) in the middle block vanishes for all $\zeta = e_4$ (e_1). This is because, if we take $\eta = \cos\theta\, p + \sin\theta\, \xi_p$ where ξ_p is the unit normal of M at $p \in M$ and $\theta = \operatorname{arccot}\lambda_4$, ($\operatorname{arccot}\lambda_1$, respectively), B_η is given by $\operatorname{diag}(1_{m_2}, 0_{m_1}, -1_{m_2})$ ($C_\eta = \operatorname{diag}(1_{m_1}, 0_{m_2}, -1_{m_1})$, respectively), (see [33]) and we should have $\bigcap_{n \in T^\perp M_-} \operatorname{Ker} B_n \ne \{0\}$, ($\bigcap_{T^\perp M_+} \operatorname{Ker} C_n \ne \{0\}$, respectively). Note that any $n \in T^\perp M_-$ ($n \in T^\perp M_+$) can be written as a combination of η and vectors in D^4 (D^1).

By the argument of 3.1 in [33], the Codazzi equation implies

$$\Lambda_{ij}^k(\lambda_j - \lambda_k) = \Lambda_{jk}^i(\lambda_k - \lambda_i) = \Lambda_{ki}^j(\lambda_i - \lambda_j)$$

hence, vanishing of Λ_{ij}^k where $\lambda_i, \lambda_j, \lambda_k$ are distinct, depends only on the set $\{i, j, k\}$. We calculate Λ_{ij}^k for necessary indices. For this we need root vectors of the symmetric Lie algebras given in section 4 of [39].

For $(m_1, m_2) = (1, k-2), (2, 2k-3), (4, 4k-5)$, the Lie algebra is

$$\mathfrak{g} = \{A \in \mathfrak{gl}(k+2, \mathbb{K}) \mid T(A) = 0, \ {}^t\bar{A}\Phi + \Phi A = 0\}$$

where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively, and

$$\Phi = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_k \end{pmatrix}.$$

The Cartan involution is given by $\sigma(A) = -{}^t\bar{A}$ hence the ± 1 eigenspaces $\mathfrak{k}$, $\mathfrak{p}$ are, respectively

$$\mathfrak{k} = \left\{ \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \,\middle|\, {}^t\bar{Y} + Y = 0, \ {}^t\bar{Z} + Z = 0 \right\}$$

$$\mathfrak{p} = \left\{ \hat{X} = \begin{pmatrix} 0 & {}^t\bar{X} \\ X & 0 \end{pmatrix} \,\middle|\, X = (x_{ij}) \in M_{k,2}(\mathbb{K}) \right\}.$$

We use a metric

$$\langle \hat{X}, \hat{Y} \rangle = \Re \operatorname{Tr}({}^t\bar{X}Y).$$

A maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{ H(\xi_1, \xi_2) = \xi_1(E_{31} + E_{13}) + \xi_2(E_{42} + E_{24}) \mid \xi_1, \xi_2 \in \mathbb{R} \},$$

where we use the standard basis E_{ij}. Following the notation in [39], Σ_*^+ consists of

$$\gamma_1 = \xi_1 - \xi_2, \ \gamma_2 = \xi_1, \ \gamma_3 = \xi_1 + \xi_2, \ \gamma_4 = \xi_2.$$

The corresponding root vectors in $M_{r+2}(\mathbb{K})$ are, respectively, given by (we omit $\hat{\ }$)

$$
\begin{aligned}
X_1 &= x_{21}(E_{41} + E_{23}) + \bar{x}_{21}(E_{14} + E_{32}), \quad (\bar{x}_{21} = x_{12}) \\
X_2^i &= x_{i-2\,1}E_{i1} + \bar{x}_{i-2\,1}E_{1i}, \quad 5 \le i \le k+2 \\
X_2' &= x_{11}(-E_{13} + E_{31}), \quad (x_{11} + \bar{x}_{11} = 0) \\
X_3 &= y_{21}(E_{41} - E_{23}) + \bar{y}_{21}(E_{14} - E_{32}) \quad (\bar{y}_{21} = -y_{12}) \\
X_4^i &= x_{i-2\,2}E_{i-2\,2} + \bar{x}_{i2}E_{2i}, \quad 5 \le i \le k+2 \\
X_4' &= x_{22}(-E_{24} + E_{42}), \quad (x_{22} + \bar{x}_{22} = 0).
\end{aligned}
$$

Here $x_{ij}, y_{ij} \in \mathbb{K}$ and noting that X_2' and X_4' are trivial for $\mathbb{K} = \mathbb{R}$, we have

$$
(m_1, m_2) = \begin{cases} (1, k-2) & \mathbb{K} = \mathbb{R} \\ (2, 2k-3) & \mathbb{K} = \mathbb{C} \\ (4, 4k-5) & \mathbb{K} = \mathbb{H}. \end{cases}
$$

Putting $T_i = [H, X_i] \in \mathfrak{k}$ up to constant, we get

$$T_1 = \bar{x}_{21}(E_{12} + E_{34}) - x_{21}(E_{21} + E_{43})$$
$$T_2^i = \bar{x}_{i-2\,1}E_{3i} - x_{i-2\,1}E_{i3}, \quad 5 \le i \le k+2$$
$$T_2' = x_{11}(E_{11} - E_{33}), \quad (x_{11} + \bar{x}_{11} = 0)$$
$$T_3 = \bar{y}_{21}(-E_{12} + E_{34}) + y_{21}(E_{21} - E_{43})$$
$$T_4^i = \bar{x}_{i-2\,2}E_{4i} - x_{i-2\,2}E_{i4}, \quad 5 \le i \le k+2$$
$$T_4' = x_{22}(E_{22} - E_{44}), \quad (x_{22} + \bar{z}_{22} = 0).$$

By the argument in [42] and in [33] and [34], we know that

$$\nabla_{X_i^j} X_k^l \sim [T_i^j, X_k^l],$$

where $\sim$ means be equal up to constant.

Noting that $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$, we obtain

$$\nabla_{X_3} X_1 \sim [T_3, X_1] = \big[\bar{y}_{21}(-E_{12} + E_{34}) + y_{21}(E_{21} - E_{43}),$$
$$x_{21}(E_{41} + E_{23}) + \bar{x}_{21}(E_{14} + E_{32})\big]$$

which vanishes only when $\mathbb{K} = \mathbb{R}$, and then we have

$$C_\zeta = c' \begin{pmatrix} 0 & 0 & C_{24} \\ 0 & 0 & 0 \\ {}^t C_{24} & 0 & 0 \end{pmatrix},$$

hence M_+ is tangentially degenerate. When $\mathbb{K} = \mathbb{C}$ and $\mathbb{H}$, noting that $\langle E_{ij}, E_{kl} \rangle = \delta_{ik}\delta_{jl}$, we know that $\nabla_{X_3} X_1$ has X_2' and X_4' components, hence M_+ is not tangentially degenerate.

Next, we have

$$\nabla_{X_2^i} X_4^j \sim [T_2^i, X_4^j] = \big[\bar{x}_{i-2\,1}E_{3i} - x_{i-2\,1}E_{i3}, x_{j-2\,2}E_{j2} + \bar{x}_{j-2\,2}E_{2j}\big]$$

which has X_1 and X_3 components if $i = j$, hence M_- is not tangentially degenerate if $\mathbb{K} = \mathbb{R}$. On the other hand, when $\mathbb{K} = \mathbb{C}$ and $\mathbb{H}$,

$$\nabla_{X_2'} X_4^j \sim [T_2', X_4^j] = \big[x_{21}(E_{11} - E_{33}), x_{j-2\,2}E_{j2} + \bar{x}_{j-2\,2}E_{2j}\big] = 0$$
$$\nabla_{X_2'} X_4' \sim [T_2', X_4'] = \big[x_{21}(E_{11} - E_{33}), x_{22}(-E_{24} + E_{42})\big] = 0$$

implies that the row in B_ζ corresponding to the vector X_2' vanishes identically, hence M_- is tangentially degenerate with $(l, r) = (2m_1 + m_2, m_1 + m_2 + 1)$. Thus we obtain Proposition 3.1.

Next we consider the case $(m_1, m_2) = (2, 2), (4, 5)$. By [39], we have

$$\mathfrak{g} = \{A \in \mathfrak{gl}(5, \mathbb{H}) \mid {}^t\bar{A}\Psi + \Psi A = 0, \ \Psi = \sqrt{-1}\,1_5\} = \mathfrak{k} + \mathfrak{p},$$

where

$$\begin{cases} \mathfrak{k} = \hat{\mathfrak{u}}(5) \\ \mathfrak{p} = \{jZ \mid Z \in M_5(\mathbb{C}), \; {}^t\bar{Z} = -Z\}, \end{cases}$$

$$\begin{cases} \mathfrak{k} = \mathfrak{o}(5) \\ \mathfrak{p} = \{\sqrt{-1}\,Z \mid Z \in M_5(\mathbb{R}), \; {}^tZ = -Z\}, \end{cases}$$

and the former corresponds to $(m_1, m_2) = (4, 5)$. We identify $jZ \mapsto Z$, and $\sqrt{-1}\,Z \mapsto Z$, respectively. We use

$$\langle Z, W \rangle = -\frac{1}{2}\Re \operatorname{Tr}(Z\bar{W}).$$

A maximal abelian subalgebra of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{H(\xi_1, \xi_2) = \xi_1(E_{21} - E_{12}) + \xi_2(E_{43} - E_{34}) \mid \xi_1, \xi_2 \in \mathbb{R}\}.$$

Then Σ_*^+ is as before and the corresponding root vectors are given by

$$X_1^1 = z_{13}(E_{13} + E_{24}) - \bar{z}_{13}(E_{31} + E_{42}),$$
$$X_1^2 = z_{14}(E_{14} - E_{23}) - \bar{z}_{14}(E_{41} - E_{32}),$$
$$X_2^1 = z_{15}E_{15} - \bar{z}_{15}E_{51}, \quad X_2^2 = z_{25}E_{25} - \bar{z}_{25}E_{52},$$
$$(X_2^3 = \sqrt{-1}\,s(-E_{11} + E_{22})),$$
$$X_3^1 = w_{13}(E_{13} - E_{24}) - \bar{w}_{13}(E_{31} - E_{42}),$$
$$X_3^2 = w_{14}(E_{14} + E_{23}) - \bar{w}_{14}(E_{41} + E_{32}),$$
$$X_4^1 = z_{35}E_{35} - \bar{z}_{35}E_{53}, \quad X_4^2 = z_{45}E_{45} - \bar{z}_{45}E_{54},$$
$$(X_4^3 = \sqrt{-1}\,t(E_{33} - E_{44})),$$

where $x_{ij}, y_{ij} \in \mathbb{R}$ or $\mathbb{C}$ and $s, t \in \mathbb{R}$, and in the real case, $X_2^3 = X_4^3 = 0$. Now putting $T_i^j = [H, X_i^j]$ up to constant, we obtain

$$T_1^1 = z_{13}(-E_{14} + E_{23}) - \bar{z}_{13}(-E_{41} + E_{32}),$$
$$T_1^2 = z_{14}(E_{13} + E_{24}) - \bar{z}_{14}(E_{31} + E_{42}),$$
$$T_2^1 = z_{15}E_{25} - \bar{z}_{15}E_{52}, \quad T_2^2 = -z_{25}E_{15} + \bar{z}_{25}E_{51},$$
$$(T_2^3 = \sqrt{-1}\,s(E_{12} + E_{21})),$$
$$T_3^1 = w_{13}(E_{14} + E_{23}) - \bar{w}_{13}(E_{41} + E_{32}),$$
$$T_3^2 = w_{14}(-E_{13} + E_{24}) - \bar{w}_{14}(-E_{31} + E_{42}),$$
$$T_4^1 = z_{35}E_{45} - \bar{z}_{35}E_{54}, \quad T_4^2 = z_{45}E_{35} - \bar{z}_{45}E_{53},$$
$$(T_4^3 = \sqrt{-1}\,t(E_{34} + E_{43})),$$

From these, we have

$$[T_3^1, X_1^1] = \big[w_{13}(E_{14} + E_{23}) - \bar{w}_{13}(E_{41} + E_{32}),$$
$$z_{13}(E_{13} + E_{24}) - \bar{z}_{13}(E_{31} + E_{42})\big] = 0$$
$$[T_3^2, X_1^2] = \big[w_{14}(-E_{13} + E_{24}) - \bar{w}_{14}(-E_{31} + E_{42}),$$
$$z_{14}(E_{14} - E_{23}) - \bar{z}_{14}(E_{41} - E_{32})\big] = 0$$
$$[T_3^1, X_1^2] = \big[w_{13}(E_{14} + E_{23}) - \bar{w}_{13}(E_{41} + E_{32}),$$
$$z_{14}(E_{14} - E_{23}) - \bar{z}_{14}(E_{41} - E_{32})\big]$$
$$= -w_{13}\bar{z}_{14}(E_{11} - E_{22}) - \bar{w}_{13}z_{14}(E_{44} - E_{33})$$
$$[T_3^2, X_1^1] = \big[w_{14}(-E_{13} + E_{24}) - \bar{w}_{14}(-E_{31} + E_{42}),$$
$$z_{13}(E_{13} + E_{24}) - \bar{z}_{13}(E_{31} + E_{42})\big]$$
$$= -w_{14}\bar{z}_{13}(-E_{11} + E_{22}) - \bar{w}_{14}z_{13}(-E_{33} + E_{44}).$$

Note that the latter two vanishes when $\mathbb{K} = \mathbb{R}$, hence M_+ is tangentially degenerate. When $\mathbb{K} = \mathbb{C}$, each vector has distinct one of X_2^3 and X_4^3 components, thus M_+ is not tangentially degenerate.

On the other hand, we obtain

$$[T_2^1, X_4^1] = z_{15}\bar{z}_{35}E_{23} + \bar{z}_{15}z_{35}E_{32},$$
$$[T_2^1, X_4^2] = -z_{15}\bar{z}_{45}E_{24} + \bar{z}_{15}z_{45}E_{42}, \quad [T_2^1, X_4^3] = 0,$$
$$[T_2^2, X_4^1] = z_{25}\bar{z}_{35}E_{13} - \bar{z}_{25}z_{35}E_{31},$$
$$[T_2^2, X_4^2] = z_{25}\bar{z}_{45}E_{14} - \bar{z}_{25}z_{45}E_{41}, \quad [T_2^2, X_4^3] = 0,$$
$$[T_2^3, X_4^1] = 0, \quad [T_2^3, X_4^2] = 0, \quad [T_2^3, X_4^3] = 0.$$

When $\mathbb{K} = \mathbb{R}$, the last three are trivial, and the four non-vanishing vectors have distinct one of X_1^1, X_1^2 and X_3^1, X_3^2 components, thus M_- is not tangentially degenerate. When $\mathbb{K} = \mathbb{C}$, from the last three, the row corresponding to X_2^3 in B_ζ vanishes, hence M_- is tangentially degenerate. This completes the proof of Proposition 3.2.

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