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On One-Parametric Families of Bäcklund Transformations

Sergei Igonin and Joseph Krasil′shchik

Abstract.

In the context of the cohomological deformation theory, infinitesimal description of one-parametric families of Bäcklund transformations of special type including classical examples is given. It is shown that any family of such a kind evolves in the direction of a nonlocal symmetry shadow in the sense of [9].

Introduction

The role of Bäcklund transformations in constructing exact solutions of nonlinear partial differential equations is well known, see [2] and [13] and relevant references therein, for example. A general scheme is illustrated by classical works by Bäcklund and Bianchi. Namely, for the sine-Gordon equation

$$u_{xy} = \sin u \tag{1}$$

Bäcklund constructed a system of differential relations $\mathcal{B}(u, v; \lambda) = 0$ depending on a real parameter $\lambda \in \mathbb{R}$ and satisfying the following property: if $u = u(x, y)$ is a solution of (1), then v is a solution of the same equation and vice versa. Using this result, Bianchi showed that if a known solution u_0 is given and solutions u_1, u_2 satisfy the relations $\mathcal{B}(u_0, u_i; \lambda_i) = 0$, $i = 1, 2$, then there exists a solution u_{12} which satisfies $\mathcal{B}(u_1, u_{12}; \lambda_2) = 0$, $\mathcal{B}(u_2, u_{12}; \lambda_1) = 0$ and is expressed in terms of u_0, u_1, u_2 in terms of relatively simple equalities. This is the so-called *Bianchi permutability theorem*, or *nonlinear superposition principle*. This scheme was successfully applied to many other "integrable" equations.

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Quite naturally, a general problem arises: given an arbitrary PDE $\mathcal{E}$, when are we able to implement a similar construction? This question is closely related to another problem of a great importance in the theory of integrable systems, the problem of insertion of a nontrivial "spectral parameter" to the initial equation. In this paper, we mainly deal with the first problem referring the reader to the yet unpublished work by M. Marvan [10], where the second problem is analyzed.

Our approach to solution lies in the framework of the geometrical theory of nonlinear PDE, and the first section of the paper contains a brief introduction to this theory, including its nonlocal aspects (the theory of coverings), see [1], [7], [8] and [9]. The second section deals with cohomological invariants of nonlinear PDE naturally associated to the equation structure. Our main concern here is the relation between this cohomology theory and deformations of the structure [4], [5] and [8]. In the third section, we give a geometrical definition of Bäcklund transformations and using cohomological techniques prove the main result of the paper describing infinitesimal part of one-parameter families of Bäcklund transformations.

§1. Equations and coverings

Let us recall basic facts from the geometry of nonlinear PDE, [1], [7] and [8].

Consider a smooth manifold M, $\dim M = n$, and a locally trivial smooth vector bundle $\pi\colon E \to M$. Denote by $\pi_k\colon J^k(\pi) \to M$, $k = 0, 1, \dots, \infty$, the corresponding bundles of jets. A *differential equation* of order k, $k < \infty$, in the bundle π is a smooth submanifold $\mathcal{E} \subset J^k(\pi)$. To any equation $\mathcal{E}$ there corresponds a series of its *prolongations* $\mathcal{E}^s \subset J^{k+s}(\pi)$ and the *infinite prolongation* $\mathcal{E}^\infty \subset J^\infty(\pi)$. We consider below *formally integrable* equations, which means that all $\mathcal{E}^s$ are smooth manifolds and the natural projection $\pi_\mathcal{E} = \pi_\infty|_{\mathcal{E}^\infty} : \mathcal{E}^\infty \to M$ is a smooth bundle. For any $s > 0$ there also exist natural bundles

$$\mathcal{E}^\infty \xrightarrow{\pi_{\mathcal{E},s}} \mathcal{E}^s \xrightarrow{\pi_{\mathcal{E},s,s-1}} \mathcal{E}^{s-1} \xrightarrow{\pi_{\mathcal{E},s-1}} M \tag{2}$$

whose composition equals $\pi_\mathcal{E}$. The space $J^\infty(\pi)$ is endowed with an integrable distribution[1] denoted by $\mathcal{C}\mathrm{D}(\pi)$. Namely, any point $\theta \in J^\infty(\pi)$ is, by definition, represented in the form $[f]_x^\infty$, $x = \pi_\infty(\theta) \in M$, where f is a (local) section of π such that the graph M_f^∞ of its infinite jet

[1]Integrability in this context means that $\mathcal{C}\mathrm{D}(\pi)$ satisfies the Frobenius condition: $[\mathcal{C}\mathrm{D}(\pi), \mathcal{C}\mathrm{D}(\pi)] \subset \mathcal{C}\mathrm{D}(\pi)$.

passes through θ while $[f]_x^\infty$ is the class of (local) sections f' such that the graph of f' is tangent to the graph of f at $f(x) \in E$ with infinite order.

Then the tangent plane $T_\theta M_f^\infty$ is independent of the choice of f and we set $\mathcal{C}\mathrm{D}(\pi)_\theta = T_\theta M_f^\infty$. The distribution $\mathcal{C}\mathrm{D}(\pi)$ is n-dimensional and is called the *Cartan distribution* on $J^\infty(\pi)$. Since, by construction, all planes of the Cartan distribution are horizontal (with respect to π_∞) and n-dimensional, a connection $\mathcal{C}\colon \mathrm{D}(M) \to \mathrm{D}(\pi)$ is determined, where $\mathrm{D}(M)$ and $\mathrm{D}(\pi)$ denote the modules of vector fields on M and $J^\infty(\pi)$ respectively. This connection is flat and is called the *Cartan connection.*

Remark 1. In fact, the bundle π_∞ possesses a stronger structure than just a flat connection. Namely, for any vector bundles ξ and η over M and a linear differential operator Δ acting from ξ to η, a linear differential operator $\mathcal{C}\Delta$ acting from the pullback $\pi_\infty^*(\xi)$ to $\pi_\infty^*(\eta)$ is defined in a natural way. The correspondence $\Delta \mapsto \mathcal{C}\Delta$ is linear, preserves composition, and the Cartan connection is its particular case.

Both the Cartan distribution and the Cartan connection are restricted to the spaces $\mathcal{E}^\infty$ and bundles $\pi_\mathcal{E}$ respectively. The corresponding objects are denoted by $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ and $\mathcal{C} = \mathcal{C}_\mathcal{E}\colon \mathrm{D}(M) \to \mathrm{D}(\mathcal{E}^\infty)$, where $\mathrm{D}(\mathcal{E}^\infty)$ is the module of vector fields on $\mathcal{E}^\infty$. The characteristic property of the Cartan distribution $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ on $\mathcal{E}^\infty$ is that its maximal integral manifolds are solutions of the equation $\mathcal{E}$ and vice versa. The connection form $U_\mathcal{E} \in \mathrm{D}(\Lambda^1(\mathcal{E}^\infty))$ of the connection $\mathcal{C}_\mathcal{E}$ is called the *structural element* of the equation $\mathcal{E}$. Here $\mathrm{D}(\Lambda^1(\mathcal{E}^\infty))$ denotes the module of derivations $C^\infty(\mathcal{E}^\infty) \to \Lambda^1(\mathcal{E}^\infty)$ with the values in the module of one-forms on $\mathcal{E}^\infty$.

Denote by $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$ the module

$$\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty) = \{X \in \mathrm{D}(\mathcal{E}^\infty) \mid [X, \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)] \subset \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)\}.$$

Then $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$ is a Lie algebra with respect to commutator of vector fields and due to integrability of the Cartan distribution $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is its ideal. The quotient Lie algebra $\operatorname{sym}\mathcal{E} = \mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)/\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is called the *algebra of* (*higher*) *symmetries* of the equation $\mathcal{E}$. Denote by $\mathrm{D}^v(\mathcal{E}^\infty)$ the module of $\pi_\mathcal{E}$-vertical vector fields on $\mathcal{E}^\infty$. Then in any coset $X \bmod \mathcal{C}\mathrm{D}(\mathcal{E}^\infty) \in \operatorname{sym}\mathcal{E}$ there exists a unique vertical element and this element is called a (*higher*) *symmetry* of $\mathcal{E}$.

Remark 2. It may so happen that a coset $X \bmod \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ contains a representative X' which is projectable to a vector field X'_s on $\mathcal{E}^s$ by $\pi_{\mathcal{E},s}$ for some $s < \infty$. In this case, X' is called a *classical* (*infinitesimal*) *symmetry* of $\mathcal{E}$ and possesses trajectories in $\mathcal{E}^\infty$. The corresponding diffeomorphisms preserve solutions of $\mathcal{E}$ and are called *finite symmetries.*

We now pass to a generalization of the above described geometrical theory, the theory of coverings [1, 8, 9]. Let $\tau\colon W \to \mathcal{E}^\infty$ be a smooth fiber bundle, the manifold W being equipped with an integrable distribution $\mathcal{C}_\tau \mathrm{D}(W) = \mathcal{C}\mathrm{D}(W) \subset \mathrm{D}(W)$ of dimension $n = \dim M$. Then τ is called a *covering* over $\mathcal{E}$ (or over $\mathcal{E}^\infty$), if for any point $\theta \in W$ one has $\tau_*(\mathcal{C}\mathrm{D}(W)_\theta) = \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)_{\tau(\theta)}$. Equivalently, a covering structure in the bundle τ is determined by a flat connection $\mathcal{C}_\tau\colon \mathrm{D}(M) \to \mathrm{D}(W)$ satisfying $\tau_* \circ \mathcal{C}_\tau = \mathcal{C}_\mathcal{E}$. Let $U_\tau \in \mathrm{D}(\Lambda^1(W))$ be the corresponding connection form. We call it the *structural element* of the covering τ.

Example 1 (see [12]). Let $\mathcal{E} \subset J^k(\pi)$ be an equation. Consider the tangent bundle $T\mathcal{E}^\infty \to \mathcal{E}^\infty$ and the subbundle $\pi_\mathcal{E}^v\colon T^v\pi_\mathcal{E} \to \mathcal{E}^\infty$, where $T^v\pi_\mathcal{E}$ consists of $\pi_\mathcal{E}$-vertical vectors. Hence, the module of sections for $\pi_\mathcal{E}^v$ consists of $\pi_\mathcal{E}$-vertical vector fields on $\mathcal{E}^\infty$.

Then $\pi_\mathcal{E}^v$ carries a natural covering structure. Namely, for any vector field $X \in \mathrm{D}(M)$ and a vertical vector field Y we set $X(Y) = [\mathcal{C}_\mathcal{E}(X), Y]$. Thus, X acts on sections of $\pi_\mathcal{E}^v$ and defines a vector field $\mathcal{C}_{\pi_\mathcal{E}^v}(X)$ on $T^v\pi_\mathcal{E}$ projected by $(\pi_\mathcal{E}^v)_*$ to $\mathcal{C}_\mathcal{E}(X)$. The connection $\mathcal{C}_{\pi_\mathcal{E}^v}$ is well defined in this way and is projected to the connection $\mathcal{C}_\mathcal{E}$.

Given two coverings $\tau_i\colon W_i \to \mathcal{E}^\infty$, $i = 1, 2$, we say that a smooth mapping $F\colon W_1 \to W_2$ is a *morphism* of τ_1 to τ_2, if

(i) F is a morphism of fiber bundles,
(ii) F_* takes the distribution $\mathcal{C}_{\tau_1}\mathrm{D}(W_1)$ to $\mathcal{C}_{\tau_2}\mathrm{D}(W_2)$ (equivalently, $F_* \circ \mathcal{C}_{\tau_1} = \mathcal{C}_{\tau_2}$).

A morphism F is said to be an *equivalence*, if it is a diffeomorphism.

Similar to the case of infinitely prolonged equations, we can define the Lie algebra $\mathrm{D}_{\mathcal{C}_\tau}(W)$ such that $\mathcal{C}_\tau\mathrm{D}(W)$ is its ideal and introduce the algebra of *nonlocal τ-symmetries* as the quotient $\mathrm{sym}_\tau \mathcal{E} = \mathrm{D}_{\mathcal{C}_\tau}(W)/\mathcal{C}_\tau\mathrm{D}(W)$. Again, in any coset $X \bmod \mathcal{C}_\tau\mathrm{D}(W) \in \mathrm{sym}_\tau \mathcal{E}$ there exists a unique $(\pi_\mathcal{E} \circ \tau)$-vertical representative and it is called a *nonlocal τ-symmetry* of the equation $\mathcal{E}$.

Obviously, one can introduce the notion of a covering over covering, etc. In particular, the subbundle $(\pi_\mathcal{E} \circ \tau)^v\colon T^v(\pi_\mathcal{E} \circ \tau) \to W$ of $(\pi_\mathcal{E} \circ \tau)$-vertical vectors (cf. Example 1) is a covering over W. It is also easily seen that the corresponding integrable distribution is tangent to the submanifold $T^v\tau \subset T^v(\pi_\mathcal{E} \circ \tau)$ of τ-vertical vectors and therefore $\tau^v\colon T^v\tau \to W$ is a covering over W as well. Note that the correspondence $\tau \Rightarrow \tau^v$ determines a covariant functor in the category of coverings.

We shall now reinterpret the concepts of a symmetry and nonlocal symmetry using the results of [12]. Namely, one has

Proposition 1. *Let $\mathcal{E}$ be an equation and $\tau\colon W \to \mathcal{E}^\infty$ be a covering over it. Then:*

(1) *There is a one-to-one correspondence between symmetries of $\mathcal{E}$ and sections $\varphi\colon \mathcal{E}^\infty \to T^v\pi_{\mathcal{E}}$ of the bundle $\pi_{\mathcal{E}}^v\colon T^v\pi_{\mathcal{E}} \to \mathcal{E}^\infty$ such that φ_* takes the Cartan distribution on $\mathcal{E}^\infty$ to that on $T^v\pi_{\mathcal{E}}$.*

(2) *There is a one-to-one correspondence between nonlocal τ-symmetries of $\mathcal{E}$ and sections ψ of the bundle $(\pi_{\mathcal{E}}\circ\tau)^v\colon T^v(\pi_{\mathcal{E}}\circ\tau) \to W$ such that ψ_* takes the Cartan distribution on W to that on $T^v(\pi_{\mathcal{E}}\circ\tau)$.*

Let us say that a mapping $s\colon W \to T^v\pi_{\mathcal{E}}$ is a τ-*shadow* of a nonlocal symmetry (cf. [1], [8] and [9]), if $\pi_{\mathcal{E}}^v \circ s = \tau$ and s_* preserves the Cartan distribution.

Example 2. Every symmetry φ considered as a section $\varphi\colon \mathcal{E}^\infty \to T^v\pi_{\mathcal{E}}$ determines a shadow $\varphi\circ\tau$.

Proposition 2 (The shadow reconstruction theorem). *For an arbitrary covering $\tau\colon W \to \mathcal{E}^\infty$ and a τ-shadow $s\colon W \to T^v\pi_{\mathcal{E}}$ there exists a covering $\tau'\colon W' \to W$ and a nonlocal $\tau\circ\tau'$-symmetry $s'\colon W' \to T^v(\pi_{\mathcal{E}}\circ\tau\circ\tau')$ such that the diagram*

$$
\begin{tikzcd}
 & T^v\pi_{\mathcal{E}} \arrow[d, "\pi_{\mathcal{E}}^v"'] & & T^v(\pi_{\mathcal{E}}\circ\tau\circ\tau') \arrow[ll, "(\tau\circ\tau')_*"'] \arrow[d, shift right, "(\pi_{\mathcal{E}}\circ\tau\circ\tau')^v"'] \\
M & \mathcal{E}^\infty \arrow[l, "\pi_{\mathcal{E}}"'] & W \arrow[l, "\tau"'] \arrow[ul, "s"'] & W' \arrow[l, "\tau'"'] \arrow[u, shift right, "s'"']
\end{tikzcd}
\tag{3}
$$

is commutative. In other words, any shadow can be reconstructed up to a nonlocal symmetry in some new covering.

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
 & T^v\pi_{\mathcal{E}} \arrow[d, "\pi_{\mathcal{E}}^v"'] & T^v(\pi_{\mathcal{E}}\circ\tau) \arrow[l] \arrow[d, "(\pi_{\mathcal{E}}\circ\tau)^v"] & T^v(\pi_{\mathcal{E}}\circ\tau\circ\tau^v) \arrow[l] \arrow[d, "(\pi_{\mathcal{E}}\circ\tau\circ\tau^v)^v"] & \cdots \arrow[l] \\
M & \mathcal{E}^\infty \arrow[l, "\pi_{\mathcal{E}}"] & W \arrow[l, "\tau"] \arrow[ul, "s"] & T^v\tau \arrow[l, "\tau^v"] \arrow[ul, "s_*"] & T^v\tau^v \arrow[l, "(\tau^v)^v"] \arrow[ul, "(s_*)_*"'] & \cdots \arrow[l]
\end{tikzcd}
$$

and let us set $\tau_0 = \tau$, $\tau_{i+1} = \tau_i^v$, $W_0 = W$, $W_i = T^v\tau_i$, $s_0 = s$, $s_{i+1} = (s_i)_*$, where $s_* = ds$. Then the above diagram is infinitely continued to the right, while by setting $\bar\tau_i = \tau_1\circ\cdots\circ\tau_i$ and passing to the inverse limit, we obtain Diagram 3 with $\tau' = \bar\tau_\infty$, $s' = s_\infty$, and $W' = W_\infty$. $\square$

§2. $\mathcal{C}$-complex and deformations

We now pass to describe a cohomological theory naturally related to covering structures and supplying their important invariants, cf. [5] and [8].

Let W be a smooth manifold and $\mathrm{D}(\Lambda^i(W))$ denote the $C^\infty(W)$-module of $\Lambda^i(W)$-valued derivations $C^\infty(W) \to \Lambda^i(W)$. For any element $\Omega \in \mathrm{D}(\Lambda^i(W))$ one can define the *inner product* operation

$$\mathrm{i}_\Omega \colon \Lambda^j(W) \to \Lambda^{i+j-1}(W),$$

also denoted by $\Omega \lrcorner \rho$, $\rho \in \Lambda^*(W)$, and the *Lie derivative* along Ω:

$$\mathrm{L}_\Omega = [\mathrm{i}_\Omega, d] \colon \Lambda^j(W) \to \Lambda^{i+j}(W),$$

where $[\mathrm{i}_\Omega, d]$ denotes the *graded* commutator.

Then for any two elements Ω, $\Theta \in \mathrm{D}(\Lambda^*(W))$ we can introduce their *Frölicher-Nijenhuis bracket* by setting

$$[\![\Omega, \Theta]\!](f) = \mathrm{L}_\Omega(\Theta(f)) - (-1)^{ij}\mathrm{L}_\Theta(\Omega(f)),$$

where $f \in C^\infty(W)$ and i, j are degrees of Ω and Θ respectively[2]. Note also that two other operations are defined: we can multiply elements of $\mathrm{D}(\Lambda^*(W))$ by forms $\rho \in \Lambda^*(W)$ and insert elements of $\mathrm{D}(\Lambda^*(W))$ into each other. Namely, set

$$(\rho \wedge \Theta)(f) = \rho \wedge (\Theta(f)), \quad (\mathrm{i}_\Omega\Theta)(f) = \mathrm{i}_\Omega(\Theta(f)).$$

The basic properties of the above introduced operations are formulated in

Proposition 3 (see [4]). *Let* $\Omega \in \mathrm{D}(\Lambda^i(W))$, $\Theta \in \mathrm{D}(\Lambda^j(W))$, $\rho \in \Lambda^k(W)$, *and* $\eta \in \Lambda^l(W)$. *Then*:

(i) $\mathrm{i}_\Omega(\rho \wedge \eta) = \mathrm{i}_\Omega(\rho) \wedge \eta + (-1)^{(i-1)k}\rho \wedge \mathrm{i}_\Omega(\eta)$;
(ii) $\mathrm{i}_\Omega(\rho \wedge \Theta) = \mathrm{i}_\Omega(\rho) \wedge \Theta + (-1)^{(i-1)k}\rho \wedge \mathrm{i}_\Omega(\Theta)$;
(iii) $[\mathrm{i}_\Omega, \mathrm{i}_\Theta] = \mathrm{i}_{[\![\Omega,\Theta]\!]^{\mathrm{rn}}}$, *where*

$$[\![\Omega, \Theta]\!]^{\mathrm{rn}} = \mathrm{i}_\Omega\Theta - (-1)^{(i-1)(j-1)}\mathrm{i}_\Theta\Omega$$

is the Richardson-Nijenhuis bracket *of* Ω *and* Θ;
(iv) $\mathrm{L}_\Omega(\rho \wedge \eta) = \mathrm{L}_\Omega(\rho) \wedge \eta + (-1)^{ik}\omega \wedge \mathrm{L}_\Omega(\eta)$;
(v) $\mathrm{L}_{\rho\wedge\Omega} = \rho \wedge \mathrm{L}_\Omega + (-1)^{i+k}d\omega \wedge \mathrm{i}_\Omega$;
(vi) $[\mathrm{L}_\Omega, d] = 0$;

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[2]We say that i is the degree of Ω, if $\Omega \in \mathrm{D}(\Lambda^i(W))$.

(vii) $[\mathrm{L}_\Omega, \mathrm{L}_\Theta] = \mathrm{L}_{[\![\Omega,\Theta]\!]}$;
(viii) $[\![\Omega, \Theta]\!] + (-1)^{ij}[\![\Theta, \Omega]\!] = 0$;
(ix) $[\![\Omega, [\![\Theta, \Xi]\!]]\!] = [\![[\![\Omega, \Theta]\!], \Xi]\!] + (-1)^{ij}[\![\Omega, [\![\Theta, \Xi]\!]]\!]$, *where* $\Xi \in \mathrm{D}(\Lambda^m(W))$;
(x) $[\mathrm{L}_\Omega, \mathrm{i}_\Theta] = \mathrm{i}_{[\![\Omega,\Theta]\!]} - (-1)^{i(j+1)}\mathrm{L}_{\Theta \lrcorner \Omega}$;
(xi) $\Xi \lrcorner [\![\Omega, \Theta]\!] = [\![\Xi \lrcorner \Omega, \Theta]\!] + (-1)^{i(m+1)}[\![\Omega, \Xi \lrcorner \Theta]\!] + (-1)^i[\![\Xi, \Omega]\!] \lrcorner \Theta - (-1)^{(i+1)j}[\![\Xi, \Theta]\!] \lrcorner \Omega$;
(xii) $[\![\Omega, \rho\wedge\Theta]\!] = (\mathrm{L}_\Omega\rho)\wedge\Theta - (-1)^{(i+1)(j+k)}d\rho\wedge\mathrm{i}_\Theta\Omega + (-1)^{ik}\rho\wedge[\![\Omega, \Theta]\!]$.

In particular, from Proposition 3 (ix) it follows that for $\Omega \in \mathrm{D}(\Lambda^1(W))$ satisfying the *integrability property*

$$[\![\Omega, \Omega]\!] = 0 \tag{4}$$

the mapping

$$\partial_\Omega = [\![\Omega, \cdot\,]\!] \colon \mathrm{D}(\Lambda^i(W)) \to \mathrm{D}(\Lambda^{i+1}(W))$$

is a differential, i.e., $\partial_\Omega \circ \partial_\Omega = 0$, and thus we obtain the complex

$$0 \to \mathrm{D}(W) \to \cdots \to \mathrm{D}(\Lambda^i(W)) \xrightarrow{\partial_\Omega} \mathrm{D}(\Lambda^{i+1}(W)) \to \cdots \tag{5}$$

Assume now that the manifold W is fibered by $\xi \colon W \to M$ and a connection ∇ is given in the bundle ξ. Then the following fact is valid:

Proposition 4 (cf. [3]).

$$[\![U_\nabla, U_\nabla]\!] = 2R_\nabla,$$

where U_∇ *is the connection form and* R_∇ *is the curvature.*

Consequently, if ∇ is a flat connection, i.e., $R_\nabla = 0$, then $\Omega = U_\nabla$ enjoys the integrability property (4) and to any flat connection a complex of the form (5) corresponds. In this case, we shall use the notation $\partial_\Omega = \partial_\nabla$.

Now, we pass to the case of our main interest: let ξ be the composition $W \xrightarrow{\tau} \mathcal{E}^\infty \xrightarrow{\pi_\mathcal{E}} M$, τ being a covering over $\mathcal{E}$, and ∇ be the Cartan connection $\mathcal{C}_\tau$ associated to the covering structure. We include in consideration the case $W = \mathcal{E}^\infty$, $\tau = \mathrm{id}$, and $\mathcal{C}_\tau = \mathcal{C}_\mathcal{E}$. Let us restrict complex (5) to *vertical* derivations, i.e., to derivations

$$\mathrm{D}^v(\Lambda^i(W)) = \{\Omega \in \mathrm{D}(\Lambda^i(W)) \mid \Omega(f) = 0,\ \forall f \in C^\infty(M)\}.$$

By construction, U_τ (or $U_\mathcal{E}$) lies in $\mathrm{D}^v(\Lambda^1(W))$ (resp., in $\mathrm{D}^v(\Lambda^1(\mathcal{E}^\infty))$), while from the definition of the Frölicher-Nijenhuis bracket it follows that the differential in (5) preserves vertical derivations. The vertical part of (5) will be denoted by

$$0 \to \mathrm{D}^v(W) \to \cdots \to \mathrm{D}^v(\Lambda^i(W)) \xrightarrow{\partial_\tau} \mathrm{D}^v(\Lambda^{i+1}(W)) \to \cdots \tag{6}$$

or

$$0 \to \mathrm{D}^v(\mathcal{E}^\infty) \to \cdots \to \mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty)) \xrightarrow{\partial_\mathcal{E}} \mathrm{D}^v(\Lambda^{i+1}(\mathcal{E}^\infty)) \to \cdots, \tag{7}$$

when the equation is considered as is. The cohomology of (6) (resp., of (7)) is denoted by $H_\mathcal{C}(\mathcal{E};\tau)$ (resp., by $H_\mathcal{C}(\mathcal{E})$) and is called the *$\mathcal{C}$-cohomology* of the covering τ (resp., of the equation $\mathcal{E}$). The following fundamental result is valid:

Theorem 1 (cf. [5]). *Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\tau\colon W \to \mathcal{E}^\infty$ be a covering over $\mathcal{E}$. Then*:

1. *The module $H^0_\mathcal{C}(\mathcal{E};\tau)$ is isomorphic to the Lie algebra* $\mathrm{sym}_\tau \mathcal{E}$ *of nonlocal τ-symmetries (resp., $H^0_\mathcal{C}(\mathcal{E})$ is isomorphic to* $\mathrm{sym}\,\mathcal{E}$*).*
2. *The module $H^1_\mathcal{C}(\mathcal{E};\tau)$ is identified with equivalence classes of nontrivial infinitesimal deformations of the covering structure U_τ (resp., of the equation structure $U_\mathcal{E}$).*
3. *The module $H^2_\mathcal{C}(\mathcal{E};\tau)$ consists of obstructions to prolongation of infinitesimal deformations up to formal ones.*

Remark 3. Of course, if U_λ is a deformation of the equation structure, the condition that $dU_\lambda/d\lambda|_{\lambda=0}$ lies in $\ker \partial_\mathcal{E}$ is not sufficient for this deformation to be trivial. Nevertheless, the following fact is obviously valid:

Proposition 5. *Let U_λ be a smooth deformation of the equation structure $U = U_\mathcal{E}$ satisfying the condition*

$$\frac{dU_\lambda}{d\lambda} = [\![X_\lambda, U_\lambda]\!], \tag{8}$$

where X_λ is a smooth vector field on $\mathcal{E}^\infty$ for any λ with smooth dependence on λ. Then U_λ is uniquely defined by (8) *and is of the form*

$$U_\lambda = \exp\Big(\int_0^\lambda X_\mu \, d\mu\Big) U,$$

where the left and right hand sides are understood as formal series. In this sense, U_λ is formally trivial.

Let us now consider the mapping $\mathrm{L}_{U_\tau}\colon \Lambda^i(W) \to \Lambda^{i+1}(W)$ and denote it by $d_\mathcal{C}$. Since the element U_τ is integrable, one has the identity

$d_{\mathcal{C}} \circ d_{\mathcal{C}} = 0$. We call $d_{\mathcal{C}}$ the *vertical*, or *Cartan differential* associated to the covering structure. Due to Proposition 3 (vi), $[d, d_{\mathcal{C}}] = 0$ and consequently the mapping $d_h = d - d_{\mathcal{C}}$ is also a differential and $[d_h, d_{\mathcal{C}}] = 0$. The differential d_h is called the *horizontal differential*, while the pair $(d_h, d_{\mathcal{C}})$ forms a bicomplex with the total differential d. The corresponding spectral sequence coincides with the Vinogradov $\mathcal{C}$-spectral sequence for the covering τ, [1], [8] and [14].

Denote by $\Lambda_h^1(W)$ the submodule in $\Lambda^1(W)$ spanned by $\operatorname{im} d_h$ and by $\mathcal{C}^1\Lambda(W)$ the submodule generated by $\operatorname{im} d_{\mathcal{C}}$. Then the direct sum decomposition $\Lambda^1(W) = \Lambda_h^1(W) \oplus \mathcal{C}\Lambda^1(W)$ takes place and generates the decomposition

$$\Lambda^i(W) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda(W) \otimes \Lambda_h^q(W) = \bigoplus_{p+q=i} \Lambda^{p,q}(W),$$

where

$$\mathcal{C}^p\Lambda(W) = \underbrace{\mathcal{C}^1\Lambda(W) \wedge \cdots \wedge \mathcal{C}^1\Lambda(W)}_{p \text{ times}}, \quad \Lambda_h^q(W) = \underbrace{\Lambda_h^1(W) \wedge \cdots \wedge \Lambda_h^1(W)}_{q \text{ times}}.$$

Then $d_{\mathcal{C}} \colon \Lambda^{p,q}(W) \to \Lambda^{p+1,q}(W)$, $d_h \colon \Lambda^{p,q}(W) \to \Lambda^{p,q+1}(W)$ and, moreover, as it follows from Proposition 3 (xi), $\partial_\tau \colon \mathrm{D}^v(\Lambda^{p,q}(W)) \to \mathrm{D}^v(\Lambda^{p,q+1}(W))$.

Remark 4. The complex $(\Lambda_h^q(W), d_h)$ is called the *horizontal complex* of the covering τ, while its cohomology is the *horizontal cohomology* of τ. It is worth to note that d_h in this case is obtained from the de Rham differential on the manifold M by applying the operation $\mathcal{C} = \mathcal{C}_\tau$ (see Remark 1). From Proposition 3 (xii) it follows that the $\mathcal{C}$-cohomology of τ is a graded module over the graded algebra of horizontal cohomology.

§3. Bäcklund transformations and the main result

Following [9], let us give a geometric definition of Bäcklund transformations. Let $\mathcal{E}_i \subset J^{k_i}(\pi_i)$, $i = 1, 2$, be two differential equations and $\tau_i \colon W \to \mathcal{E}_i^\infty$ be coverings with the same total space W endowed with an integrable distribution $\mathcal{C}\mathrm{D}(W)$. Then the diagram

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{\tau_1} & & \searrow^{\tau_2} & \\ \mathcal{E}_1^\infty & & & & \mathcal{E}_2^\infty \end{array}$$

is called a *Bäcklund transformation* between the equations $\mathcal{E}_1^\infty$ and $\mathcal{E}_2^\infty$. A point $w \in W$ is called *generic*, if the intersection of the tangent planes to the fibres of τ_1 and τ_2 passing through w is trivial. Now, if $s \subset \mathcal{E}_1^\infty$ is a solution of $\mathcal{E}_1$ and $\tau_1^{-1}(s)$ contains a generic point, then there exists a neighborhood $\mathcal{U}$ of this point such that $\tau_2(\mathcal{U} \cap \tau_1^{-1}(s))$ is fibered by solutions of $\mathcal{E}_2$. Thus, Bäcklund transformations really determine a correspondence between solutions.

The following construction is equivalent to the definition of Bäcklund transformations. Let $\tau_i \colon W_i \to \mathcal{E}_i^\infty$, $i = 1, 2$, be two coverings and $F \colon W_1 \to W_2$ be a diffeomorphism taking the distribution $\mathcal{C}_{\tau_1}\mathrm{D}(W)$ to $\mathcal{C}_{\tau_2}\mathrm{D}(W)$. Then $\mathcal{B} = (W, \tau_1, \tau_2 \circ F, \mathcal{E}_1, \mathcal{E}_2)$ is a Bäcklund transformation and any Bäcklund transformations is obtained in this way.

Remark 5. It is important to stress here that if F is an isomorphism of coverings, then the Bäcklund transformation obtained in such a way is *trivial* in the sense of its action on solutions. Thus, we are interested in mappings F which are isomorphisms of manifolds with distributions, but not morphisms of coverings.

Assume now that a smooth family $F_\lambda \colon W_1 \to W_2$ of diffeomorphisms is given, then it generates a family $\mathcal{B}_\lambda$ of Bäcklund transformations. Our aim is to describe such families in sufficiently efficient terms. One way to construct these objects is given by the following

Example 3 (see [9]). Consider an equation $\mathcal{E}$, a covering $\tau \colon W \to \mathcal{E}^\infty$ over it, and a finite symmetry $A \colon \mathcal{E}^\infty \to \mathcal{E}^\infty$. Let $\bar{A} \colon W \to W$ be a diffeomorphic lifting of A to W such that

$$\tau \circ \bar{A} = A \circ \tau. \tag{9}$$

Denote by $\bar{A}_*\mathcal{C}_\tau\mathrm{D}(W)$ the image of the distribution $\mathcal{C}_\tau\mathrm{D}(W)$ under $\bar{A}$. Then, by obvious reasons, $\bar{A}_*\mathcal{C}_\tau\mathrm{D}(W)$ determines a covering structure $U_\tau^{\bar{A}}$ in W and if $\tilde{A}$ is another lifting of A, then the structures $U_\tau^{\bar{A}}$ and $U_\tau^{\tilde{A}}$ are equivalent. Thus, $\mathcal{B}_A = (W, \tau, A \circ \tau, \mathcal{E})$ is a Bäcklund transformation for $\mathcal{E}$.

Let X be a classical infinitesimal symmetry of $\mathcal{E}$ and $A_\lambda = \exp(\lambda X) \colon \mathcal{E}^\infty \to \mathcal{E}^\infty$ the corresponding one-parameter group of transformations lifted to $\mathcal{E}^\infty$. Then, using the above construction, we obtain a one-parametric family of Bäcklund transformations $\mathcal{B}_\lambda = \mathcal{B}_{A_\lambda}$.

In fact, the families of Bäcklund transformations obtained in the previous example are in a sense "counterfeit", since, due to (9), their action on solutions reduces to the action of symmetries A_λ. To get a "real" Bäcklund transformation, one needs to add into considerations an additional diffeomorphism $F\colon W \to W$ such that $\tau \circ F$ is a covering, i.e., τ_* projects $F_*(\mathcal{C}_\tau \mathrm{D}(W))$ onto $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$, but F does not preserve the fibres of τ.

Example 4. Consider the infinite prolongation $\mathcal{E}^\infty$ of the sine-Gordon equation

$$u_{xy} = \sin u$$

and the trivial bundle

$$\tau\colon W = \mathcal{E}^\infty \otimes \mathbb{R} \to \mathcal{E}^\infty$$

with a coordinate v along fibres. Then the distribution $\mathcal{C}_\lambda \mathrm{D}(W)$ spanned by the vector fields $D_x + X$ and $D_t + T$, where $D_x = \mathcal{C}(\partial/\partial x)$, $D_t = \mathcal{C}(\partial/\partial t)$ are total derivatives and

$$X = \left(-u_x + 2\lambda \sin\frac{u-v}{2}\right)\frac{\partial}{\partial v},$$
$$T = \left(u_t + \frac{2}{\lambda}\sin\frac{u+v}{2}\right)\frac{\partial}{\partial v},$$

$\lambda \neq 0$, determines a one-dimensional covering structure on the bundle τ. The manifold W with this distribution is isomorphic to the infinite prolongation of the system

$$\begin{aligned} v_x &= -u_x + 2\lambda \sin\frac{u-v}{2}, \\ v_t &= u_t + \frac{2}{\lambda}\sin\frac{u+v}{2}. \end{aligned} \tag{10}$$

This system has a finite symmetry $F_\lambda\colon W \to W$ acting on the coordinates as follows $F_\lambda^*(u) = v$, $F_\lambda^*(v) = u$, $F_\lambda^*(x) = -x$, $F_\lambda^*(t) = -t$. Consider also the group $A_\lambda\colon x \mapsto \lambda x$, $t \mapsto \lambda^{-1} t$ of scale symmetries of the sine-Gordon equation and denote by $\bar{A}_\lambda\colon W \to W$ the diffeomorphic lifting of A_λ acting trivially on the coordinate v. Then

$$\mathcal{C}_\lambda \mathrm{D}(W) = \bar{A}_{\lambda,*}(\mathcal{C}_1 \mathrm{D}(W)), \quad F_\lambda = \bar{A}_\lambda \circ F_1 \circ \bar{A}_{\lambda^{-1}},$$

and $(W, \tau, \tau \circ \bar{A}_{-1} \circ F_\lambda, \mathcal{E}^\infty)$ is the family of the classical Bäcklund transformations: if $u(x,t)$ is a solution of $\mathcal{E}$ then every solution $v(x,t)$ of (10) satisfies the sine-Gordon equation as well.

Example 5. Consider now the potential KdV equation $\mathcal{E}$

$$u_t = -u_{xxx} - 3u_x^2$$

and the bundle

$$\tau\colon W = \mathcal{E}^\infty \otimes \mathbb{R} \to \mathcal{E}^\infty$$

with a coordinate v along fibres. The distribution $\mathcal{C}_\lambda\mathrm{D}(W)$ generated by $D_x + X$ and $D_t + T$, where

$$X = -\Big(u_x + \frac{1}{2}(v-u)^2 + 2\lambda\Big)\frac{\partial}{\partial v},$$
$$T = \big(u_{xxx} + u_x^2 - 4\lambda u_x - 8\lambda^2 + 2u_{xx}(u-v) + (u_x - 2\lambda)(u-v)^2\big)\frac{\partial}{\partial v},$$

$\lambda \in \mathbb{R}$, defines a one-dimensional covering structure on τ. There is a finite symmetry $F_\lambda\colon W \to W$, $F_\lambda^*(u) = v$, $F_\lambda^*(v) = u$, preserving $\mathcal{C}_\lambda\mathrm{D}(W)$. Then $(W, \tau, \tau\circ F_\lambda, \mathcal{E}^\infty)$ is the one-parameter family of Bäcklund transformations

$$v_x = -u_x - \frac{1}{2}(v-u)^2 - 2\lambda,$$
$$v_t = u_{xxx} + u_x^2 - 4\lambda u_x - 8\lambda^2 + 2u_{xx}(u-v) + (u_x - 2\lambda)(u-v)^2$$

constructed by Wahlquist and Estabrook [15]. Consider the group

$$A_\lambda\colon u(x,t) \mapsto u(x - 6\lambda t, t) + \lambda x - 3\lambda^2 t \tag{11}$$

of symmetries of the potential KdV equation and denote by $\bar{A}_\lambda\colon W \to W$ the diffeomorphic lifting of A_λ acting on v as follows

$$\bar{A}_\lambda^*(v) = v - (\lambda x - 3\lambda^2 t).$$

Then we similarly have

$$\mathcal{C}_\lambda\mathrm{D}(W) = \bar{A}_{\lambda,*}(\mathcal{C}_0\mathrm{D}(W)) \text{ and } F_\lambda = \bar{A}_\lambda \circ F_0 \circ \bar{A}_{-\lambda}.$$

Let us denote by

$$\mathrm{D}^g(\Lambda^i(W)) = \{\Omega \in \mathrm{D}^v(\Lambda^i(W)) \mid \Omega(f) = 0,\ \forall f \in C^\infty(\mathcal{E}^\infty)\}$$

the module of τ-vertical derivations.

Lemma 1. *The modules $\mathrm{D}^g(\Lambda^i(W))$ are invariant with respect to the differential ∂_τ:*

$$\partial_\tau\big(\mathrm{D}^g(\Lambda^i(W))\big) \subset \mathrm{D}^g(\Lambda^{i+1}(W)).$$

Proof. Let $\Omega \in \mathrm{D}^g(\Lambda^i(W))$ and $f \in C^\infty(\mathcal{E}^\infty)$. Then due to the definition of the Frölicher-Nijenhuis bracket one has

$$(\partial_\tau(\Omega))(f) = [\![U_\tau, \Omega]\!](f) = \mathrm{L}_{U_\tau}(\Omega(f)) - (-1)^\Omega \mathrm{L}_\Omega(U_\tau(f)).$$

The first summand vanishes, since $\Omega \in \mathrm{D}^g(\Lambda^i(W))$. On the other hand, $U_\tau(f) = U_\mathcal{E}(f)$ and consequently is a one-form on $\mathcal{E}^\infty$. Hence, the second summand vanishes as well. □

Denote by $\partial_g\colon \mathrm{D}^g(\Lambda^i(W)) \to \mathrm{D}^g(\Lambda^{i+1}(W))$ the restriction of ∂_τ to $\mathrm{D}^g(\Lambda^i(W))$ and by

$$\partial_s\colon \mathrm{D}^s(\Lambda^i(W)) \to \mathrm{D}^s(\Lambda^{i+1}(W))$$

the corresponding quotient complex, where, by definition,

$$\mathrm{D}^s(\Lambda^i(W)) = \mathrm{D}^v(\Lambda^i(W))/\mathrm{D}^g(\Lambda^i(W)).$$

Then the short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & & 0 & & 0 \\
 & \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
0 \to & \mathrm{D}^g(W) & \xrightarrow{\partial_g} & \mathrm{D}^g(\Lambda^1(W)) & \to \cdots \to & & \mathrm{D}^g(\Lambda^i(W)) & \xrightarrow{\partial_g} & \mathrm{D}^g(\Lambda^{i+1}(W)) \to \cdots \\
 & \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
0 \to & \mathrm{D}^v(W) & \xrightarrow{\partial_\tau} & \mathrm{D}^v(\Lambda^1(W)) & \to \cdots \to & & \mathrm{D}^v(\Lambda^i(W)) & \xrightarrow{\partial_\tau} & \mathrm{D}^v(\Lambda^{i+1}(W)) \to \cdots \\
 & \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
0 \to & \mathrm{D}^s(W) & \xrightarrow{\partial_s} & \mathrm{D}^s(\Lambda^1(W)) & \to \cdots \to & & \mathrm{D}^s(\Lambda^i(W)) & \xrightarrow{\partial_s} & \mathrm{D}^s(\Lambda^{i+1}(W)) \to \cdots \\
 & \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
 & 0 & & 0 & & & 0 & & 0
\end{array}$$

is defined.

Denote by $H^i_g(\mathcal{E};\tau)$ and $H^i_s(\mathcal{E};\tau)$ the cohomology of the top and bottom lines respectively. Then one has the long exact cohomology sequence

$$\begin{aligned}
&0 \to H^0_g(\mathcal{E};\tau) \to H^0_{\mathcal{C}}(\mathcal{E};\tau) \to H^0_s(\mathcal{E};\tau) \\
&\xrightarrow{\phi} H^1_g(\mathcal{E};\tau) \to H^1_{\mathcal{C}}(\mathcal{E};\tau) \to H^1_s(\mathcal{E};\tau) \to \cdots \\
&\to H^i_g(\mathcal{E};\tau) \to H^i_{\mathcal{C}}(\mathcal{E};\tau) \to H^i_s(\mathcal{E};\tau) \to \cdots ,
\end{aligned} \tag{12}$$

where ϕ is the connecting homomorphism.

Similar to Theorem 1, we have the following result:

Proposition 6. *In the situation above one has:*

(1) *The module $H^0_g(\mathcal{E};\tau)$ consists of "gauge" symmetries in the covering τ, i.e., of nonlocal τ-symmetries vertical with respect to the projection τ.*
(2) *The module $H^0_s(\mathcal{E};\tau)$ coincides with the set of τ-shadows in the covering τ.*
(3) *The module $H^1_g(\mathcal{E};\tau)$ consists of equivalence classes of deformations of the covering structure U_τ acting trivially on the equation structure $U_{\mathcal{E}}$.*

Now, combining the last result with exact sequence (12), we obtain the following fundamental theorem:

Theorem 2. *Let $\tau\colon W \to \mathcal{E}$ be a covering and $A_\lambda\colon W \to W$ be a smooth family of diffeomorphisms such that $A_0 = \mathrm{id}$ and $\tau_\lambda = \tau \circ A_\lambda\colon W \to \mathcal{E}$ is a covering for any $\lambda \in \mathbb{R}$. Then U_{τ_λ} is of the form*

$$U_{\tau_\lambda} = U_\tau + \lambda[\![U_\tau, X]\!] + O(\lambda^2), \tag{13}$$

where X is a τ-shadow, i.e., all smooth families corresponding to the covering τ are infinitesimally identified with $\operatorname{im}\partial_s$.

Proof. The family of coverings τ_λ is a deformation of τ. Since we work with deformations which leave the equation structure unchanged, then, by Proposition 6, their infinitesimal parts are elements of $H^1_g(\mathcal{E};\tau)$. Let Ω be such an element.

Now, by Remark 5, the deformation we are dealing with is to be trivial as a deformation of W endowed with the structure U_τ. On the infinitesimal level, this means that the image of Ω in $H^1(\mathcal{E};\tau)$ should vanish. But by exactness of (12) we see that $\Omega = \phi(X)$ for some $X \in H^0_s(\mathcal{E};\tau)$. It now suffices to note that by construction of the connecting homomorphism, $\phi(X) = [\![U_\tau, X]\!]$.

The family A_λ allows us to find the shadow explicitly. Namely, we obviously have

$$\begin{aligned}\left.\frac{d}{d\lambda}\right|_{\lambda=0} U_{\tau_\lambda} &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} A_{\lambda,*}(\mathrm{L}_{U_\tau})\\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} A^*_\lambda \circ \mathrm{L}_{U_\tau} \circ (A^*_\lambda)^{-1} = [\mathrm{L}_Y, \mathrm{L}_{U_\tau}] = \mathrm{L}_{[\![Y,U_\tau]\!]},\end{aligned}$$

where

$$Y = \left.\frac{dA_\lambda}{d\lambda}\right|_{\lambda=0} \in \mathrm{D}(W).$$

Hence, infinitesimal action is given by the Frölicher-Nijenhuis bracket. In the coset $Y \bmod \mathcal{C}_\tau\mathrm{D}(W)$ there exists a unique $(\pi_{\mathcal{E}} \circ \tau)$-vertical representative X, and the corresponding element $[X] \in H^0_s(\mathcal{E};\tau)$ is the required shadow. □

Remark 6. Consider the one-parameter families of coverings τ_λ and τ'_λ from Examples 4 and 5 respectively. The classical infinitesimal symmetries corresponding to the one-parameter groups A_λ of finite symmetries are

$$x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} \qquad \text{for the sine-Gordon equation,}$$
$$x\frac{\partial}{\partial u} - 6t\frac{\partial}{\partial x} \qquad \text{for the potential KdV equation.}$$

The corresponding higher symmetries are shadows in τ_1 and τ_0' respectively (see Example 2). These shadows determine the infinitesimal parts of the families U_{τ_λ} and $U_{\tau_\lambda'}$ according to Theorem 2.

Remark 7. Denote by $\mathcal{C}ov(\tau)$ the "manifold" of all coverings obtained from the covering τ by the above described way. Then from exactness of (12) it follows that the "tangent plane" to $\mathcal{C}ov(\tau)$ at τ is identified with the space $\mathrm{shad}_\tau \mathcal{E} / \overline{\mathrm{sym}}_\tau \mathcal{E}$, where $\mathrm{shad}_\tau \mathcal{E} = H_s^0(\mathcal{E};\tau)$ is the space of all τ-shadows. Finally, the space $\overline{\mathrm{sym}}_\tau \mathcal{E} = \mathrm{sym}_\tau \mathcal{E} / \mathrm{sym}_\tau^g \mathcal{E}$ is the quotient of all τ-symmetries over gauge ones.

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Sergei Igonin
Independent University of Moscow,
University of Twente
the Netherlands
igonin@mccme.ru

Joseph Krasil′shchik
The Diffiety Institute
the Independent University of Moscow
Correspondence to:
1st Tverskoy-Yamskoy per. 14, Apt. 45
125047 Moscow
Russia
josephk@online.ru